Solution to the practice problems

(1) (1) True. We can simply plug in \( f(t) \) into the differential equation to check that the two sides are equal.

(2) False. We have \((y_1 + y_2)' = p(t)(y_1 + y_2) + 2q(t)\), hence it does not solve the same equation unless \( g(t) \) is always 0.

(3) True. Note that \( 2 + e^{ty} > 0 \) for any \( t \) and \( y \), hence any solution must satisfy \( y'(t) > 0 \), which means \( y \) is an increasing function of \( t \).

(4) False. Just \( f \) being continuous is not enough; we also need \( \frac{\partial f}{\partial y} \) to be continuous around \((t_0, y_0)\).

(2) (1) This equation is separable (since right hand side can be written as \( e^{2x}e^y \)), so we can solve it as

\[
\int e^{-y}dy = \int e^{2x}dx + C \implies -e^{-y} = \frac{1}{2}e^{2x} + C.
\]

Plugging in \( x = 0, y = 0 \) gives \( C = -3/2 \), hence \( y = -\ln(\frac{3}{2} - \frac{1}{2}e^{2x}) \).

(2) \( x \frac{dy}{dx} = x^5 \ln x + 3y \), \( x > 0 \). This is a first order linear equation, and it is \( y' - 3y = x^4 \ln x \) in standard form. Therefore its integrating factor is \( \mu(x) = e^{\int -\frac{3}{x}dx} = x^{-3} \), and multiplying it to both sides leads to \((xy^{-3})' = x \ln x \). Integrating it yields \( xy^{-3} = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C \), hence the general solution is \( y = \frac{x^5 \ln x}{2} - \frac{x^5}{4} + Cx^3 \).

(3) This equation is exact since \( \frac{\partial}{\partial y}(\cos x + y^2) = 2y \), and \( \frac{\partial}{\partial x}(2xy) = 2y \) too. So we only need to look for a function \( \psi \) such that \( \frac{\partial \psi}{\partial x} = \cos x + y^2 \), and \( \frac{\partial \psi}{\partial y} = 2xy \). We have \( \psi(x) = \int (\cos x + y^2)dx + h(y) = \sin x + xy^2 + h(y) \), and taking the partial derivative in \( y \) and comparing it with \( 2xy \) gives us \( h(y) = 0 \). Hence the general solution is \( \sin x + xy^2 = C \), and plugging in \( x = \pi, y = -1 \) gives \( C = \pi \). Finally, the explicit solution is \( y = -\sqrt{\frac{2}{\sin x}} \). (We take the negative sign in front of the square root, since \( y(\pi) \) should be negative rather than positive).

(3) This is a first-order linear equation, and we first rewrite it into the standard form

\[
y' + \frac{t^2}{t^2 - 4}y = \frac{e^t}{t^2 - 4}.
\]

The two functions \( \frac{t^2}{t^2 - 4} \) and \( \frac{e^t}{t^2 - 4} \) are continuous in \((-\infty, -2), (-2, 2) \) and \((2, +\infty) \). Here the initial condition is \( y(-3) = 4 \), hence \((-\infty, -2) \) is the interval containing \(-3 \). By the theorem we learned in class, \((-\infty, -2) \) is the largest interval in which a unique solution is guaranteed to exist.

(4) By Theorem 2.4.2, for a nonlinear equation \( y' = f(t, y) \), if both \( f \) and \( \frac{\partial f}{\partial y} \) are continuous in some rectangle containing \((t_0, y_0)\), then the solution with initial condition \( y(t_0) = y_0 \) is certain to exist in some small interval containing \( t_0 \). For this question, we have \( f(t, y) = y^{1/3} \). Note that \( f \) is continuous everywhere except \( t = -1 \), and \( \frac{\partial f}{\partial y} = \frac{y^{-2/3}}{3(1+y)} \) is continuous everywhere except on \( y = 0 \) and \( t = -1 \). Therefore as long as \( t_0 \neq -1 \) and \( y_0 \neq 0 \), a unique solution is certain to exist in some interval containing \( t_0 \).

