ASYMPTOTIC PAIRS, STABLE SETS AND CHAOS IN POSITIVE ENTROPY SYSTEMS

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Abstract. We consider positive entropy $G$-systems for certain countable, discrete, infinite left-orderable amenable groups $G$. By undertaking local analysis, the existence of asymptotic pairs and chaotic sets will be studied in connecting with the stable sets. Examples are given for the case of integer lattice groups, the Heizenberg group, and the groups of integral unipotent upper triangular matrices.

1. Introduction

Throughout the paper, we let $G$ be a countable, discrete, infinite amenable group. Recall that the group $G$ is said to be amenable if there exists an invariant mean on it, or equivalently, if there exists a sequence of finite subsets $F_n \subset G$, such that, for every $g \in G$,

\[
\lim_{n \to +\infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0.
\]

A sequence satisfying condition (1.1) is called a Følner sequence (see [12]). For instance, when $G = \mathbb{Z}^d$ for some $d \in \mathbb{N}$, \{\[F_n = [0, n]^d : n \in \mathbb{N}\}\} is a Følner sequence of $G$.

A $G$-system $(X, G)$ is such that $X$ is a compact metric space and $G$ continuously acts on $X$. Let $(X, G)$ be a $G$-system and $S$ be an infinite subset of $G$, where $X$ is endowed with the metric $d$. A pair $(x, y) \in X \times X$ is called a $S$-asymptotic pair if for each $\epsilon > 0$, there are only finitely many elements $s \in S$ with $d(sx, sy) > \epsilon$. For a point $x \in X$, the set

\[W_S(x, G) = \{y \in X : (x, y) \text{ is an } S\text{-asymptotic pair}\}\]

is called the $S$-stable set of $x$. Let $\delta > 0$. A pair $(x, y) \in X \times X$ is called a $(S, \delta)$-Li-Yorke pair if $(x, y)$ is $S'$-asymptotic pair for some infinite subset $S'$ of $S$ and \{\[s \in S : d(sx, sy) > \delta\}\} is an infinite subset of $S$. A subset $E$ of $X$ is said to be $(S, \delta)$-chaotic if $(x, y)$ is a $(S, \delta)$-Li-Yorke pair for every $x \neq y \in E$.

Given a $G$-system $(X, G)$, one can define in the usual way its topological entropy $h_{\top}(G, X)$ as well as measure-theoretic entropy with respect to an invariant Borel probability measure, lying in $[0, +\infty]$ (see Section 2 for detail). One fundamental question to ask is the relationship between the positivity of the entropies of $(X, G)$ and the chaotic behavior of its orbits. A well-known result in this direction is given by Blanchard and co-authors in [4] for the case $G = \mathbb{Z}$. They showed via ergodic-theoretic method that if the $\mathbb{Z}$-system is generated by a...
homeomorphism $T : X \to X$ and $h_{\text{top}}(Z, X) > 0$, then there exists an uncountable $(Z_+, \delta)$-chaotic subset $E$ of $X$ for some $\delta > 0$, where $Z_+ = \{0, 1, 2, \ldots\}$ (see also [21] for an alternative proof using combinatorial method). The result in [4] is later generalized to the case of sofic group actions by Kerr and Li ([22, Corollary 8.4]).

Another fundamental question to ask is the relationship between the positivity of the entropies of a $G$-system $(X, G)$ and the existence of asymptotic pairs. On one hand, Lind and Schmidt ([25, Example 3.4]) constructed examples of $Z$-actions generated by toral automorphisms which have positive entropy but admit no non-diagonal $Z$-asymptotic pairs. In fact, by using co-induction techniques introduced in [10, 11], such examples can be constructed for general $G$-actions if $G$ is an infinite countable discrete amenable group containing a subgroup $\Gamma$ isomorphic to $Z$. Thus, additional conditions are needed for a positive entropy $G$-system to admit non-diagonal $G$-asymptotic pairs. Indeed, it was shown by Schmidt (see [31, Proposition 2.1]) for the case $G = Z^d$ for some $d \in \mathbb{N}$ that every subshift of finite type with positive entropy has non-diagonal $G$-asymptotic pairs. For an expansive action of $Z^d$ by (continuous) automorphisms of a compact abelian group, Lind and Schmidt [25] proved that the action has positive entropy if and only if there exists non-diagonal $Z^d$-asymptotic pairs. Recently, Chung and Li [6] extended this result in [25] for a larger class of amenable groups.

On the other hand, it was shown by Blanchard et al in [3] for the case $G = Z$ that if the $Z$-system is generated by a homeomorphism $T : X \to X$ and if $\mu$ is a $T$-invariant ergodic Borel probability measure on $X$ with positive entropy, then there exists $\delta > 0$ such that for $\mu$-a.e. $x \in X$, there exists an uncountable subset $F_x \subseteq W_{Z_+}(x, Z)$ with the property that $(x, y)$ is a $(Z_-, \delta)$-Li-Yorke pair for each $y \in F_x$, where $Z_- = \{0, -1, -2, \ldots\}$. We refer the readers to [35] for relativized versions of the results in [3, 4], to [19, 20, 33] for more precise characterizations on chaotic phenomenon appearing in stable sets of positive entropy $Z$-systems, and to [8] for dimension analysis of these sets.

With the fundamental questions above in mind, the aim of the present paper is to investigate the connections among the positivity of the entropies, chaotic behavior, and the existence of asymptotical pairs in a general $G$-system when $G$ is a countable discrete infinite amenable group with the algebraic past $\Phi$. More precisely, let $G$ be a group with the unit $e_G$. $G$ is said to be left-orderable if there exists a linear ordering in $G$ which is invariant under left translations. The group $G$ is left-orderable if and only if it contains a subset $\Phi$, called an algebraic past of $G$ ([30]), with the following properties:

1. $\Phi \cap \Phi^{-1} = \emptyset$,
2. $\Phi \cup \Phi^{-1} \cup \{e_G\} = G$,
3. $\Phi \cdot \Phi \subseteq \Phi$.

Indeed, with respect to the algebraic past $\Phi$, one obtains the desired linear ordering on $G$ as follows: $g_1$ is less than $g_2$ (write $g_1 <_\Phi g_2$), if $g_2^{-1}g_1 \in \Phi$. Let $(g_i)_{i \geq 1}$ be a sequence in $G$. We say that a sequence $(g_i)_{i \geq 1}$ increasingly goes to infinity with respect to $\Phi$ (write $g_i \nearrow \infty$ w.r.t $\Phi$) if $g_i <_\Phi g_{i+1}$ for each $i \geq 1$ and for each element $g \in G$, $\# \{i \in \mathbb{N} : g_i <_\Phi g\} < +\infty$. We say that a sequence $(g_i)_{i \geq 1}$ decreasingly goes to infinity with respect to $\Phi$ (write $g_i \searrow \infty$ w.r.t $\Phi$) if $g_i >_\Phi g_{i+1}$ for each $i \geq 1$ and for each element $g \in G$, $\# \{i \in \mathbb{N} : g_i >_\Phi g\} < +\infty$. 
The theory of left-orderable groups is a well developed subject in group theory which can be traced back to the late nineteenth century. For more details of this theory, we refer the readers to widely known modern books [5], [24]. It is well-known that a nontrivial left-orderable group must be torsion-free, and the category of left-orderable groups include torsion-free nilpotent groups and free groups.

One main result of the paper concerns the existence of asymptotic pairs in a positive entropy $G$-system as follows.

**Theorem 1.1.** Let $G$ be a countable discrete infinite amenable group with the algebraic past $\Phi$, $f_n \not\rightarrow \infty$ w.r.t $\Phi$ with $f_n \Phi f_n^{-1} = \Phi$ for $n \geq 1$, and $S$ be an infinite subset of $G$ such that $\sharp \{ s \in S : s \not< \Phi f_n \} < \infty$ for each $n \geq 1$. Then any positive entropy $G$-system has proper $S$-asymptotic pairs. More precisely, if $(X, G)$ is a $G$-system and $\mu$ is a positive entropy ergodic $G$-invariant Borel probability measure on $X$, then $W_S(x, G) \setminus \{ x \} \neq \emptyset$ for $\mu$-a.e. $x \in X$.

Let $G$ be a countable discrete infinite amenable group with the algebraic past $\Phi$. If $S$ is a subset of a group $G$, then $\langle S \rangle$, the subgroup generated by $S$, is the smallest subgroup of $G$ containing $S$. A semigroup $S$ of $G$ is called $\Phi$-admissible if $S \subseteq \Phi^{-1} \cup \{e_G\}$, $\langle S \rangle = G$ and there exist $f_n \not\rightarrow \infty$ w.r.t $\Phi$ and $h_n \not\rightarrow \infty$ w.r.t $\Phi$ such that $\sharp \{ s \in S : s \not< \Phi f_n \} < \infty$, $h_n \in S^{-1}$ and $f_n \Phi f_n^{-1} = \Phi$ for each $n \geq 1$. It is clear that a $\Phi$-admissible semigroup $S$ of $G$ is always an infinite semigroup since $G$ is torsion-free.

Another main result of the paper concerns the existence of certain chaotic sets in a positive entropy $G$-system as follows.

