5. Let \( x_0, x_1, \ldots, x_n \) be distinct real numbers and \( l_k(x) \) be the Lagrange's basis function. \( \psi_n = \prod_{k=0}^{n} (x-x_k) \). Prove that

(A) \( \sum_{k=0}^{n} l_k(x) = 1 \); 
(B) \( \sum_{k=0}^{n} x_k^j l_k(x) = x_j \), for \( j = 0, 1, \ldots, n \);
(C) \( \sum_{k=0}^{n} (x_k-x)^j l_k(x) = 0 \) for \( j = 0, 1, 2, \ldots, n \);

(D) Let \( p(x) \) be any polynomial of degree \( n+1 \) with its highest degree coefficient \( a_{n+1} = 1 \). Then \( p(x) = \sum_{k=0}^{n} p(x_k) l_k(x) = \psi_n(x) \).

(E) Let \( p_n(x) \) interpolate \( f(x) \) at \( x_0, \ldots, x_n \). Then

\[
\hat{p}_n(x) = \frac{\sum_{k=0}^{n} f(x_k) / \left[ \psi_n'(x_k) (x-x_k) \right]}{\sum_{k=0}^{n} 1 / \left[ \psi_n'(x_k) (x-x_k) \right]}
\]

This is the barycentric representation of \( p_n(x) \).

6. Let \( f(x) = a_m x^m + \text{lower degree terms} \) be a polynomial. Show that

\[
f[x_0, \ldots, x_n, x] = \begin{cases} \text{polynomials of degree } m-n-1, & n < m-1 \\ a_m & n = m-1 \\ 0 & n > m-1 \end{cases}
\]

7. Consider the following table of values for a function \( f_0(x) \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_0(x) )</td>
<td>1.0000</td>
<td>0.99833</td>
<td>0.99355</td>
<td>0.98507</td>
<td>0.97355</td>
<td>0.95885</td>
<td>0.94107</td>
<td>0.9203</td>
<td>0.89670</td>
<td>0.87036</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.0</th>
<th>1.1</th>
<th>1.2</th>
<th>1.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_0(x) )</td>
<td>0.84147</td>
<td>0.81019</td>
<td>0.77670</td>
<td>0.74120</td>
</tr>
</tbody>
</table>

What should be the maximum degree of polynomial interpolation used with the table? Hint: Use the forward difference table to detect the influence of the rounding errors.