Separation of Variables for a Finite String

(1) Set up the Equations for Harmonic Oscillating Modes

(PROBLEM) Consider a finite vibrating string with fixed ends:

(1) \[ u_{tt} - c^2 u_{xx} = 0 \quad 0 < x < L, t \in \mathbb{R}, \]
(2) \[ u(0, t) = 0, u(L, t) = 0 \quad t \in \mathbb{R}. \]

(IDEA: Separation of Variables) We look for solutions whose spatial oscillating structure remains invariant with time. In these solutions only the amplitudes of the oscillation may change with time. Mathematically, we try to classify all solutions of the following form:

\[ u(x, t) = T(t)X(x). \]

It turns out that such product solutions are exactly the harmonic modes. This not only gives us a particularly interesting family of special solutions, but it also helps to solve the whole initial-boundary value problem. The general solutions are obtained as linear superpositions of these harmonic modes.

(REWRITING THE EQUATIONS FOR PRODUCT SOLUTIONS) The nontrivial product solutions we are looking for are of the form:

\[ u(x, t) = T(t)X(x) \neq 0. \]

Plugging this in equations (1)-(2) we see: There is a constant \( \alpha \) such that

(3) \[ T''(t) - \alpha c^2 T(t) = 0 \quad t \in \mathbb{R}; \]
(4) \[ X''(x) - \alpha X(x) = 0 \quad 0 < x < L, \]
(5) \[ X(0) = 0, X(L) = 0. \]

We look for solutions of (3)-(4)-(5) that are \( \neq 0 \). Here, (3) and (4) are deduced from (1), and (5) follows from (2).

(EIGENVALUES AND EIGENFUNCTIONS) Now focus on the boundary value problem (4)-(5) for \( X(x) \). The solution structure of this problem depends on the parameter value \( \alpha \). It can be shown that for most choices of \( \alpha \), (4)-(5) only has the trivial solution \( X(x) \equiv 0 \). The special values of \( \alpha \) admitting nontrivial solutions are called eigenvalues of (4)-(5), and in that case, the corresponding nontrivial solutions \( X(x) \neq 0 \) are called eigenfunctions. In this particular problem, it can be shown that the eigenvalues of (4)-(5) are

\[ \alpha_n = -(n\pi/L)^2 \quad n = 1, 2, \cdots, \]

and the corresponding eigenfunctions are nonzero constant multiples of

\[ \phi_n(x) = \sin(n\pi x/L). \]

(We will explain how to compute eigenvalues in next lecture.)

Let \( \alpha \) be one of the eigenvalues, say, \( \alpha = \alpha_n = -(n\pi/L)^2 \). We now solve equation (3), which becomes

\[ T''(t) + (n\pi c/L)^2 T(t) = 0 \quad t \in \mathbb{R}. \]
The solutions are

\[ T(t) = a \cos(n\pi ct/L) + b \sin(n\pi ct/L). \]

**CLASSIFICATION OF PRODUCT SOLUTIONS** Thus, the product solutions are completely classified: a product solution of (1)-(2) must be of the form

\[ \sin(n\pi x/L) \left[ a \cos(n\pi ct/L) + b \sin(n\pi ct/L) \right]. \]

These solutions correspond to the harmonic oscillations.

**FROM PRODUCT SOLUTIONS TO GENERAL SOLUTIONS** The general oscillations of a finite vibrating string are linear superpositions of harmonic modes; or, the general solutions are linear combinations of the product solutions.

The general solutions of problem (1)-(2) are of the form

\[ (6) \quad u(x, t) = \sum_{n=1}^{\infty} \sin(n\pi x/L) \left[ a_n \cos(n\pi ct/L) + b_n \sin(n\pi ct/L) \right], \]

where \( a_n \) and \( b_n \) are constants.

**SOLVE THE INITIAL-BOUNDARY VALUE PROBLEM** This allows us to solve the initial-boundary value problem for a finite vibrating string with fixed ends:

\[ \begin{align*}
(7) & \quad u_{tt} - c^2 u_{xx} = 0 \quad 0 < x < L, t \in \mathbb{R}, \\
(8) & \quad u(0, t) = 0, u(L, t) = 0 \quad t \in \mathbb{R}, \\
(9) & \quad u(x, 0) = f(x), u_t(x, 0) = g(x) \quad 0 \leq x \leq L.
\end{align*} \]

We only need to match equation (6) with the initial conditions (9):

\[ \begin{align*}
(10) & \quad f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L), \\
(11) & \quad g(x) = \sum_{n=1}^{\infty} b_n (n\pi c/L) \sin(n\pi x/L).
\end{align*} \]

These are Fourier series expansions of \( f \) and \( g \). In order (10)-(11) to hold, we must have

\[ a_k = \frac{2}{L} \int_0^L f(x) \sin(k\pi x/L) dx, \quad b_k (k\pi c/L) = \frac{2}{L} \int_0^L g(x) \sin(k\pi x/L) dx, \]

for every \( k \). (The derivation of the formula for \( a_k \): multiply the two sides of (10) by \( \sin(k\pi x/L) \) and then integrate the two sides on \([0, L]\).)