**Theorem 1.2.** Let $G$ be a countable discrete infinite amenable group with the algebraic past $\Phi$ and $S$ be a $\Phi$-admissible semigroup of $G$. If $(X, G)$ is a $G$-system and $\mu$ is a positive entropy ergodic $G$-invariant Borel probability measure on $X$, then there exists $\delta > 0$ such that for $\mu$-a.e. $x \in X$, the stable set $W_S(x, G)$ contains a $(S^{-1}, \delta)$-chaotic set which is a Cantor set.

To demonstrate applications of Theorem 1.2, we give below three examples of $\Phi$-admissible semigroups $S$ for some special left-orderable amenable groups $G$. For the first example, we consider $G = \mathbb{Z}^d$ for some $d \in \mathbb{N}$ and let $S = \mathbb{Z}^d_+ := \{(n_1, \cdots, n_d) \in \mathbb{Z}^d : n_i \geq 0, 1 \leq i \leq d\}$,

$$\Phi = \{(n_1, \cdots, n_d) \in \mathbb{Z}^d : \exists j \in \{0, 1, 2, \cdots, d-1\} such that \sum_{i=1}^{d-k} n_t = 0 for k = 0, \cdots, j-1 and \sum_{i=1}^{d-j} n_t < 0\},$$

and $f_n = (n, n, \cdots, n) \in \mathbb{Z}^d$, $n \in \mathbb{N}$. Then it is not difficult to see that $\Phi$ is an algebraic past of $G$, $\langle S \rangle = G$, $f_n \not\rightarrow \infty$ w.r.t $\Phi$, $f_n^{-1} \not\rightarrow \infty$ w.r.t $\Phi$, and $\sharp \{ s \in S : s \not< \Phi f_n \} < \infty$, $f_n \Phi f_n^{-1} = \Phi$ for each $n \geq 1$. Thus $S$ is $\Phi$-admissible. By applying Theorem 1.2, we immediately obtain the following result.

**Theorem 1.3.** Any positive entropy $\mathbb{Z}^d$-system has proper $\mathbb{Z}^d_+$-asymptotic pairs. More precisely, if $(X, \mathbb{Z}^d)$ is a $\mathbb{Z}^d$-system and $\mu$ is a positive entropy ergodic $\mathbb{Z}^d$-invariant Borel probability measure on $X$, then there exists $\delta > 0$ such that for $\mu$-a.e. $x \in X$, the stable set
$W_{Z^d}(x, Z^d)$ contains a $(Z^d, \delta)$-chaotic set which is a Cantor set, where $Z^d := \{(n_1, \cdots, n_d) \in \mathbb{Z}^d : n_i \leq 0, 1 \leq i \leq d\}$.

Our next example treats the case of Heizenberg group – the two-step nilpotent countable matrix group

\[(1.2) \quad G = \left\{ \begin{pmatrix} 1 & m_3 & m_1 \\ 0 & 1 & m_2 \\ 0 & 0 & 1 \end{pmatrix} : m_1, m_2, m_3 \in \mathbb{Z} \right\}. \]

We fix the generators

\[T_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad T_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \]

Then

\[T_3^{n_3}T_2^{n_2}T_1^{n_1} = \begin{pmatrix} 1 & n_2 & n_1 \\ 0 & 1 & n_3 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } n_1, n_2, n_3 \in \mathbb{Z}. \]

Here $T_1$ generates the center $Z$ of $G$. Define the linear order relation on the above generators by setting $T_3 > T_2 > T_1$, together with the associated lexicographic linear order relation on $G$, i.e., $T_3^{j_3}T_2^{j_2}T_1^{j_1} > T_3^{k_3}T_2^{k_2}T_1^{k_1}$ if and only if $(j_3, j_2, j_1)$ is lexicographically less than $(k_3, k_2, k_1)$. This order relation is invariant with respect to the left translations of $G$, so we obtain an algebraic past $\Phi$ in $G$ defined as a subset of all elements of $G$ which are less than the identity $I_3$. Since $G$ is nilpotent, it is amenable ([28]). Thus the Heizenberg group $G$ is a countable discrete infinite amenable group with algebraic past $\Phi$.

Let $f_n = T_3^n, h_n = T_3^{-n}$ for $n \in \mathbb{N}$ and

\[(1.3) \quad S := \{T_3^{n_3}T_2^{n_2}T_1^{n_1} : (n_3, n_2, n_1) \in \mathbb{Z}^3 \text{ with } n_3 \geq n_2 \geq 0 \text{ and } n_3^2 \geq n_1 \geq 0\}. \]

Then $S \subset \Phi^{-1} \cup \{I_3\}$ is a semigroup of $G$, $< S > = < T_3, T_3T_2, T_3T_1 > = G$ and $f_n \not\rightarrow \infty$ w.r.t. $\Phi$ and $h_n \not\rightarrow \infty$ w.r.t. $\Phi$. It is also not hard to see that

\[\sharp\{s \in S : s <_\Phi f_n\} \leq \sharp\{(n_3, n_2, n_1) : n \geq n_3 \geq n_2 \geq 0 \text{ and } n_3^2 \geq n_1 \geq 0\} \leq n^4 < \infty, \]

$f_n\Phi f_n^{-1} = \Phi$ and $h_n \in S^{-1}$ for $n \geq 1$. Summarizing up, $S$ is a $\Phi$-admissible semigroup of the Heizenberg group $G$. Therefore, an application of Theorem 1.2 yields the following result.

**Theorem 1.4.** Let $G$ be the Heizenberg group defined in (1.2) and $S$ be the semigroup defined in (1.3). Then any positive entropy $G$-system has proper $S$-asymptotic pairs. More precisely, if $(X, G)$ is a $G$-system and $\mu$ is a positive entropy ergodic $G$-invariant Borel probability measure on $X$, then there exists $\delta > 0$ such that for $\mu$-a.e. $x \in X$, the stable set $W_S(x, G)$ contains a $(S^{-1}, \delta)$-chaotic set which is a Cantor set.
We now turn to a more general case of the group of integral unipotent upper triangular matrices. Given \( d \in \mathbb{N} \), consider the matrix-valued function

\[
M(a) = \begin{pmatrix}
1 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\
0 & 1 & a_1 & \cdots & a_{d-2} & a_{d-1} \\
0 & 0 & 1 & \cdots & a_2 & a_1 \\
0 & 0 & 0 & \cdots & a_{d-1} & a_{d-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & a_1 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix},
\]

\( a = (a_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{Z}^{d(d+1)/2} \). Then the group

\[
G =: U_{d+1}(\mathbb{Z}) = \{ M(a) : a = (a_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{Z}^{d(d+1)/2} \}
\]

of integral unipotent upper triangular matrices is in fact a \( d \)-step nilpotent group. It is clear that for any \( A \in U_{d+1}(\mathbb{Z}) \) there exists a unique \( a = (a_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{Z}^{d(d+1)/2} \) such that \( A = M(a) \). Moreover, for any \( a = (a_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{Z}^{d(d+1)/2} \) and \( b = (b_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{Z}^{d(d+1)/2} \), if \( c = (c_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{Z}^{d(d+1)/2} \) is such that \( M(c) = M(a)M(b) \), then

\[
c_i^k = a_i^k + \left( \sum_{j=1}^{k-1} a_i^{k-j} b_{i+j-k}^j \right) + b_i^k
\]

for \( 1 \leq k \leq d \) and \( 1 \leq i \leq d-k+1 \).

Consider the following linear order relation on \( U_{d+1}(\mathbb{Z}) \): \( M(a) < M(b) \) if and only if

\[
(a_1^1, a_1^{d-1}, \ldots, a_1^1; a_2^1; a_3^{d-2}; \ldots, a_3^1; \ldots; a_2^1; a_1^{d-1}; a_1^1)
\]

is lexicographically less than

\[
(b_1^1, b_1^{d-1}, \ldots, b_1^1; b_2^1; b_3^{d-2}; \ldots, b_3^1; \ldots; b_2^1; b_1^{d-1}; b_1^1),
\]

where \( a = (a_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{Z}^{d(d+1)/2} \) and \( b = (b_i^k)_{1 \leq k \leq d, 1 \leq i \leq d-k+1} \in \mathbb{Z}^{d(d+1)/2} \). This order relation is invariant with respect to the translations of \( U_{d+1} \) by (1.4), so we can define an algebraic past \( \Phi \) in \( G \) as a subset of all elements of \( U_{d+1}(\mathbb{Z}) \) which are less than the identity \( I_{d+1} \), i.e.,

\[
\Phi := \{ M(a) \in U_{d+1}(\mathbb{Z}) : M(a) < I_{d+1} \}.
\]

As before, since \( G \) is nilpotent, it is amenable. Thus \( G =: U_{d+1}(\mathbb{Z}) \) is a countable discrete infinite amenable group with algebraic past \( \Phi \).