(5) The first step is to figure out what is the differential equation here. Let \( y = f(x) \) be the equation of the curve. Since its slope at \((x, y)\) should be \( xy \), it means it satisfies \( y'(x) = xy \). This is a separable equation, whose general solution is \( y = Ce^{x^2/2} \). Since we are also given that \( y(x) \) passes through \((1, 2) \), it means \( y(1) = 2 \), hence \( C = 2/e^{1/2} \). Finally the equation of the curve is

\[
y = \frac{2}{e^{1/2}}e^{x^2/2}.
\]

Note that the exponent is \( x^2/2 \) is the smallest when \( x = 0 \), hence the function attains its minimum value \( \frac{2}{e^{1/2}} \) when \( x = 0 \).
(6) We are looking for an equation of the form \( y' = f(y) \). To find a formula of \( f \), it would be helpful to use the phase line method to figure out the sign of \( f \) in the intervals. One possible choice would be \( y' = -(x - 1)(x - 2)(x - 3) \).

Of course, the correct answer here is not unique and your function \( f \) may differ from mine. As long as your function \( f \) has 1,2,3 as zeros and satisfies \( f > 0 \) in \( (-\infty, 1) \cup (2, 3) \) and \( f < 0 \) in \( (1, 2) \cup (3, +\infty) \), it would be a correct answer.

(7) The equilibrium solutions are \( y = 1 \) and \( y = 5 \). Using the phase line method (see the graph below on the left), we know \( y = 1 \) is unstable and \( y = 5 \) is asymptotically stable. For a sketch of the solution curves corresponding to initial conditions \( y(0) = 1,3,5,7 \), see the graphs on the right.

\[
\begin{align*}
\int \frac{dy}{(y-1)(y-5)} &= \int -1 dt + C.
\end{align*}
\]

We can use partial fraction decomposition to write \( \frac{1}{(y-1)(y-5)} \) as \( \frac{1}{y-1} - \frac{1}{y-5} \), hence integrating it gives
\[
\frac{1}{4} \left( \ln |y-5| - \ln |y-1| \right) = -t + C \Rightarrow \ln \left| \frac{y-5}{y-1} \right| = -4t + C \Rightarrow y-5 = Ce^{-4t}.
\]

If you are asked to make it explicit, then we solve for \( y \), which gives
\[
y - 5 = Ce^{-4t}(y - 1) \Rightarrow (1 - Ce^{-4t})y = 5 - Ce^{-4t} \Rightarrow y = \frac{5 - Ce^{-4t}}{1 - Ce^{-4t}}.
\]

(8) Let \( M(x,y) = 3xy + 1 \), and \( N(x,y) = x^2 + \frac{y}{x} \). Note that \( \frac{\partial M}{\partial y} = 3x \) and \( \frac{\partial N}{\partial x} = 2x - \frac{y}{x^2} \). They are not equal, so the equation is not exact. However, when both \( M \) and \( N \) are multiplied by \( x \), we have \( \frac{\partial M}{\partial y} = 3x^2 \) and \( \frac{\partial N}{\partial x} = 3x^2 \), so the equation becomes exact.

To solve for it, we look for a function \( \psi \) such that \( \frac{\partial \psi}{\partial x} = 3x^2y + x \), and \( \frac{\partial \psi}{\partial y} = x^3 + y \). We have
\[
\psi(x) = \int (3x^2y + x) dx + h(y) = x^3y + \frac{x^2}{2} + h(y),
\]

and taking the partial derivative in \( y \) and comparing it with \( x^3 + y \) gives us \( h(y) = \frac{y^2}{2} \). Hence the general solution is
\[
x^3y + \frac{x^2}{2} + \frac{y^2}{2} = C,
\]

and plugging in the initial condition \( x = 1, y = 1 \) gives \( C = 2 \). Thus \( x^3y + \frac{x^2}{2} + \frac{y^2}{2} = 2 \). Let’s write it into a quadratic equation of \( y \):
\[
y^2 + 2x^3y + (x^2 - 4) = 0.
\]