**SOLUTION FORMULA FOR THE INIT-BDRY VAL PROBLEM** In summary, the solution of the initial-boundary value problem (7)-(8)-(9) is

\[ (12) \quad u(x, t) = \sum_{n=1}^{\infty} \sin(n\pi x/L) \left[ a_n \cos(n\pi ct/L) + b_n \sin(n\pi ct/L) \right], \]

where

\[ \begin{align*}
(13) & \quad a_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L) dx, \quad b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin(n\pi x/L) dx.
\end{align*} \]
EXERCISES

[1] (a) Let \( \alpha = -(n\pi/L)^2 \) where \( n \) is a positive integer. Verify that the solutions of (4)-(5) are \( X(x) = C \sin(n\pi x/L) \).

(b) Let \( \alpha = 0 \). Verify that the only solution of (4)-(5) is \( X(x) \equiv 0 \).

(c) Let \( \alpha > 0 \). Verify that the only solution of (4)-(5) is \( X(x) \equiv 0 \).

[2] Graph the following product solutions of (1)-(2) for several different times:

(a) \( u_1(x, t) = \sin(\pi x/L) \cos(\pi ct/L) \);

(b) \( u_3(x, t) = \sin(3\pi x/L) \sin(3\pi ct/L) \);

(c) \( u_4(x, t) = \sin(4\pi x/L) \left[ \cos(4\pi ct/L) + \sin(4\pi ct/L) \right] \).

(The animation would be even better to see the solution behavior. All these solutions are time periodic oscillations. Try to observe each solution over one oscillating period.)

[3] Consider (7)-(8)-(9) with the following initial conditions. Find the solution \( u(x, t) \) expressed in a Fourier series.

(a) \( f(x) = 0 \), \( g(x) = v_0 \) (constant).

(b) \( f(x) = 3 \sin(4\pi x/L) \), \( g(x) = 0 \).

(c) \( f(x) = \begin{cases} 4hx/L & \text{for } 0 \leq x < L/4, \\ 2h - 4hx/L & \text{for } L/4 \leq x < L/2, \\ 0 & \text{for } L/2 \leq x \leq L. \end{cases} \), \( g(x) = 0 \).

[4] Consider the initial boundary value problem for a finite vibrating string with free ends:

\[
\begin{align*}
  &u_{tt} - c^2 u_{xx} = 0, & 0 < x < L, \ t \in \mathbb{R}, \\
  &u_x(0, t) = 0, \ u_x(L, t) = 0, & t \in \mathbb{R}.
\end{align*}
\]

Set up the equations for all nontrivial product solutions, similar to equations (3)-(4)-(5). (Don’t need to solve them until next lecture.)

[5] Consider the following initial boundary value problem:

\[
\begin{align*}
  &u_{tt} + u_t - u_{xx} = 0, & 0 < x < 1, \ t \in \mathbb{R}, \\
  &u_x(0, t) = 0, \ u(1, t) = 0, & t \in \mathbb{R}.
\end{align*}
\]

Set up the equations for all nontrivial product solutions, similar to equations (3)-(4)-(5). (Don’t need to solve them until next lecture.)

(See next page for the answers)
**Answers:**

[1] (a) When \( \alpha = -(n\pi/L)^2 \), equation becomes:

\[
X''(x) + (n\pi/L)^2 X(x) = 0.
\]

Write down the general solutions of this equation and examine the boundary conditions (5).

(b) and (c) are similar.


[3] (a) \( u(x, t) = \sum_{n=1}^{\infty} \frac{2v_0 L}{n^2 \pi^2 c} (1 - \cos n\pi) \sin(n\pi ct/L) \sin(n\pi x/L) \)

(b) \( u(x, t) = 3 \sin(4\pi x/L) \cos(4\pi ct/L) \)

(c) \( u(x, t) = \sum_{n=1}^{\infty} \frac{8h}{n^2 \pi^2} \left(2 \sin \frac{n\pi}{4} - \sin \frac{n\pi}{2} \right) \cos(n\pi ct/L) \sin(n\pi x/L) \)

[4] \[
T''(t) - \alpha c^2 T(t) = 0 \quad t \in \mathbb{R};
\]
\[
X''(x) - \alpha X(x) = 0 \quad 0 < x < L,
\]
\[
X'(0) = 0, X'(L) = 0.
\]

[5] \[
T''(t) + T'(t) - \alpha T(t) = 0 \quad t \in \mathbb{R};
\]
\[
X''(x) - \alpha X(x) = 0 \quad 0 < x < 1,
\]
\[
X'(0) = 0, X(1) = 0.
\]