Consider the set of generators for \( U_{d+1}(\mathbb{Z}) \) formed by the matrices \( T_{i,j} = I_{d+1} + u_{i,j}, i < j, 1 \leq i, j \leq d + 1 \), where each \( u_{i,j} \) is a matrix unit – the matrix whose \( (k,p) \)-entry equals \( \delta_{i_k} \delta_{j_p} \) for all \( 1 \leq k, p \leq d + 1 \). Let \( f_n = T_{d,d+1}^n, h_n = T_{d+1,d}^{-n}, n \in \mathbb{N} \), and

\[
S := \{ M(a) : (a_i^1)^k \geq a_i^k \geq 0 \text{ for } 1 \leq k \leq d, 1 \leq i \leq d-k+1 \}.
\]

Then \( S \) is a semigroup of \( U_{d+1}(\mathbb{Z}) \), \( S \subset \Phi^{-1} \cup \{ I_{d+1} \} \) by (1.4) and (1.5),

\[
< S >= \{ T_{d,d+1} \cup \{ T_{d+1,i} : i < j, 1 \leq i, j \leq d+1 \} \} = G,
\]
and \( f_n \not\to \infty \) w.r.t. \( \Phi \) and \( h_n \not\to \infty \) w.r.t. \( \Phi \). It is not hard to see that
\[
\sharp \{ s \in S : s <_\Phi f_n \} \\
\leq \sharp \{ (a_i^k)_{1 \leq k \leq d, 1 \leq i \leq d - k + 1} : n^k \geq a_i^k \geq 0 \text{ for } 1 \leq k \leq d, 1 \leq i \leq d - k + 1 \}
\leq \infty,
\]
f\( n \Phi f_n^{-1} = \Phi, f_n \in S \) and \( h_n \in S^{-1} \) for \( n \geq 1 \). Summarizing up, \( S \) is a \( \Phi \)-admissible semigroup of the group \( G \), hence an application of Theorem 1.2 yields the following result.

**Theorem 1.5.** Let \( S \) be the semigroup defined in (1.5). Then any positive entropy \( U_{d+1}(\mathbb{Z}) \)-system has proper \( S \)-asymptotic pairs. More precisely, if \((X, U_{d+1}(\mathbb{Z}))\) is a \( U_{d+1}(\mathbb{Z}) \)-system and \( \mu \) is a positive entropy ergodic \( U_{d+1}(\mathbb{Z}) \)-invariant Borel probability measure on \( X \), then there exists \( \delta > 0 \) such that for \( \mu \)-a.e. \( x \in X \), the stable set \( W_S(x, U_{d+1}(\mathbb{Z})) \) contains a \((S^{-1}, \delta)\)-chaotic set which is a Cantor set.

The paper is organized as follows. In Section 2, we review some basic dynamical properties for \( G \)-systems. In Section 3, we establish a Pinsker formula for a countable discrete infinite amenable group \( G \) with the algebraic past \( \Phi \). In Section 4, we introduce Pinsker \( \sigma \)-algebra and investigate some basic properties of it. The proof of the main results is given in Section 5.

## 2. Preliminary

In this section, we review some basic notions and fundamental properties of \( G \)-systems mostly taking from [16]. Let \( G \) be a countable discrete infinite amenable group with unit element \( e_G \). By a \( G \)-system \((X, G)\) we mean that \( X \) is a compact metric space and \( \Gamma : G \times X \to X, (g, x) \mapsto gx \) is a continuous mapping satisfying

1. \( \Gamma(e_G, x) = x \) for each \( x \in X \),
2. \( \Gamma(g_1, \Gamma(g_2, x)) = \Gamma(g_1g_2, x) \) for each \( g_1, g_2 \in G \) and \( x \in X \).

In the following, we fix a \( G \)-system \((X, G)\). Then under the induced continuous action on \( X \times X : g(x_1, x_2) := (gx_1, gx_2), g \in G, (x_1, x_2) \in X \times X \), \((X \times X, G)\) is also a \( G \)-system.

### 2.1. Følner sequence and Entropy

Let \( F(G) \) be the set of all finite non-empty subsets of \( G \). If \( (F_n)_{n \geq 1} \) is a Følner sequence of \( G \), then

\[
\lim_{n \to \infty} \frac{|F_n \Delta KK_n|}{|F_n|} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{|\{g \in F_n : Kg \subset F_n\}|}{|F_n|} = 1
\]

for every \( K \in F(G) \).

A **cover** of \( X \) is a finite family of subsets of \( X \) whose union is \( X \). A **partition** of \( X \) is a cover of \( X \) whose elements are pairwise disjoint. Denote by \( C_X \) (resp. \( C_X^k \)) the set of all covers (resp. finite open covers) of \( X \) and by \( P_X \) (resp. \( P_X^k \)) the set of all partitions of \( X \) (resp. finite Borel partition).

Given two covers \( \mathcal{U}, \mathcal{V} \in C_X \), \( \mathcal{U} \) is said to be **finer** than \( \mathcal{V} \) (denoted by \( \mathcal{U} \succeq \mathcal{V} \)) if each element of \( \mathcal{U} \) is contained in some element of \( \mathcal{V} \). Let \( \mathcal{U} \lor \mathcal{V} = \{ U \cap V : U \in \mathcal{U}, V \in \mathcal{V} \} \).
Denote by $N(\mathcal{U})$ the number of sets in a subcover of $\mathcal{U}$ of minimal cardinality. The entropy of $\mathcal{U} \in \mathcal{C}_X$ with respect to $G$ is defined by

$$h_{top}(G, \mathcal{U}) = \lim_{n \to +\infty} \frac{1}{|F_n|} \log N(\bigvee_{g \in F_n} g^{-1} \mathcal{U}),$$

where $F_n$ is a Følner sequence in the group $G$. As is shown in [26, Theorem 6.1] (see also [23, 28]) the limit exists and is independent of Følner sequences. The topological entropy of $(X, G)$ is then defined by

$$h_{top}(G) = h_{top}(G, X) = \sup_{\mathcal{U} \in \mathcal{C}_X} h_{top}(G, \mathcal{U}).$$

Denote by $\mathcal{B}_X$ the collection of all Borel subset of $X$ and $\mathcal{M}(X)$ the set of all Borel probability measures on $X$. For $\mu \in \mathcal{M}(X)$, denote by $\text{supp}(\mu)$ the support of $\mu$, i.e., the smallest closed subset $W \subseteq X$ such that $\mu(W) = 1$. $\mu \in \mathcal{M}(X)$ is called $G$-invariant if $g\mu = \mu$ for each $g \in G$, and called ergodic if it is $G$-invariant and $\mu(\bigcup_{g \in G} gA) = 0$ or 1 for any $A \in \mathcal{B}_X$. Denote by $\mathcal{M}(X, G)$ (resp. $\mathcal{M}^e(X, G)$) the set of all $G$-invariant (resp. ergodic $G$-invariant) elements in $\mathcal{M}(X)$. Note that the amenability of $G$ ensures that $\mathcal{M}^e(X, G) \neq \emptyset$ and both $\mathcal{M}(X)$ and $\mathcal{M}(X, G)$ are convex compact metric spaces when they are endowed with the weak*-topology.

For $\mu \in \mathcal{M}(X)$, denote by $\mathcal{B}^\mu_X$ the completion of $\mathcal{B}_X$ under $\mu$, and define

$$\mathcal{P}^\mu_X = \{ \alpha \in \mathcal{P}_X : \text{each element in } \alpha \text{ belongs to } \mathcal{B}^\mu_X \}.$$ 

Given $\alpha \in \mathcal{P}^\mu_X$ and a sub-$\sigma$-algebra $\mathcal{A}$ of $\mathcal{B}^\mu_X$, define

$$H_\mu(\alpha|\mathcal{A}) = \sum_{A \in \alpha} \int_X -\mathbb{E}_\mu(1_A|\mathcal{A}) \log \mathbb{E}_\mu(1_A|\mathcal{A}) d\mu,$$

where $\mathbb{E}_\mu(1_A|\mathcal{A})$ is the expectation of $1_A$ with respect to $\mathcal{A}$. One standard fact is that $H_\mu(\alpha|\mathcal{A})$ increases with respect to $\alpha$ and decreases with respect to $\mathcal{A}$. Set $\mathcal{N} = \{\emptyset, X\}$ and define

$$H_\mu(\alpha) = H_\mu(\alpha|\mathcal{N}) = \sum_{A \in \alpha} -\mu(A) \log \mu(A).$$

Note that any $\beta \in \mathcal{P}^\mu_X$ naturally generates a sub-$\sigma$-algebra $\mathcal{F}(\beta)$ of $\mathcal{B}^\mu_X$. We then define

$$H_\mu(\alpha|\beta) = H_\mu(\alpha|\mathcal{F}(\beta)) = H_\mu(\alpha \vee \beta) - H_\mu(\beta).$$

Given $\mu \in \mathcal{M}(X, G)$ and $\alpha \in \mathcal{P}^\mu_X$. The measure-theoretic entropy of $\mu$ relative to $\alpha$ is defined by

$$h_\mu(G, \alpha) = \lim_{n \to +\infty} \frac{1}{|F_n|} H_\mu(\bigvee_{g \in F_n} g^{-1} \alpha),$$

where $F_n$ is a Følner sequence in the group $G$. As is shown in [26, Theorem 6.1] (see also [23, 28]), the limit exists and is independent of Følner sequences. The measure-theoretic entropy of $\mu$ is defined by

$$h_\mu(G) = h_\mu(G, X) = \sup_{\alpha \in \mathcal{P}^\mu_X} h_\mu(G, \alpha).$$
A sub-$\sigma$-algebra $A$ of $B_X^\mu$ is said to be $G$-invariant if $gA = A$ for any $g \in G$. For the conditional entropy of $\alpha \in P_X^\mu$ with respect to a $G$-invariant sub-$\sigma$-algebra $A$ of $B_X^\mu$ we define

$$h_\mu(G, \alpha|A) = \lim_{n \to +\infty} \frac{1}{|F_n|} H_\mu(\bigvee_{g \in F_n} g^{-1}\alpha|A),$$

where $F_n$ is a Følner sequence in the group $G$. One can deduce the existence of this limit and its independence of the sequence $F_n$ from [26, Theorem 6.1] (see also [23, 28] and [34] for an extension of the approach of [23] to the conditional case). The conditional entropy of $\mu$ with respect to $A$ is defined by

$$h_\mu(G|A) = \sup_{\alpha \in P_X^\mu} h_\mu(G, \alpha|A).$$

It is well known that $h_{top}(G) = \sup_{\mu \in M(X,G)} h_\mu(G)$ (see e.g. [27, 32]).

