SOLUTION TO THE PRACTICE PROBLEMS

(1) True. To check it, plug \( y(t) \) into the equation to check whether it satisfies the equation.

(2) False. To apply the existence and uniqueness theorem, we have to first rewrite the equation into the standard form \( y'' + \frac{3}{7}y' - \frac{e^t}{t}y = 0 \), where the IVP is only certain to have a solution in \((0, \infty)\).

(3) False. If the approximating linear system have two different eigenvalues with non-zero real parts, then the type and stability of the critical point in the nonlinear system must be the same; but if the eigenvalues do not satisfy this condition, either the type and stability may be inconclusive. (See Thm 7.2.2)

(4) False. If a critical point is asymptotically stable, only trajectories starting from sufficiently close points are guaranteed to be attracted to the critical point.

(5) False. For example, the system of equations \( x'(t) = 1, y'(t) = 1 \) does not have any critical point.

(6) True. When \( A \) has repeated eigenvalue \( \lambda = -2 \), the origin is either a stable proper node or a stable improper node. In either case, using the formula of the general solution, we can see all solutions converge to 0 as \( t \to \infty \).

(7) True. For the new system with negative right hand sides, the trajectories follow the same curves as the original system, except that all the arrows are reversed. Hence if \((1,2)\) is originally a spiral sink, after reversing all the arrows it would become a spiral source.

(2) (1) Its eigenvalue is \( \lambda = \pm 4i \). For \( \lambda = 4i \), its corresponding eigenvector is \((5, 2 - 4i)\). So the origin is a center, and it is stable (but not asymptotically stable). To find the general solution, first note that we have a solution

\[
e^{4it} \left( \begin{array}{c} 5 \\ 2 - 4i \end{array} \right) = (\cos(4t) + i \sin(4t)) \left( \begin{array}{c} 5 \\ 2 - 4i \end{array} \right) = \left( \begin{array}{c} 5 \cos(4t) \\ 2 \cos(4t) + 4 \sin(4t) \end{array} \right) + i \left( \begin{array}{c} 5 \sin(4t) \\ -4 \cos(4t) + 2 \sin(4t) \end{array} \right)
\]

We have learned that by taking its real and imaginary part respectively, we get a fundamental set of solutions. Hence the general solution is

\[
y(t) = C_1 \left( \begin{array}{c} 5 \cos(4t) \\ 2 \cos(4t) + 4 \sin(4t) \end{array} \right) + C_2 \left( \begin{array}{c} 5 \sin(4t) \\ -4 \cos(4t) + 2 \sin(4t) \end{array} \right).
\]

To sketch the phase portrait, we need to determine in which direction the trajectories circle around the origin. Let us plug in a point \((0,1)\), which gives us \( x' = (-5, -2) \), which means the trajectories are going counterclockwise. Hence the phase portrait looks like
The matrix has repeated eigenvalue \(-4\), with \(v = \begin{pmatrix} 3 \\ -1 \end{pmatrix}\) being the only eigenvalue. In this case, we have the general solution is \(y(t) = C_1e^{4t} + C_2e^{4t}(tv + w)\), where \(w\) is any solution to \((A - \lambda I)w = v\), that is,

\[
\begin{pmatrix} -7 - (-4) & -9 \\ 1 & -1 - (-4) \end{pmatrix} w = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.
\]

One of the possible solutions is \(w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\). Thus the general solution is

\[x(t) = C_1e^{4t} \begin{pmatrix} 3 \\ -1 \end{pmatrix} + C_2e^{4t} \left( t \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right),\]

and its phase portrait looks like the following:

(3) Its eigenvalue is \(\lambda = 1 + 2i\). For \(\lambda = 1 + 2i\), its corresponding eigenvector is \((2, i)\). So the origin is an (unstable) spiral source. To find the general solution, we start with the solution

\[e^{(1+2i)t} \begin{pmatrix} 2 \\ i \end{pmatrix} = e^t(\cos(2t) + i\sin(2t)) \begin{pmatrix} 2 \\ i \end{pmatrix} = e^t \begin{pmatrix} 2\cos(2t) \\ -i\sin(2t) \end{pmatrix} + ie^t \begin{pmatrix} 2\sin(2t) \\ \cos(2t) \end{pmatrix}\]

By taking its real and imaginary part respectively as a fundamental set of solutions, we know the general solution is

\[y(t) = C_1e^t \begin{pmatrix} 2\cos(2t) \\ -\sin(2t) \end{pmatrix} + C_2e^t \begin{pmatrix} 2\sin(2t) \\ \cos(2t) \end{pmatrix}.
\]

To sketch the phase portrait, we need to determine in which direction the trajectories circle around the origin. Let us plug in a point \((0, 1)\), which gives us \(x' = (4, 1)\), which means the trajectories are going clockwise. Hence the phase portrait looks like a spiral source, whose trajectories are going clockwise around the origin. (The sketch is omitted).

