1. True or false:

(1) The function \( y(t) = t \sin t \) is a solution of \( y'' + y = 2 \cos t \). TRUE FALSE

(2) The equation \( ty'' + 3e^t y = 0 \) with initial condition \( y(1) = 0, y'(1) = 2 \) is certain to have a unique solution in \((-\infty, \infty)\). TRUE FALSE

(3) For a nonlinear autonomous system, the type and stability of a critical point is always the same as those of its approximating linear system. TRUE FALSE

(4) If \( (x_0, y_0) \) is an asymptotically stable critical point for an autonomous system, then every solution \( (x(t), y(t)) \) must converge to \( (x_0, y_0) \) as \( t \to +\infty \). TRUE FALSE

(5) Every first order autonomous system must have at least one critical point. TRUE FALSE

(6) If a \( 2 \times 2 \) linear system \( \mathbf{x}' = A \mathbf{x} \) has repeated eigenvalue \( \lambda = -2 \), then all solutions \( \mathbf{x}(t) \) must converge to the origin as \( t \to +\infty \). TRUE FALSE

(7) If the point \( (1, 2) \) is a spiral sink for the system \( x' = F(x, y), y' = G(x, y) \), then the system \( x' = -F(x, y), y' = -G(x, y) \) must have \( (1, 2) \) as a spiral source. TRUE FALSE

2. For each of the following system of equations, find the general solutions (in the case of complex eigenvalues, real solutions are preferred), identify the type and stability of the critical point \((0, 0)\), and sketch the phase portrait.

(1) \[
\begin{cases} 
  x'(t) = 2x - 5y, \\
  y'(t) = 4x - 2y. 
\end{cases}
\]

(2) \[
\begin{cases} 
  x'(t) = -7x - 9y, \\
  y'(t) = x - y. 
\end{cases}
\]

(3) \[
\begin{cases} 
  x'(t) = x + 4y, \\
  y'(t) = -x + y. 
\end{cases}
\]

3. Consider the initial value problem

\[
(x^2 - 9)y'' - y' + (\ln |x|)y = e^t, \quad y(2) = 4, y'(2) = 0.
\]

According to the existence and uniqueness theorem, what is the largest interval in which a unique solution is guaranteed to exist?

(Correction: in the previous version there was a typo on the initial condition, and it is fixed above.)

4. Solve the initial value problem \( y'' + 7y' + 10y = 0, y(0) = 2, y'(0) = -1 \). As \( t \to +\infty \), what is the long time behavior of \( y(t) \)?

5. Verify that \( y_1(t) = t^3 \) and \( y_2(t) = t^2 \) are both solutions to the equation \( t^2y'' - 4ty' + 6y = 0 \) in the interval \( t > 0 \). Using their Wronskian, check whether they form a fundamental set of solutions in this interval. Then solve the initial value problem with \( y(1) = 0, y'(1) = 2 \).
6. Formulate the second order equation \( x''(t) + x'(t) + c \sin(x(t)) = 0 \) into a first order system, and find all its critical points. In order for the critical point \((0, 0)\) to be a nodal sink, what is the range of \(c\)?

7. Consider the following system of differential equations with parameter \(a\):
   \[
   \begin{align*}
   \mathbf{x}' &= \begin{pmatrix} a & -2 \\ 2 & 1 \end{pmatrix} \mathbf{x}.
   \end{align*}
   \]
   Determine all values of \(a\) (if any) such that the critical point \((0, 0)\) is of each of the following type:
   (1) Saddle point
   (2) Nodal source
   (3) Nodal sink
   (4) Spiral sink
   (5) Spiral source

8. Consider the system of equations
   \[
   \begin{align*}
   x'(t) &= x^2 - y^2 \\
   y'(t) &= x^2 + y^2 - 8.
   \end{align*}
   \]
   (1) Find all critical points of this system.
   (2) For the critical point in the \(\{x < 0, y < 0\}\) quadrant, find the approximating linear system near it.
   (3) What is the type and stability of the critical point in (b)? Sketch the phase portrait near this critical point.

9. Assume the motion of a mass-spring system is given by \(y'' + \gamma y' + 9y = 0\), with initial condition \(y(0) = 0, y'(0) = 3\). Here \(\gamma \geq 0\) is the damping constant.
   (1) If \(\gamma = 0\), find the amplitude and period of the oscillation.
   (2) For what range of \(\gamma\) will the system be over-damped?

10. On a strange island, two native species, the polar bears and the penguins, participate in a very predictable relationship. The polar bear population (in thousands) is described over time by a function \(x(t)\), where \(t\) is measured in months; the penguin population is described by a function \(y(t)\). Naturalists coming to the island observe that the rates of growth of the two species are given by the equations
    \[
    \begin{align*}
    x'(t) &= -4x + 0.2xy \\
    y'(t) &= 4y - 0.1y^2 - 0.1xy,
    \end{align*}
    \]
    (1) What is the relationship between the polar bears and penguins? Choose one: (a) competition, (b) mutual cooperation, (c) predator-prey (if this is case, specify which plays which role). Justify your answer by explaining how the equations show what effect each species has on the other.
    (2) Find all critical points to the above system.
    (3) There is only one critical point where both species have positive population. What is the type and stability of this critical point?