2.2. Measurable partition and Relatively independent squares. Let $\mu \in M(X,G)$. Then $\mu \in M(X,G)$ is a Lebesgue system. If $\{\alpha_i\}_{i \in I}$ is a countable family of finite Borel partitions of $X$, then the partition $\alpha = \bigvee_{i \in I} \alpha_i$ is called a measurable partition. The sets $A \in B_X^\mu$, which are unions of atoms of $\alpha$, form a sub-$\sigma$-algebra of $B_X^\mu$, to be denoted by $\hat{\alpha}$ or $\alpha$ if there is no ambiguity. In fact, every sub-$\sigma$-algebra of $B_X^\mu$ coincides with a $\sigma$-algebra constructed in the way above (mod $\mu$). The following result is well-known (see e.g., [16]).

**Theorem 2.1.** (Martingale Theorem) Let $(A_n)_{n \geq 1}$ be a decreasing sequence (resp. an increasing sequence) of sub-$\sigma$-algebras of $B_X^\mu$ and let $A = \bigcap_{n \geq 1} A_n$ (resp. $A = \bigvee_{n \geq 1} A_n$). Then for every $f \in L^2(X, B_X^\mu, \mu)$, $\mathbb{E}_\mu(f|A_n) \to \mathbb{E}_\mu(f|A)$ in $L^2(\mu)$ and also $\mu$-almost everywhere.

Let $\mathcal{F}$ be a sub-$\sigma$-algebra $B_X^\mu$ and $\alpha$ be the measurable partition of $X$ with $\hat{\alpha} = \mathcal{F}$ (mod $\mu$). $\mu$ can be disintegrated over $\mathcal{F}$ as $\mu = \int_X \mu_x d\mu(x)$ where $\mu_x \in M(X)$ and $\mu_x(\alpha(x)) = 1$ for $\mu$-a.e. $x \in X$. The disintegration is characterized by the properties (2.2) and (2.3) below:

(2.2) For every $f \in L^1(X, B_X^\mu, \mu)$, $f \in L^1(X, B_X^\mu, \mu_x)$ for $\mu$-a.e. $x \in X$,

and the map $x \mapsto \int_X f(y) d\mu_x(y)$ is in $L^1(X, \mathcal{F}, \mu)$;

(2.3) For every $f \in L^1(X, B_X^\mu, \mu)$, $\mathbb{E}_\mu(f|\mathcal{F})(x) = \int_X f d\mu_x$ for $\mu$-a.e. $x \in X$.

Then for any $f \in L^1(X, B_X^\mu, \mu)$,

$$\int_X \left( \int_X f d\mu_x \right) d\mu(x) = \int_X f d\mu.$$

Let $\mathcal{F}$ be sub-$\sigma$-algebra of $B_X^\mu$ and $\mu = \int_X \mu_x d\mu(x)$ be the disintegration of $\mu$ over $\mathcal{F}$. According to [13, 14], the conditional square (or conditional product) $\mu \times_{\mathcal{F}} \mu$ of $\mu$ relatively to $\mathcal{F}$ is the Borel probability measure $\mu \times_{\mathcal{F}} \mu$ on $X \times X$ such that

(2.4) $\mu \times_{\mathcal{F}} \mu(A \times B) = \int_X \mathbb{E}_\mu(1_A|\mathcal{F})(x)\mathbb{E}_\mu(1_B|\mathcal{F})(x) d\mu(x) = \int_X \mu_x(A)\mu_x(B) d\mu(x)$

for all $A, B \in B_X$. It is clear that both projections of $\mu \times_{\mathcal{F}} \mu$ to $X$ are equal to $\mu$. 

By standard arguments, for every pair of bounded Borel functions $f, h$ on $X$ one has
\[ \int_{X \times X} f(x)h(y)d(\mu \times \mu)(x,y) = \int_X \mathbb{E}_\mu(f|\mathcal{F})(x)\mathbb{E}_\mu(h|\mathcal{F})(x)d\mu(x). \]
By (2.4), it is not hard to see that
\[ (\mu \times g\mathcal{F}) = g(\mu \times \mathcal{F}) \]
for any $g \in G$ and any sub-$\sigma$-algebra $\mathcal{F}$ of $\mathcal{B}_X^\mu$, where $g(\mu \times \mathcal{F})(E) := \mu \times \mathcal{F}(g^{-1}E)$ for any Borel subset $E$ of $X \times X$.

3. Pinsker formula

In this section, we will establish the following Pinsker formula for a countable discrete infinite amenable group $G$ with the algebraic past $\Phi$.

**Theorem 3.1.** (Pinsker formula) Let $G$ be a countable discrete infinite amenable group with algebraic past $\Phi$, $(X, G)$ be a $G$-system, $\mu \in \mathcal{M}(X, G)$, and $\mathcal{A}$ be a $G$-invariant sub-$\sigma$-algebra of $\mathcal{B}_X^\mu$. Then for any $\alpha, \beta \in \mathcal{P}_X^\mu$,
\[ h_\mu(G, \alpha \vee \beta|\mathcal{A}) = h_\mu(G, \beta|\mathcal{A}) + H_\mu(\alpha|\beta_G \vee \alpha_\Phi \vee \mathcal{A}), \]
where $\beta_G = \bigvee_{g \in G} g\beta$ and $\alpha_\Phi = \bigvee_{h \in \Phi} g\alpha$. In particular, $h_\mu(G, \alpha|\mathcal{A}) = H_\mu(\alpha|\alpha_\Phi \vee \mathcal{A})$.

**Proof.** Let $(F_n)_{n \geq 1}$ be a Følner sequence of $G$. Denote $F_n^{-1} = \{g_{n,j}\}_{j=1}^{\lfloor F_n \rfloor}$ such that
\[ g_{n,1} < \Phi g_{n,2} \cdots < \Phi g_{n,\lfloor F_n \rfloor}. \]
Then
\[ h_\mu(G, \alpha \vee \beta|\mathcal{A}) = \lim_{n \to \infty} \frac{1}{|F_n|} H_\mu(\bigvee_{g \in F_n} g^{-1}(\alpha \vee \beta)|\mathcal{A}) \]
\[ = \lim_{n \to \infty} \frac{1}{|F_n|} H_\mu(\bigvee_{g \in F_n} g^{-1}\beta|\mathcal{A}) + H_\mu(\bigvee_{g \in F_n} g^{-1}\alpha|\bigvee_{g \in F_n} g^{-1}\beta \vee \mathcal{A})) \]
\[ (3.1) = h_\mu(G, \beta|\mathcal{A}) + \lim_{n \to \infty} \frac{1}{|F_n|} H_\mu(\bigvee_{g \in F_n} g^{-1}\alpha|\bigvee_{g \in F_n} g^{-1}\beta \vee \mathcal{A}). \]
Note that
\[ \frac{1}{|F_n|} H_\mu(\bigvee_{g \in F_n} g^{-1}\alpha|\bigvee_{g \in F_n} g^{-1}\beta \vee \mathcal{A}) = \frac{1}{|F_n|} \sum_{i=1}^{\lfloor F_n \rfloor} H_\mu(g_{n,i}\alpha|\bigvee_{j=1}^{i-1} g_{n,j}\alpha \vee \bigvee_{g \in F_n} g^{-1}\beta \vee \mathcal{A}) \]
\[ = \frac{1}{|F_n|} \sum_{i=1}^{\lfloor F_n \rfloor} H_\mu(\alpha|\bigvee_{j=1}^{i-1} g_{n,j}^{-1}\alpha \vee \bigvee_{g \in F_n} g_{n,j}^{-1}\beta \vee \mathcal{A}) \]
\[ \geq \frac{1}{|F_n|} \sum_{i=1}^{\lfloor F_n \rfloor} H_\mu(\alpha|\alpha_\Phi \vee \beta_G \vee \mathcal{A}) \]
\[ = H_\mu(\alpha|\alpha_\Phi \vee \beta_G \vee \mathcal{A}). \]
Combining this with (3.1), we have
\[ (3.2) h_\mu(G, \alpha \vee \beta|\mathcal{A}) \geq h_\mu(G, \beta) + H_\mu(\alpha|\alpha_\Phi \vee \beta_G \vee \mathcal{A}). \]
Since \((F_n)_{n \geq 1}\) is a Følner sequence of \(G\), it follows from (2.1) that, for given \(\epsilon > 0\) and \(M, L \in F(G)\) with \(M \subset \Phi, L \subset G\), there exists a natural number \(N\), such that, whenever \(n \geq N\),

\[
\{|g \in F_n| M^{-1}g \subset F_n| \geq (1-\epsilon)|F_n| \text{ and } |g \in F_n| L^{-1}g \subset F_n| \geq (1-\epsilon)|F_n|.
\]