(3) (Correction: in the previous version there was a typo on the initial condition, and please fix it into \(y(2) = 4, y'(2) = 0\). This is a second-order linear equation, and we first rewrite it into the standard form

\[y'' - \frac{1}{x^2 - 9} y' + \frac{\ln|x|}{x^2 - 9} y = \frac{e^t}{x^2 - 9}.\]

Now we look for all the intervals in which \(\frac{1}{x^2 - 9}, \frac{\ln|x|}{x^2 - 9}\) and \(\frac{e^t}{x^2 - 9}\) are continuous. They are continuous except at \(x = \pm 3\) and \(x = 0\). (The point \(x = 0\) is bad since \(\ln|x|\) is not defined there). Since the initial condition is taken at \(t_0 = 2\), the largest interval in which a unique solution is guaranteed to exist is \((0, 3)\).

(4) The corresponding characteristic equation is \(\lambda^2 + 7\lambda + 10 = 0\), which has solution \(\lambda_1 = -5\) and \(\lambda_2 = -2\). Hence the general solution to the 2nd order equation is

\[y(t) = C_1e^{-5t} + C_2e^{-2t}.
\]
The initial condition gives $2 = C_1 + C_2$, $-1 = -5C_1 - 2C_2$. Solving this system gives $C_1 = -1, C_2 = 3$, hence the solution to the IVP is

$$y(t) = -e^{-5t} + 3e^{-2t}.$$ 

As $t \to \infty$, we have $y(t)$ approaches 0.

(5) To check they are both solutions, simply plug each of them in the equation and verify that they satisfy the differential equation. (The computation is omitted here.) To check whether they form a fundamental solution for the interval $t > 0$, we can compute their Wronskian and check whether it is non-zero.

$$W[y_1, y_2](t) = \begin{vmatrix} t^3 & t^2 \\ 3t^2 & 2t \end{vmatrix} = -t^4,$$

which is always nonzero for $t > 0$, hence they form a fundamental solution in the interval $t > 0$. This implies the general solution to the differential equation is

$$y(t) = C_1 t^2 + C_2 t^3.$$

Let’s now use the initial condition to solve for $C_1$ and $C_2$: $0 = y(1) = C_1 + C_2$, $2 = y'(1) = 2C_2 + 3C_2$. This gives $C_1 = -2$ and $C_2 = 2$, hence $y(t) = -2t^2 + 2t^3$.

(6) By setting $y(t) = x'(t)$, the 2nd order equation can be transformed into a first order system

\[
\begin{align*}
x'(t) &= y, \\
y'(t) &= -c \sin x - y.
\end{align*}
\]

If $c \neq 0$, its critical points are $(k\pi, 0)$ for any integer $k$. (If $c = 0$, all points of the form $(a, 0)$ with any $a$ would be a critical point).

Its Jacobian is $J(x, y) = \begin{pmatrix} 0 & 1 \\ -c \cos x & -1 \end{pmatrix}$, hence its Jacobian at $(0, 0)$ is $J(0, 0) = \begin{pmatrix} 0 & 1 \\ -c & -1 \end{pmatrix}$, whose eigenvalues are given by $\lambda^2 + \lambda + c = 0$. So its eigenvalues are $\lambda = -\frac{1}{2} \pm \frac{\sqrt{1 - 4c}}{2}$. In order for $(0, 0)$ to be a nodal sink, we need two different negative real eigenvalues, hence we need to have $1 - 4c > 0$ and $c > 0$, which gives $0 < c < \frac{1}{4}$.

(7) The eigenvalues satisfy the quadratic equation $\lambda^2 - (a + 1)\lambda + (a + 4) = 0$. So its eigenvalues are

$$\lambda = \frac{a + 1}{2} \pm \frac{\sqrt{(a+1)^2 - 4(a+4)}}{2} = \frac{a + 1}{2} \pm \frac{\sqrt{a^2 - 2a - 15}}{2} = \frac{a + 1}{2} \pm \frac{\sqrt{(a - 5)(a + 3)}}{2}.$$ 

In addition, recall that for an equation $\lambda^2 + p\lambda + q = 0$, the product of two solutions is $q$, and the sum of two solutions is $-p$. So in our situation, the product of the two eigenvalues is $a + 4$, and the sum of the two eigenvalues is $a + 1$.

