NONLINEAR INSTABILITY OF IDEAL PLANE FLOWS

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Abstract. We study nonlinear instability of stationary ideal plane flows. For any bounded domain and very general steady flows, we showed that if the linearized equation has an exponentially growing solution then the steady flow is nonlinearly unstable. The nonlinear instability is in the sense that we can find an initial perturbation arbitrarily close to the steady flow such that the $L^p$ norm of the velocity perturbation grows exponentially beyond a fixed value. The same result is also proved for the Charney-Hasegawa-Mima equation.

1. Introduction

We consider a two-dimensional incompressible inviscid flow satisfying Euler equations

\begin{align}
\frac{\partial}{\partial t} u + (u \cdot \nabla u) + \nabla p &= 0, \\
\nabla \cdot u &= 0,
\end{align}

in a bounded domain $\Omega$ of class $C^2$ with boundary $\partial \Omega$ composed of a finite number of connected components $\Lambda_i$. The boundary condition is

\begin{align}
\frac{\partial}{\partial n} u &= 0 \quad \text{on} \quad \partial \Omega,
\end{align}

where $n$ stands for the unit outer normal of $\partial \Omega$. The vorticity form of (1.1) is given by

\begin{align}
\frac{\partial}{\partial t} \omega + \text{curl}^{-1}(\omega) \cdot \nabla \omega &= 0,
\end{align}

where $u=\text{curl}^{-1} \omega$ is defined for given circulations

\[
\int_{\Lambda_i} u \cdot dl = \Gamma_i.
\]

Using the stream function $\psi$ for $u$, we can rewrite (1.2) as

\begin{align}
\frac{\partial}{\partial t} \omega - \frac{\partial_2 \psi}{\partial_1 \omega} + \frac{\partial_1 \psi}{\partial_2 \omega} &= 0,
\end{align}

The boundary conditions associated with (1.3) are given by

\begin{align}
\psi|_{\Lambda_i} &= \Psi_i, \\
\int_{\Lambda_i} \frac{\partial \psi}{\partial n} &= \Gamma_i,
\end{align}

with $\Psi_i$ depending on time only.

2000 Mathematics Subject Classification. Primary 76E09,76E30; Secondary 35Q35,47N20.

Key words and phrases. Nonlinear instability, Liapunov exponent.
A steady flow satisfying (1.3) and boundary conditions (1.4), (1.5) has a stream function \( \psi_0 \) satisfying
\[
- \partial_2 \psi_0 \partial_1 \omega_0 + \partial_1 \psi_0 \partial_2 \omega_0 = 0,
\]
where \( \omega_0 \equiv -\Delta \psi_0 \) is the associated vorticity and \( u_0 = \nabla^\perp \psi_0 = (\partial_2 \psi_0, -\partial_1 \psi_0) \) is the steady velocity field. An important class of steady flows are those \( \psi_0 \) satisfying the following elliptic equation
\[
-\Delta \psi = g(\psi)
\]
with boundary conditions (1.4), (1.5), where \( g \) is some differentiable function.

In this paper we consider nonlinear instability of an arbitrary steady flow in \( \Omega \). To study instability of a steady flow, usually we start with finding an exponentially growing solution to the linearized equation or a discrete unstable eigenvalue of the linearized operator. The linearized equation of (1.1) is
\[
\partial_t v + u_0 \cdot \nabla v + v \cdot \nabla u_0 + \nabla p = 0, \quad \text{in} \ \Omega
\]
with boundary conditions (1.1b), (1.1c). The problem of linear instability has a long history dated back to 19th century ([22]). But until now very few sufficient conditions for instability are known, though many efforts have been devoted to it. Most of the investigations are about finding necessary conditions for linear instability and are restricted to shear flows and rotating flows. See [3],[4] and the references therein. Recently in [15] [18] we got some sufficient instability conditions for shear flows and rotating flows, some of which are sharp and cover many known results. In [16] we obtained the first sufficient condition for linear instability of general flows satisfying (1.7) in an arbitrary bounded domain.