Thus for each \(n \geq N\), there are at least \((1-2\epsilon)|F_n|\) indices \(i\) in \(\{1, 2, \ldots, |F_n|\}\) satisfying \(M^{-1} \subseteq F_n g_{n,i}\) and \(L^{-1} \subseteq F_n g_{n,i}\). Since \(M^{-1} \subseteq \Phi^{-1}\) and \(\Phi^{-1} \cap F_n g_{n,i} = \{g_{n,j} : 1 \leq j \leq i-1\}\), we have that, for each \(n \geq N\), there are at least \((1-2\epsilon)|F_n|\) indices \(i\) in \(\{1, 2, \ldots, |F_n|\}\) satisfying \(M^{-1} \subseteq \bigcup_{j=1}^{i-1} \{g_{n,j}^{-1}g_{n,i}\}\) and \(L^{-1} \subseteq F_n g_{n,i}\), or equivalently, \(M \subseteq \bigcup_{j=1}^{i-1} \{g_{n,j}^{-1}g_{n,i}\}\) and \(L \subseteq g_{n,i}^{-1}F_n^{-1}\).

For every \(n \geq N\), we have

\[
\frac{1}{|F_n|} H_{\mu}(\bigvee_{g \in F_n} g^{-1}\alpha \bigvee_{g \in F_n} g^{-1}\beta \vee A) = \frac{1}{|F_n|} \sum_{i=1}^{|F_n|} H_{\mu}(\alpha \bigvee_{j=1}^{i-1} g_{n,j}^{-1}g_{n,j}\alpha \bigvee_{g \in F_n} g_{n,i}^{-1}g^{-1}\beta \vee A) \leq \frac{1}{|F_n|} ((1-2\epsilon)|F_n| H_{\mu}(\alpha \bigvee_{g \in M} g\alpha \vee \bigvee_{g \in L} g\beta \vee A) + 2\epsilon |F_n| H_{\mu}(\alpha|A)) = (1-2\epsilon) H_{\mu}(\alpha \bigvee_{g \in M} g\alpha \vee \bigvee_{g \in L} g\beta \vee A) + 2\epsilon H_{\mu}(\alpha|A).
\]

By letting \(N \to \infty\) in the above inequality and (3.1), we have

\[
h_{\mu}(G, \alpha \vee \beta|A) \leq h_{\mu}(G, \beta|A) + (1-2\epsilon) H_{\mu}(\alpha \bigvee_{g \in M} g\alpha \vee \bigvee_{g \in L} g\beta \vee A) + 2\epsilon H_{\mu}(\alpha|A).
\]

For fixed \(M, L \in F(G)\) with \(M \subset \Phi, L \subset G\), letting \(\epsilon \searrow 0\) yields that

\[
h_{\mu}(G, \alpha \vee \beta|A) \leq h_{\mu}(G, \beta|A) + H_{\mu}(\alpha \bigvee_{g \in M} g\alpha \vee \bigvee_{g \in L} g\beta \vee A).
\]

Letting \(M \to \Phi\) and \(L \to G\) in the above further yields that

\[
h_{\mu}(G, \alpha \vee \beta|A) \leq h_{\mu}(G, \beta|A) + H_{\mu}(\alpha|A) \vee \beta G \vee A).
\]

Combing this with (3.2), we have \(h_{\mu}(G, \alpha \vee \beta|A) = h_{\mu}(G, \beta|A) + H_{\mu}(\alpha|A) \vee \beta G \vee A).\) This completes the proof.
Proposition 3.1. Let $G$ be a countable infinite amenable group with the algebraic past $\Phi$ and $f_n \not\to_{\mu} \infty$ w.r.t $\Phi$ with $f_n \Phi f_n^{-1} = \Phi$ for each $n \geq 1$. Also let $(X, G)$ be a $G$-system, $\mu \in \mathcal{M}(X, G)$, $A$ be a $G$-invariant sub-$\sigma$-algebra of $\mathcal{B}_X^\mu$, and $\alpha, \beta, \gamma \in \mathcal{P}^\mu_X$ with $\alpha \leq \beta$. Then

$$
\lim_{n \to \infty} H_\mu(\alpha|\beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A) = H_\mu(\alpha|\beta_\Phi \vee A).
$$

Proof. It follows from Theorem 3.1 and the fact $f_n \Phi f_n^{-1} = \Phi$ that

$$
H_\mu(\beta|\beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A)
= H_\mu(\beta \vee f_n^{-1}\gamma|\beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A) - H_\mu(f_n^{-1}\gamma|\beta \vee \beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A)
= h_\mu(G, \beta \vee f_n^{-1}\gamma|A) - H_\mu(f_n^{-1}\gamma|\beta \vee \beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A)
= H_\mu(\beta|\beta_\Phi \vee A) + H_\mu(f_n^{-1}\gamma|\beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A) - H_\mu(f_n^{-1}\gamma|\beta \vee \beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A)
= H_\mu(\beta|\beta_\Phi \vee A) + H_\mu(\gamma|\beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A) - H_\mu(\gamma|\beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A)
= H_\mu(\beta|\beta_\Phi \vee A).
$$

Since $f_n \not\to_{\mu} \infty$ w.r.t $\Phi$, $f_n \beta \vee f_n \beta_\Phi \not\to_{\mu} \beta_\Phi$ as $n \to \infty$. Moreover, $\lim_{n \to \infty} H_\mu(\gamma|f_n \beta \vee f_n \beta_\Phi \vee \gamma_\Phi \vee A) = H_\mu(\gamma|\beta_\Phi \vee \gamma_\Phi \vee A)$ by the Martingale Theorem. Thus, by letting $n \to \infty$ in (3.3), we have

$$
\lim_{n \to \infty} H_\mu(\beta|\beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A)
= \lim_{n \to \infty} H_\mu(\beta|\beta_\Phi \vee A) + H_\mu(\gamma|\beta_\Phi \vee \gamma_\Phi \vee A) - H_\mu(\gamma|f_n \beta \vee f_n \beta_\Phi \vee \gamma_\Phi \vee A)
= H_\mu(\beta|\beta_\Phi \vee A).
$$

Combining this with the identity

$$
H_\mu(\alpha|\beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A) = H_\mu(\beta|\beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A) - H_\mu(\beta|\alpha \vee \beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A),
$$

we have

$$
\liminf_{n \to \infty} H_\mu(\alpha|\beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A) \geq \liminf_{n \to \infty} H_\mu(\beta|\beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A) - H_\mu(\beta|\alpha \vee \beta_\Phi \vee A)
= H_\mu(\beta|\beta_\Phi \vee A) - H_\mu(\beta|\alpha \vee \beta_\Phi \vee A)
= H_\mu(\alpha|\beta_\Phi \vee A).
$$

Since $\limsup_{n \to \infty} H_\mu(\alpha|\beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A) \leq H_\mu(\alpha|\beta_\Phi \vee A)$,

$$
\lim_{n \to \infty} H_\mu(\alpha|\beta_\Phi \vee (f_n^{-1}\gamma)_\Phi \vee A) = H_\mu(\alpha|\beta_\Phi \vee A).
$$

This completes the proof. \hfill \Box

4. Pinsker $\sigma$-algebra

In this section, we will introduce Pinsker $\sigma$-algebra and investigate some of its basic properties to be used in the proof of our main results. Let $G$ be a countable infinite amenable group and $(X, G)$ be a $G$-system. For $\mu \in \mathcal{M}(X, G)$ and a $G$-invariant sub-$\sigma$-algebra $A$ of $\mathcal{B}_X^\mu$, denote

$$
P_\mu(G|A) = \{ A \in \mathcal{B}_X^\mu : h_\mu(G, \{ A \setminus \gamma \}|A) = 0 \}.
$$

It follows from Theorem 3.1 or [17, Lemma 1.1] that $P_\mu(G|A)$ must be a $G$-invariant sub-$\sigma$-algebra of $\mathcal{B}_X^\mu$ containing $A$. We call this $\sigma$-algebra the Pinsker $\sigma$-algebra of the system.
\( (X, \mathcal{B}_X, \mu, G) \) relative to \( A \). We simply refer to the Pinsker \( \sigma \)-algebra of \( (X, \mathcal{B}_X, \mu, G) \) relative to the trivial algebra as the \textit{Pinsker} \( \sigma \)-algebra, denoted by \( P_\mu(G) \).

Descriptions of Pinsker algebras are already given in [7] for \( \mathbb{Z}^d \)-actions and in [18] for actions of a finitely-generated nilpotent group. Also, some new properties of Pinsker algebras related to disjointness and quasi-factors are discovered in [17], followed by a relativized version given in [9] in which a notion of entropy for cocycles is introduced.