(1) In order for $(0, 0)$ to be a saddle point, we need solutions to be of opposite sign, so $a + 4 < 0$, or in other words $a < -4$. (Once this is satisfied, the terms in the square root must be positive, so we indeed have two real solutions.)

(2) In order for $(0, 0)$ to be a nodal source, we need two different positive eigenvalues. So we need (1) $a + 4 > 0$; (2) $a + 1 > 0$; (3) $(a - 5)(a + 3) > 0$. Note that (3) means either $a > 5$ or $a < -3$.

Putting all three conditions together, $a > 5$ is the only interval satisfying all conditions.

(3) In order for $(0, 0)$ to be a nodal sink, we need two different negative eigenvalues. So we need (1) $a + 4 > 0$; (2) $a + 1 < 0$; (3) $(a - 5)(a + 3) > 0$. Putting them together, the only interval satisfying all three conditions is $-4 < a < -3$.

(4) In order for $(0, 0)$ to be a spiral sink, we need two complex eigenvalues with negative real parts. So we need (1) $(a - 5)(a + 3) < 0$; (2) $a + 1 < 0$. These two conditions give $-3 < a < 1$.

(5) In order for $(0, 0)$ to be a spiral source, we need two complex eigenvalues with positive real parts. So we need (1) $(a - 5)(a + 3) < 0$; (2) $a + 1 > 0$. These two conditions give $1 < a < 5$. 


There are four critical points in total, and they are $(2, 2), (-2, 2), (2, -2)$ and $(-2, -2)$.

The only critical point in the $\{x < 0, y < 0\}$ quadrant is $(-2, -2)$. The Jacobian at this point is 
\[
J(-2, -2) = \begin{pmatrix} -4 & 4 \\ -4 & -4 \end{pmatrix}.
\]
Its eigenvalues are $\lambda = -4 \pm 4i$, which corresponds to a spiral sink. To sketch the phase portrait, we plug in a point $(0, 1)$ into the approximating linear system
\[
\mathbf{u}'(t) = \begin{pmatrix} -4 & 4 \\ -4 & -4 \end{pmatrix} \mathbf{u},
\]
so the trajectory at $(0, 1)$ is going in the direction $(4, -4)$. It is going towards the right, so it means the trajectories of the spiral is going clockwise. So for the original nonlinear system, the trajectories near $(-2, -2)$ is a spiral sink going clockwise. (The sketch is omitted here).

If $\gamma = 0$, the characteristic equation is $\lambda^2 + 9 = 0$, hence $\lambda = \pm 3i$. As a result, the general solution is $y(t) = C_1 \cos(3t) + C_2 \sin(3t)$. Applying the initial condition, we have $C_1 = 0, C_2 = 1$. Hence $y(t) = \sin(3t)$ is the solution, where the amplitude is 1 and period is $\frac{2\pi}{3}$.

The characteristic equation is $\lambda^2 + \gamma \lambda + 9 = 0$, and its eigenvalues are $\lambda = -\frac{\gamma}{2} \pm \frac{\sqrt{\gamma^2 - 36}}{2}$. If the system is over-damped, it must have two real negative eigenvalues, which corresponds to the case $\gamma > 6$.

The relationship is predator-prey, and polar bear (with corresponding population $x$) is the predator, while penguin is the prey. To see this, note that $y$ has a positive effect on the growth of $x$, while $x$ has a negative effect on the growth of $y$.

By setting the right hand sides equal to zero, we have
\[
\begin{cases}
x(-4 + 0.2y) = 0 \\
y(4 - 0.1y - 0.1x) = 0,
\end{cases}
\]
which has three solutions $(0, 0), (0, 40), (20, 20)$.

The only critical point where both species have positive population is $(20, 20)$. The Jacobian is 
\[
J(x, y) = \begin{pmatrix} -4 + 0.2y & 0.2x \\ -0.1y & 4 - 0.2y - 0.1x \end{pmatrix},
\]
hence $J(20, 20) = \begin{pmatrix} 0 & 4 \\ -2 & -2 \end{pmatrix}$. Its eigenvalues are $\lambda = -1 \pm \sqrt{7}i$, hence it is a (stable) spiral sink.