Using the quadratic formula gives
\[
y = \frac{-2x^3 \pm \sqrt{4x^6 - 4(x^2 - 4)}}{2} = -x^3 \pm \sqrt{x^6 - x^2 + 4}.
\]

We should pick the plus sign to ensure that \( y(1) = 1 \), so finally the solution is \( y = -x^3 + \sqrt{x^6 - x^2 + 4} \).
(9) Let $y(t)$ be the population of rats after $t$ years. Then we have $y(0) = 200$, and

$$y'(t) = \frac{1}{2} y(t) - 250.$$ 

You can use your favorite method to solve it (this equation is both separable and first order linear, so you can use either method). The solution to this IVP is $y(t) = 500 - 300e^{t/2}$, which will become 0 at $t = 2\ln \frac{5}{3}$. So the rats will become extinct in finite time.

(10) Let $y(t)$ be the amount of salt (in kg) in the tank $t$ minutes after noon. The initial condition is $y(0) = 0$. Then we have

$$y'(t) = \text{rate in} - \text{rate out},$$

where the rate in is 0 (since pure water is flowing in), and rate out is equal to $10y(t)/V(t)$, where $V(t)$ is the volume of liquid in the tank at time $t$. Since water is entering the tank at a rate 8 L/min and leaving at rate 10 L/min, it means $V(t) = 100 + 8t - 10t = 100 - 2t$. Therefore we have

$$y'(t) = -\frac{10y}{100 - 2t} = \frac{5y}{t - 50}.$$ 

This is a separable equation (and also first order linear), so you can use either way to solve it. The general solution is

$$y(t) = C(t - 50)^5,$$

And since $y(0) = 10$ we have $C = 10/(-50)^5$. Plugging it in gives $y(25) = \frac{10 \cdot (-25)^5}{(-50)^5} = \frac{10}{25} = \frac{5}{16}$.

(11) Let $x_1 = y$, and $x_2 = y'$. Then the second order equation can be transformed into

$$x_1' = x_2$$
$$x_2' = 4x_1 + 3x_2,$$

which can be written into matrix form as

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix} \mathbf{x}.$$ 

The eigenvalue of this matrix is $\lambda_1 = 4$ and $\lambda_2 = -1$, and their corresponding eigenvectors are

$v_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Therefore the general solution of the system is

$$\mathbf{x}(t) = C_1 e^{4t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$ 

The phase portrait is as below, where $(0, 0)$ is a saddle point (hence unstable).
(12) (1) When \( c = 3 \), the eigenvalue of the matrix is \( \lambda_1 = -5 \) and \( \lambda_2 = -1 \), and their corresponding eigenvectors are

\[ \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \]

Since both eigenvalues are negative, the phase portrait is a nodal sink, and its phase portrait looks like the following. (The red trajectory is the trajectory with initial condition \((2,4)\)).

To solve for the IVP with initial condition \((2,4)\), first note that the general solution is

\[ \mathbf{x}(t) = C_1 e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \]

Plugging in \( t = 0 \) and applying the initial condition gives

\[ C_1 + 3C_2 = 2, \quad -C_1 + C_2 = 4. \]

The solution to this system of equations is \( C_1 = -5/2, C_2 = 3/2 \), hence the solution to the IVP is

\[ \mathbf{x}(t) = -\frac{5}{2} e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{3}{2} e^{-t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \]

(2) For a general \( c \), the eigenvalues satisfies the equation

\[ (-2 - \lambda)(-4 - \lambda) - c \cdot 1 = 0, \]

which becomes \( \lambda^2 + 6\lambda + (8 - c) = 0 \). In order for \((0,0)\) to become a saddle point, the two solutions must have different signs. Since the product of the two solutions to the quadratic equation is \( 8 - c \), it means we need \( 8 - c < 0 \), hence \( c > 8 \).