The following result is well-known. For the sake of completeness, we provide a proof below.

**Lemma 4.1.** Let \( G \) be a countable infinite amenable group with the past \( \Phi, (X, G) \) be a \( G \)-system, and \( \mu \in \mathcal{M}(X, G) \). Then the Pinsker \( \sigma \)-algebra of the system \( (X, \mathcal{B}_X, \mu, G) \) relative to \( P_\mu(G) \) agrees with \( P_\mu(G) \), i.e.,

\[
P_\mu(G|P_\mu(G)) = P_\mu(G).
\]

**Proof.** It is clear that \( P_\mu(G|P_\mu(G)) \supseteq P_\mu(G) \). Now we take an increasing sequence of finite Borel partitions \( (\beta_n)_{n \geq 1} \) such that \( \beta_n \not\subseteq P_\mu(G) \) as \( n \to +\infty \). For a given \( \alpha \in \mathcal{P}_X^\mu \), we have by Theorem 3.1 and the Martingale Theorem that

\[
h_\mu(G, \alpha) \geq h_\mu(G, \alpha|P_\mu(G)) = H_\mu(\alpha|\alpha_\Phi \cup P_\mu(G))
\]

\[
\quad = \lim_{n \to +\infty} H_\mu(\alpha|\alpha_\Phi \cup (\beta_n))
\]

\[
\quad = \lim_{n \to +\infty} (h_\mu(G, \alpha \cup \beta_n) - h_\mu(G, \beta_n))
\]

\[
\quad = \lim_{n \to +\infty} h_\mu(G, \alpha \cup \beta_n)
\]

\[
\quad \geq h_\mu(G, \alpha).
\]

Thus \( h_\mu(G, \alpha|P_\mu(G)) = h_\mu(G, \alpha) \). This implies \( P_\mu(G|P_\mu(G)) = P_\mu(G) \). \( \Box \)

With Lemma 4.1, the following result follows from Theorem 4 in [17].

**Lemma 4.2.** Let \( G \) be a countable infinite amenable group with the past \( \Phi, (X, G) \) be a \( G \)-system and \( \mu \in \mathcal{M}_c(X, G) \). If \( \lambda = \mu \times P_\mu(G) \), and \( \pi : X \times X \to X \) is the canonical projection to the first factor, then \( P_\lambda(G|\pi^{-1}(P_\mu(G))) = \pi^{-1}(P_\mu(G)) \) (mod \( \lambda \)).

**Lemma 4.3.** Let \( G \) be a countable infinite amenable group with the past \( \Phi, (X, G) \) be a \( G \)-system, and \( \mu \in \mathcal{M}_c(X, G) \). Denote \( \lambda = \mu \times P_\mu(G) \) and \( \Delta_X = \{(x, x) : x \in X\} \). Then the following holds:

1) \( \lambda \in \mathcal{M}_c(X \times X, G) \);
2) If \( h_\mu(G) > 0 \), then \( \lambda(\Delta_X) = 0 \).

**Proof.** 1) Let \( A \) be a Borel subset of \( X \times X \) and \( B := \bigcup_{g \in G} gA \). Then \( gB = B \) for any \( g \in G \) and thus \( h_\lambda(G, \{B, X \times X \setminus B\}) = 0 \), i.e., \( B \in P_\lambda(G) \). Let \( \pi : X \times X \to X \) be the canonical projection to the first factor. By Lemma 4.2, we have \( P_\lambda(G|\pi^{-1}(P_\mu(G))) = \pi^{-1}(P_\mu(G)) \) (mod \( \lambda \)). Thus \( P_\lambda(G) \subseteq P_\lambda(G|\pi^{-1}(P_\mu(G))) = \pi^{-1}(P_\mu(G)) \) (mod \( \lambda \)). Let \( C \in P_\mu(G) \) be such that \( B = \pi^{-1}(C) \) (mod \( \lambda \)). Using the fact that \( gB = B \), \( g \in G \), we have \( \lambda(g\pi^{-1}(C)\Delta\pi^{-1}C) = 0 \), \( g \in G \). Moreover,

\[
\mu(gC\Delta C) = \lambda(\pi^{-1}(gC\Delta C)) = \lambda(g\pi^{-1}(C)\Delta\pi^{-1}C) = 0, \quad g \in G.
\]
Since \( \mu \) is ergodic, \( \mu(C) = 0 \) or 1. Thus \( \lambda(B) = \mu(C) = 0 \) or 1, i.e., \( \lambda \) is ergodic.

2) Assume that \( h_\mu(G) > 0 \). If \( \lambda(\Delta_X) > 0 \), then \( \lambda(\Delta_X) = 1 \) since \( \lambda \) is ergodic and \( \Delta_X \) is \( G \)-invariant. Now for any \( A \in \mathcal{B}_X \), we have

\[
0 = \lambda(A \setminus (X \setminus A)) = \int_X \mathbb{E}_\mu(1_A|P_\mu(G))(x)\mathbb{E}_\mu(1_{X \setminus A}|P_\mu(G))(x)\,d\mu(x).
\]

Thus the product of the two conditional expectations is equal to 0 a.e. As the sum of these two functions is equal to 1, each of them is equal to 0 or 1 a.e. It follows that \( \mathbb{E}_\mu(1_A|P_\mu(G)) = 1 \) a.e., and \( A \in P_\mu(G) \pmod{\mu} \). Thus the \( \sigma \)-algebras \( \mathcal{B}_X \) and \( P_\mu(G) \) are equal up to null sets. This implies \( h_\mu(G) = 0 \), a contradiction. \( \square \)

The next lemma establishes a connection among asymptotic pairs, Pinsker \( \sigma \)-algebra, and entropy.

**Lemma 4.4.** Let \( G \) be a countable discrete infinite amenable group with the algebraic past \( \Phi \), \( f_n \not\to \infty \) w.r.t. \( \Phi \) with \( f_n\Phi f_n^{-1} = \Phi \) for each \( n \geq 1 \), and \( S \) be a infinite subset of \( G \) such that \( \sharp\{s \in S : s <_\Phi f_n\} < \infty \) for each \( n \geq 1 \). Also let \( (X, G) \) be a \( G \)-system and \( \mu \in \mathcal{M}(X, G) \).

Then there exists a measurable partition \( \mathcal{P} \) of \( (X, G, \mu) \) such that the following holds:

1. \( \overline{P}_\Phi(x) \subseteq W_S(x, G) \) for any \( x \in X \), where \( \overline{P}_\Phi = \bigvee_{g \in \Phi} g\mathcal{P} \) and \( \overline{P}_\Phi(x) \) is the atom of \( \overline{P}_\Phi \) containing \( x \).

2. \( \bigcap_{h \in \Phi} h(\overline{P}_\Phi \vee P_\mu(G)) = P_\mu(G) \).

3. If, in addition, \( h_\mu(G) > 0 \), then \( \overline{P}_\Phi \neq \mathcal{B}_X^\mu (\mod \mu) \).

**Proof.** Denote the metric on \( X \) by \( d \). Let \( \{\alpha_n\}_{n \geq 1} \) be an increasing sequence of finite Borel partitions of \( X \) such that \( \lim_{n \to \infty} \text{diam}(\alpha_n) = 0 \). Applying Proposition 3.1 inductively, we can find a sequence \( k_1, k_2, \ldots \) such that \( k_i \not\to \infty \) and for each \( q \geq 2 \),

\[
H_\mu(P_k((P_{q-1})_\Phi \vee P_\mu(G))) - H_\mu(P_k((P_q)_\Phi \vee P_\mu(G))) < \frac{1}{k^{2q-1}}, \ k = 1, 2, \ldots, q - 1,
\]

where \( P_j = \bigvee_{i=1}^j f_{k_i}^{-1} \alpha_i \). We want to show that the measurable partition \( \mathcal{P} =: \bigvee_{i=1}^\infty P_i \) satisfies the properties (1)-(3) above.

1. Let \( x \in X \) and \( y \in \overline{P}_\Phi(x) \). For a given \( \epsilon > 0 \), since \( \lim_{n \to \infty} \text{diam}(\alpha_n) = 0 \), there exists \( i \in \mathbb{N} \) such that \( \text{diam}(\alpha_i) \leq \epsilon \). Now for any \( s \in S \) with \( s >_\Phi f_{k_i} \), one has \( s^{-1} f_{k_i} \in \Phi \) and hence \( y \in \overline{P}_\Phi(x) \subseteq (s^{-1} f_{k_i} f_{k_i}^{-1} \alpha_i)(x) = (s^{-1} \alpha_i)(x) \). Thus \( sy \in \alpha_i(sx) \). It follows that \( d(sx, sy) \leq \text{diam}(\alpha_i) \leq \epsilon \) for any \( s \in S \) with \( s >_\Phi f_{k_i} \). Combing this with the fact that \( \sharp\{s \in S : s <_\Phi f_{k_i}\} < \infty \), we know that there are only finitely many \( s \in S \) with \( d(sx, sy) > \epsilon \). Since \( \epsilon \) is arbitrary, \( (x, y) \) is \( S \)-asymptotic pair. Thus \( y \in W_S(x, G) \).

