Convergence in Mean ($L^2$ Convergence) of Fourier Series

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- Example: $L^2$ approximations by truncated Fourier series.
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- Theorem: $L^2$ convergence (Convergence in mean).
Euclidean Distance Between Discrete Signals

Given two sequences \[ \{u_1, \ u_2, \ \cdots, \ u_n; \ \text{and} \ \ v_1, \ v_2, \ \cdots, \ v_n, \] the Euclidean distance between them is \[ \{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2\}^{1/2}. \]
• **Euclidean Distance Between Discrete Signals**

Given two sequences \( \{u_1, u_2, \ldots, u_n\}; \) and \( \{v_1, v_2, \ldots, v_n\}; \) the Euclidean distance between them is

\[
\left\{ (u_1 - v_1)^2 + \cdots + (u_n - v_n)^2 \right\}^{1/2}.
\]

• **\( L^2 \) Distance Between Functions**

Given two functions \( u(x) \) and \( v(x) \) on \([-a, a]\), the \( L^2 \) distance between them is

\[
\left\{ \int_{-a}^{a} [u(x) - v(x)]^2 \, dx \right\}^{1/2}.
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• **Euclidean Distance Between Discrete Signals**

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• **Error in Mean**

When a function \( u(x) \) is approximated by \( v(x) \),

\[
\text{(Error in Mean)} = (L^2 \text{ distance})^2.
\]

In other words,

\[
\text{(Error in Mean)} = \int_{-a}^{a} [u(x) - v(x)]^2 \, dx.
\]
Example. $f(x)$ is $2\pi$ periodic, $f(x) = 1 + x/\pi$ ($-\pi \leq x < 0$), and $f(x) = -1$ ($0 \leq x < \pi$). The Fourier series of $f(x)$ is

$$
-\frac{1}{4} + \sum_{n=1}^{\infty} \left[ \frac{1-(-1)^n}{n^2\pi^2} \cos(nx) + \frac{-2+(-1)^n}{n\pi} \sin(nx) \right].
$$

Approximate $f(x)$ by truncating the F-series at $N = 3$:

$$
S_3(x) = -\frac{1}{4} + \sum_{n=1}^{3} \left[ \frac{1-(-1)^n}{n^2\pi^2} \cos(nx) + \frac{-2+(-1)^n}{n\pi} \sin(nx) \right].
$$

(Error in Mean) $= \int_{-\pi}^{\pi} [f(x) - S_3(x)]^2 \approx 0.4028159855$

- Error in Mean decreases with $N$.
- Error in Mean $\to 0$, as $N \to \infty$. 

(Error in Mean) $\approx 0.1572187764$

(Error in Mean) $\approx 0.07913602023$
Can we choose other coefficients, to get better approximation?

For $N = 10$, approximate $f(x)$ by $S_{10}(x)$:

$$S_{10}(x) = -\frac{1}{4} + \sum_{n=1}^{10} \left[ \frac{1 - (-1)^n}{n^2\pi^2} \cos(nx) + \frac{-2 + (-1)^n}{n\pi} \sin(nx) \right].$$

(The Error in Mean of $S_{10}$) $\approx 0.1572187764$

If we replace some Fourier coefficients by, say,

$$a_3 = \frac{1}{50}, b_5 = -\frac{1}{4}, a_9 = \frac{1}{100},$$

(just my random choices)

to form a new trig polynomial $T_{10}(x)$,

( the Error in Mean of $T_{10}$) $\approx 0.1683563939$.

- Whatever coefficients you try, you can never beat $S_{10}$.
- Fourier coefficients are our best choices, in minimizing the error in mean.
- $S_{10}(x)$ is the best $L^2$ approx of $f(x)$, among all trig polynomials of degree 10.
**Assumptions:** \( f(x) \) is \( 2a \) periodic and \( \int_{-a}^{a} f(x)^2 dx < \infty \).

Let \( a_0, a_n, b_n \) be the Fourier coefficients of \( f(x) \).

Let \( S_N(x) = a_0 + \sum_{n=1}^{N} \left[ a_n \cos \left( \frac{n\pi x}{a} \right) + b_n \sin \left( \frac{n\pi x}{a} \right) \right] \) (the truncated Fourier series of degree \( N \)).

**Theorem (Best \( L^2 \) approximation)**

\( S_N(x) \) is the best \( L^2 \) approx of \( f(x) \), among all trig polynomials of degree \( N \).

More precisely, for any trig polynomial \( T_N(x) \) of degree \( N \),

\[(\text{the error in mean of } S_N) \leq (\text{the error in mean of } T_N).\]

**Theorem (Convergence in mean. \( L^2 \) convergence.)**

The error in mean of \( S_N \) decays to 0, as \( N \to \infty \).

In other words, \( S_N(x) \) converges to \( f(x) \) in mean, as \( N \to \infty \).

**Formula (Parseval’s equality)**

\[ \int_{-a}^{a} f(x)^2 dx = a \left[ 2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]. \]

**Formula (Error in mean of \( S_N \))**

\[ (\text{The error in mean of } S_N) = \int_{-a}^{a} f(x)^2 dx - a \left[ 2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right] \]
\[ = a \left[ \sum_{n=N+1}^{\infty} (a_n^2 + b_n^2) \right]. \]