Once we have a growing mode, the next important issue is to prove nonlinear instability from the linear instability. In this paper we are able to solve this problem rather completely.

We call a real number \( \rho \) a critical value of \( \psi_0 \in C^2(\Omega) \) if \( \psi_0 \) takes the value \( \rho \) at some critical point. Our main theorem is the following.

**Theorem 1.** Consider a bounded domain \( \Omega \) of class \( C^2 \) with \( \partial \Omega = \bigcup \Lambda_i \) and a steady flow \( u_0 \) with \( \omega_0, \psi_0 \in C^2(\Omega) \). Assume (a) \( \psi_0 \) has a finite number of critical values in \( \Omega \) and (b) \( \omega_0|_{\Lambda_i} \) is constant. Suppose there exists an exponentially growing solution \( e^{\mu t} v \) \( (x^1, x^2) \) \( (\Re \lambda > 0) \) to the linearized equation (1.8) with \( v = (v_1, v_2) \in H^1(\Omega)^2 \). Let \( \omega = \text{curl} v \in L^2(\Omega) \) and \( \mu \) be the Liapunov exponent of the steady velocity field \( u_0 \). Then we have

(i) (Regularity of growing modes) \( \omega \in W^{1,p} \cap L^{p_1} \) for all \( 1 \leq p < p^* \) and \( 1 \leq p_1 < \infty \). Here \( p^* = \infty \) if \( \mu \leq \Re \lambda \) and \( p^* = \frac{\mu}{\mu - \Re \lambda} \) if \( \mu > \Re \lambda \).

(ii) (Nonlinear instability) For any \( p_0 > 1, p_1 \in [1, p^*), p_2 \in [1, \infty) \) there exists positive constants \( C, \theta_0, \delta_0 \) and a family of solutions \( \{v_\delta, 0 \leq \delta < \delta_0\} \) to the Euler equation (1.1) satisfying
\[
||\omega_\delta(0) - \omega_0||_{p_2} + ||\partial_t (\omega_\delta(0) - \omega_0)||_{p_1} \leq \delta
\]
where \( \omega_\delta = \text{curl} v_\delta \), and
\[
\sup_{0 < \tau \leq C \ln |\delta|} ||v_\delta(t) - u_0||_{p_0} \geq \theta_0.
\]

Note that Theorem 1 has very little restriction on the steady flows. In particular it applies to all flows with a finite number of critical points and flows satisfying
(1.7). Before discussing the main ideas of the proof, we review related previous results and indicate some difficulties for proving nonlinear instability. Nonlinear instability of 2D Euler is subtle for several reasons. First, the choice of the norm for instability is important. For example, in [16], we constructed a steady flow stable (both linearly and nonlinearly) in the $L^2$ norm of vorticity but linearly unstable in the $L^2$ norm of velocity. We will discuss more about this issue in Section 6. The second difficulty comes from the fact that the linearized operator might have unstable essential spectrum. The linearized operator for (1.1) is

$$L_0 v = -u_0 \cdot \nabla v - v \cdot \nabla u_0 - \nabla p$$

with $v$ satisfying boundary conditions (1.1b) and (1.1c). Define $L^2_{sol} (\Omega) = \{ v = (v_1, v_2) \in L^2 (\Omega)^2 | \nabla \cdot v = 0, v \cdot n = 0 \text{ on } \partial \Omega \}$. Starting with Echhoff in the 1970s ([5]), there are lots of papers (e.g. [6] [14]) using geometric optics or the WKB asymptotic method to treat the local linear instability or the essential spectrum of $L_0$ on $L^2_{sol}$. In [10], when $\Omega$ is a torus $T^2$ it was showed that $\sigma_{L^2_{sol}} (L_0) = \{ z \in \mathbb{C} | 0 \leq \Re z \leq \mu \}$. Here the number $\mu$ is the Liapunov exponent of the flow generated by the steady velocity field $u_0$. Note that $\mu > 0$ is equivalent to that $\psi_0$ has a nondegenerate saddle point. So steady flows with a positive Liapunov exponent are very common. Due to the presence of unstable essential spectrum, it is difficult to prove nonlinear instability from the existence of a growing mode. In [7] a spectrum gap of $L_0$ is required to prove nonlinear instability in $H^s (s > 2)$ norm of velocity and the gap condition is satisfied for flows with $\mu = 0$. In [11] Grenier proved nonlinear instability in the $L^2$ norm of velocity, for shear flows. His argument requires the growing mode to be smooth, which can only be verified for flows with $\mu = 0$ ([8]). So far the best nonlinear instability result is due to Bardos, Guo and Strauss ([1]). They showed that for any bounded domain and general steady flows, linear instability implies nonlinear instability provided the growth rate of the linearized system exceeds the Liapunov exponent of the flow. Their instability result is in terms of $L^2$ norm of the vorticity. Instead of working on velocity equation (1.1), they prove instability for vorticity equation (1.2). The advantage of this approach is that the linearized operator