2. We note by the \( G \)-invariance of \( P_\mu(G) \) that \( gP_\mu(G) = P_\mu(G) \) for all \( g \in G \). By also noting that \( \Phi \Phi \subseteq \Phi \), we have

\[
\overline{P}_\Phi \vee P_\mu(G) \supseteq \bigcup_{g \in \Phi} g(\overline{P}_\Phi \vee P_\mu(G)) \supseteq \bigcap_{g \in \Phi} g(\overline{P}_\Phi \vee P_\mu(G)) \supseteq P_\mu(G).
\]

For any \( A \in \bigcap_{h \in \Phi} h(\overline{P}_\Phi \vee P_\mu(G)) \), we let \( \xi = \{A, X \setminus A\} \). Given \( g \in G \), if \( g \notin \Phi \cup \{e_G\} \), then \( g\xi \subseteq g(\overline{P}_\Phi \vee P_\mu(G)) \subseteq \overline{P}_\Phi \vee P_\mu(G) \). If \( g \in \Phi^{-1} \), then \( g^{-1} \notin \Phi \) and

\[
g^{-1} \subseteq gg^{-1}(\overline{P}_\Phi \vee P_\mu(G)) = \overline{P}_\Phi \vee P_\mu(G).
\]
Hence \( g\hat{E} \subseteq \hat{P}_\Phi \vee P_\mu(G) \) for any \( g \in G \). Thus \( \hat{E}_G = \bigvee_{g \in G} g\hat{E} \subseteq \hat{P}_\Phi \vee P_\mu(G) \). Moreover, by Theorem 3.1 and (4.1), we have

\[
H_\mu(\xi|\Phi \vee P_\mu(G)) = H_\mu(\xi|\Phi \vee P_\mu(G)) - H_\mu(P_i|\Phi \vee \xi G \vee P_\mu(G)) \\
\leq H_\mu(\xi|\Phi \vee P_\mu(G)) + \sum_{j=i}^{\infty} (H_\mu(P_i|\Phi \vee P_\mu(G)) - H_\mu(P_i|\Phi \vee P_\mu(G))) \\
\leq H_\mu(\xi|\Phi \vee P_\mu(G)) + \sum_{j=i}^{\infty} \frac{1}{2j+1} = H_\mu(\xi|\Phi \vee P_\mu(G)) + \frac{1}{i}.
\]

Letting \( i \to \infty \) in the above yields that

\( h_\mu(G, \xi) = H_\mu(\xi|\Phi \vee P_\mu(G)) \leq H_\mu(\xi|\Phi \vee P_\mu(G)) = 0. \)

Hence \( A \in P_\mu(G) \). Since \( A \) is arbitrary,

\[
\bigcap_{h \in \Phi} h(\hat{P}_\Phi \vee P_\mu(G)) \subset P_\mu(G).
\]

This, together with (4.2), proves (2).

(3) Suppose for contradiction that \( \hat{P}_\Phi = B_\mu^\Phi \mod \mu \). Then \( h\hat{P}_\Phi = B_\mu^\Phi \mod \mu \) for any \( h \in G \). Hence \( \bigcap_{h \in \Phi} h(\hat{P}_\Phi \vee P_\mu(G)) = B_\mu^\Phi \mod \mu \). It follows from (2) that \( P_\mu(G) = B_\mu^\Phi \mod \mu \), and consequently, \( h_\mu(G) = 0 \), a contradiction. \( \square \)

5. PROOF OF MAIN RESULTS

We first prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \((X,G)\) be a \(G\)-system and \(\mu \in \mathcal{M}^e(X,G)\) with \(h_\mu(G) > 0\). Then by Lemma 4.4 there exists a measurable partition \(\mathcal{P}\) of \((X,B_\mu^\Phi,G,\mu)\) such that

- \(\mathcal{P}_\Phi(x) \subseteq W_S(x,G)\) for any \(x \in X\), and
- \(\hat{P}_\Phi \neq B_\mu^\Phi \mod \mu\).

Let

\[
E = \{(x,y) \in X \times X : (x,y) \text{ is } S\text{-asymptotic pair}\}.
\]

Then \(E\) is a Borel subset of \(X \times X\). Let

\[
J = \pi(E \setminus \Delta_X),
\]

where \(\pi : X \times X \to X\) denotes the projection onto the first factor and \(\Delta_X := \{(x,x) : x \in X\}\), i.e., \(J\) is the set of points of \(X\) which belong to a proper \(S\)-asymptotic pair. Then \(J \in B_\mu^\Phi\) and it is \(G\)-invariant. By ergodicity of \(\mu\), \(\mu(J) = 0\) or 1.

If \(\mu(J) = 0\), then \(W_S(x,G) = \{x\}\) for \(\mu\)-a.e. \(x \in X\). Thus \(\mathcal{P}_\Phi(x) = \{x\}\) for \(\mu\)-a.e. \(x \in X\). This implies that \(B_\mu^\Phi = \hat{P}_\Phi \mod \mu\), a contradiction to the fact that \(h_\mu(X,G) > 0\). Hence \(\mu(J) = 1\), which implies that \(W_S(x,G) \setminus \{x\} \neq \emptyset\) for \(\mu\)-a.e. \(x \in X\). \( \square \)

The proof of Theorem 1.2 will need the following result due to Mycielski (see e.g. [1, Theorem 5.10] and [2, Theorem 6.32]).
Lemma 5.1. (Mycielski) Let $Y$ be a perfect compact metric space and $C$ be a symmetric dense $G_{δ}$ subset of $Y \times Y$. Then there exists a dense subset $K \subseteq Y$ which is a union of countably many Cantor sets such that $K \times K \subseteq C \cup \Delta_Y$, where $\Delta_Y = \{(y, y) : y \in Y\}$.

Proof of Theorem 1.2. Let $(X, G)$ be a $G$-system and $μ ∈ M^c(X, G)$ with $h_μ(G) > 0$, where $X$ is endowed with the metric $d$. Denote $λ = μ × P_μ(G) μ$. Then by Lemma 4.3, $λ ∈ M^c(X × X, G)$ and

$$\text{supp}(λ) \subseteq \Delta_X \text{ and } \text{supp}(λ) \cap \Delta_X \neq ∅.$$ 

Take $(x_0, y_0), (z_0, z_0) ∈ \text{supp}(λ)$ such that $d(x_0, y_0) > 0$. For each $i ∈ N$, let $U_i$ (resp. $V_i$) be the open ball centered at $(x_0, y_0)$ (resp. $(z_0, z_0)$) and of radius $\frac{1}{i}$. Since $S$ is a $Φ$-admissible semigroup and $G$ is an infinite torsion free group, $S$ is an infinite set. For each $g ∈ G$, we define

$$U_{i,g} = \bigcup_{t ∈ S \setminus \{e_G\}} tg(U_i) \quad \text{(resp. } V_{i,g} = \bigcup_{t ∈ S \setminus \{e_G\}} tg(V_i)).$$

Let $δ := \frac{1}{2}d(x_0, y_0)$ and

$$W = \bigcap_{i ≥ 1} \left( \left( \bigcap_{g ∈ S} U_{i,g} \right) \cap \left( \bigcap_{g ∈ S} V_{i,g} \right) \right).$$

Claim 1. Any $(x, y) ∈ W$ is a $(S^{-1}, δ)$-Li-Yorke pair.

Let $(x, y) ∈ W$ and set $E = \{s ∈ S : d(s^{-1}x, s^{-1}y) > δ\}$. We first show that $E$ is an infinite set. Take $r ∈ N$ such that $d(u, v) > δ$ for any $(u, v) ∈ U_r$. Suppose for contradiction that $E$ is not an infinite set. Then, on one hand, there exists $a ∈ E$ such that $ba^{-1} ∈ Φ \cup \{e_G\}$ for all $b ∈ E$. On the other hand, since $(x, y) ∈ U_{r,s}$, there exists a $t(a) ∈ S \setminus \{e_G\}$ such that $(x, y) ∈ t(a) U_r$, and consequently, $t(a)a \in E$. Now, $t(a) = (t(a)a)a^{-1} ∈ Φ \cup \{e_G\}$, a contradiction to the fact that $t(a) ∈ Φ^{-1}$. This shows that $E$ is an infinite set.

Next, consider the sets $S_i = \{s ∈ S : d(s^{-1}x, s^{-1}y) < \frac{1}{i}\}$, $i ∈ N$. Since $(x, y) ∈ \bigcap_{g ∈ S} V_{2i,g}$, each $S_i$ is a non-empty set. If there exists $s ∈ \bigcap_{i=1}^{∞} S_i$, then $s^{-1}x = s^{-1}y$, i.e., $x = y$, a contradiction. Hence $\bigcap_{i=1}^{∞} S_i = ∅$. This implies that each $S_i$ is an infinite set because $S_1 ⊇ S_2 ⊇ S_3 ⋯$. Now we take $s_1 ∈ S_1$ and $s_i ∈ S_i \setminus \{s_1, s_2, \cdots, s_{i-1}\}$ for $i ≥ 2$. Let $S' = \{s_1^{-1}, s_2^{-1}, \cdots\}$. Then $S'$ is an infinite subset of $S^{-1}$ and $(x, y)$ is $S'$-asymptotic pair. This proves the claim.

Since $S$ is a $Φ$-admissible semigroup, $G, S$ satisfy conditions of Lemma 4.4. It follows that there exists a measurable partition $P$ of $(X, G, μ)$ such that

- $P_Φ(x) ⊆ W_S(x, G)$ for any $x ∈ X$, where $P_Φ = \bigvee_{g ∈ Φ} gP$ and $P_Φ(x)$ is the atom of $P_Φ$ containing $x$, and

- $\bigcap_{g ∈ Φ, g(P_Φ \lor P_μ(G)) = P_μ(G)}$.

Let $F = P_Φ \lor P_μ(G)$ and $λ_0 = μ × F μ$.

Claim 2: For every closed (resp. open) subset $F$ (resp. $U$) of $X × X$ with $h(F) ≥ F$ (resp. $h(U) ≤ U$) for any $h ∈ S$, one has $λ(F) ≥ λ_0(F)$ (resp. $λ(U) ≤ λ_0(U)$).

Since $S$ is a $Φ$-admissible semigroup, there exists $(g_n)_{n ≥ 1} ⊆ S^{-1}$ such that $g_i ≺ φ g_{i+1}$ for each $i ≥ 1$ and for each element $g ∈ G$, $\#\{i ∈ N : g_i ≺ φ g\} < +∞$. Let $F_n = g_n(F)$ for $n ∈ N$. 

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As $\Phi \Phi \subseteq \Phi$,

$$\tag{5.1} u\mathcal{F} \subseteq \mathcal{F}$$

for any $u \in \Phi$. For any $g \in \Phi$, since $\#\{i \in \mathbb{N} : g_i > \Phi g\} < +\infty$, we can find $m \in \mathbb{N}$ such that $g > \Phi g_m$, i.e., $g^{-1}g_m \in \Phi$. Thus $g^{-1}g_m \mathcal{F} \subseteq \mathcal{F}$, or equivalently, $\mathcal{F}_m \subseteq g\mathcal{F}$. Hence

$$\tag{5.2} \bigcap_{n \geq 1} \mathcal{F}_n \subseteq g\mathcal{F}$$

for any $g \in \Phi$.

Note that $g_n \in S^{-1} \subset \Phi \cup \{e_G\}$ and $g_n^{-1}g_{n+k} \in \Phi$ for $n, k \in \mathbb{N}$. Hence $(\mathcal{F}_n)_{n \geq 1}$ is a decreasing sequence of sub-$\sigma$-algebras of $\mathcal{F}$, and

$$\bigcap_{n \geq 1} \mathcal{F}_n = \bigcap_{g \in \Phi} g\mathcal{F} = P_\mu(G)$$

by (5.1) and (5.2).

Let $\lambda_n = \mu \times \mathcal{F}_n \mu$, $n \in \mathbb{N}$. Then by (2.5),

$$\tag{5.3} \lambda_n = g_n(\lambda_0), \ n \in \mathbb{N}.$$

It follows from the Martingale Theorem that, for any bounded Borel measurable functions $f, h$ on $X$,

$$\int_{X \times X} f(x)h(y)d\lambda_n(x, y) = \int_X E_\mu(f|\mathcal{F}_n)(x)E_\mu(h|\mathcal{F}_n)(x)d\mu(x)$$

$$\rightarrow \int_X E_\mu(f|P_\mu(G))(x)E_\mu(h|P_\mu(G))(x)d\mu(x)$$

$$= \int_{X \times X} f(x)h(y)d\lambda(x, y).$$

The family $\mathcal{H}$ of continuous functions $H$ on $X \times X$ such that

$$\tag{5.5} \int_{X \times X} H(x, y)d\lambda_n(x, y) \rightarrow \int_{X \times X} H(x, y)d\lambda(x, y)$$

is a closed subspace of $C(X \times X)$. By (5.4), it contains the set $\mathcal{H}_s$ of all linear combinations of functions of the form $f(x)h(y)$ such that $f, g \in C(X)$. Since $\mathcal{H}_s$ is dense in $C(X \times X)$, $\mathcal{H} = C(X \times X)$, i.e., (5.5) holds for all $H \in C(X \times X)$, or equivalently, $\lambda_n$ converges weakly to $\lambda$ as $n \to \infty$.

Now let $F$ be a closed subset of $X \times X$ with $hF \supseteq F$ for any $h \in S$. Then $g_n^{-1}(F) \supseteq F$ for $n \in \mathbb{N}$. Since $F$ is closed and $\lambda_n \rightarrow \lambda$ weakly, we have by (5.3) that

$$\lambda(F) \geq \limsup_{n \to \infty} \lambda_n(F) = \limsup_{n \to \infty} \lambda_0(g_n^{-1}(F)) \geq \lambda_0(F).$$

This proves the claim because the case of an open subset $U$ of $X \times X$ simply follows by setting $U = X \setminus F$.

**Claim 3:** $\lambda_0(W) = 1$.

Since $S$ is semigroup and $S \setminus \{e_G\} \subseteq \Phi^{-1}$, we have

$$S(S \setminus \{e_G\}) \subseteq S \cap (\Phi^{-1} \cup \{e_G\}) \Phi^{-1} \subseteq S \cap \Phi^{-1} = S \setminus \{e_G\},$$
i.e., $S(S \setminus \{e_G\}) \subset S \setminus \{e_G\}$. For given $g \in G$ and $i \in \mathbb{N}$, by the construction of $U_{i,g}$ and $V_{i,g}$, we know that

$$(5.6) \quad h(U_{i,g}) \subseteq U_{i,g} \text{ and } h(V_{i,g}) \subseteq V_{i,g}, \quad h \in S.$$

Since $\lambda$ is $G$-invariant, we have

$$\lambda(h(V_{i,g}) \Delta V_{i,g}) = 0 \text{ and } \lambda(h(U_{i,g}) \Delta U_{i,g}) = 0, \quad h \in S.$$ 

Let

$$G_* := \{ h \in G : \lambda(h(V_{i,g}) \Delta V_{i,g}) = 0 \text{ and } \lambda(h(U_{i,g}) \Delta U_{i,g}) = 0 \}.$$ 

Then $G_*$ is a subgroup of $G$ and $S \subseteq G_*$. It follows from the fact $< S > = G$ that $G_* = G$. Thus,

$$\lambda(h(V_{i,g}) \Delta V_{i,g}) = 0 \text{ and } \lambda(h(U_{i,g}) \Delta U_{i,g}) = 0, \quad h \in G.$$ 

By the ergodicity of $\lambda$ and noting that $\lambda(U_{i,g}) \geq \lambda(U_i) > 0$ and $\lambda(U_{i,g}) \geq \lambda(V_i) > 0$, we have

$$(5.7) \quad \lambda(U_{i,g}) = \lambda(V_{i,g}) = 1.$$ 

Combining Claim 2 with (5.6) and (5.7), we have $\lambda_0(U_{i,g}) = \lambda_0(U_{i,g}) = 1$. Since $g, i$ are arbitrary, the claim follows.

To proceed with the proof of the theorem, we let $\mu = \int_X \mu_x d\mu(x)$ be the disintegration of $\mu$ over $\mathcal{F}$. Then by Claim 3,

$$1 = \lambda_0(W) = \int_X \mu_x \times \mu_x(W) d\mu(x).$$

Thus for $\mu$-a.e. $x \in X$,

$$(5.8) \quad \mu_x \times \mu_x(W) = 1.$$ 

Since $h_\mu(G) > 0$, we have by Lemma 4.3 2) that $\lambda(\Delta_X) = 0$. Note that $h(\Delta_X) = \Delta_X$, $h \in S$. We then have by Claim 2 that $\lambda_0(\Delta_X) \leq \lambda(\Delta_X) = 0$. Thus,

$$\int_X \mu_x \times \mu_x(\Delta_X) d\mu(x) = \lambda_0(\Delta_X) = 0,$$

i.e., for $\mu$-a.e. $x \in X$, $\mu_x \times \mu_x(\Delta_X) = 0$. This implies that $\mu_x$ is non-atomic for $\mu$-a.e. $x \in X$. Combing this with (5.8), we have that for $\mu$-a.e. $x \in X$, $W \cap \text{supp}(\mu_x) \times \text{supp}(\mu_x)$ is a dense $G_\delta$ subset of $\text{supp}(\mu_x) \times \text{supp}(\mu_x)$ and $\text{supp}(\mu_x)$ is a perfect set. By Lemma 5.1 and Claim 1, we have that for $\mu$-a.e. $x \in X$, $\text{supp}(\mu_x)$ contains a $(S^{-1}, \delta)$-Li-Yorke set which is a union of countably many Cantor sets. But since $\mu_x(\mathcal{P}_\delta(x)) = 1$ for $\mu$-a.e. $x \in X$, $\text{supp}(\mu_x) \subseteq \overline{\mathcal{P}_\delta(x)} \subseteq W_S(x, G)$ for $\mu$-a.e. $x \in X$. In other words, for $\mu$-a.e. $x \in X$, $W_S(x, G)$ contains a $(S^{-1}, \delta)$-Li-Yorke set which is a union of countably many Cantor sets. This completes the proof. \qed
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