$$M_0 \omega := -u_0 \cdot \nabla \omega - curl^{-1} (\omega) \cdot \nabla \omega_0$$

for (1.2) has a much simpler structure compared with $L_0$, in particular its essential spectrum lies in the imaginary axis and there is no unstable essential spectrum. The main difficulty there is that the nonlinear term $curl^{-1} (\omega) \cdot \nabla \omega$ contains derivative and is unbounded in any $L^p$ space of $\omega$. This is a typical difficulty in many other problems, and there is no general theory to handle it ([24]). In a series of papers starting with [12], Y. Guo and W. Strauss developed a bootstrap technique to prove nonlinear instability for Vlasov models in plasma physics. Their basis idea is to derive a growth estimate for the derivative of the perturbation from the growth of the perturbation itself, in a time period during which the perturbation is exponentially growing while the amplitude is kept small. In [1], Bardos, Guo and Strauss used this idea to prove nonlinear instability of general ideal plane flows. However the key bootstrap result of $\nabla \omega$ from $\omega$ can only be obtained under the assumption that the growth rate exceeds the Liapunov exponent. After the paper [1] appeared, Friedlander and Vishik ([26]) studied the velocity equation (1.1) on a torus by using the bootstrap argument in [1]. It turned out that the nonlinear term $v \cdot \nabla v$ of (1.1) is easier to handle. The bootstrap result of $\nabla v$ from $v$ can be obtained with no
assumption on the growth rate. But in order to prove nonlinear instability, they still need to assume that the growth rate is greater than the Liapunov exponent due to the unstable essential spectrum of $L_0$. It should be pointed out that a good understanding of $\sigma^{ess}_{L_0}$ and the semigroup property of $e^{tL_0}$ for a domain other than a torus is unavailable by now. This makes it difficult to work on (1.1) in a general domain.

In summary, so far all the known nonlinear instability proofs have to assume the condition that the growth rate exceeds the Liapunov exponent. The novelty of this paper is that we can prove nonlinear instability without this assumption, for general bounded domain and general steady flows. Our nonlinear instability result is in term of any $L^p$ norm of velocity, which is also more physical. Particularly for $p = 2$ case, our result shows that even if the discrete unstable eigenvalue might lie deeply inside the essential spectrum of $L_0$, we can still prove nonlinear instability.

To our knowledge, this is the first result that nonlinear instability can be proved in this situation.

To prove Theorem 1, we need to introduce several new ideas.

First, we introduce what we call the averaging Liapunov exponent $\mu_{av}$. It is defined in the following way

$$\mu_{av} := \lim_{t \to \infty} \frac{1}{t} \ln \int \int_{\Omega} \left| \frac{\partial X_0}{\partial x} \right|(t) \, dx \, dx.$$  

Here $X_0(t; x)$ is the flow induced by the steady velocity field $u_0$, that is

$$\frac{\partial X_0}{\partial x} = u_0(X_0), \quad X_0(0, x) = x,$$

and $\frac{\partial X_0}{\partial x}$ denotes the $2 \times 2$ matrix $(\partial X_0^{i}/\partial x^{j})$ with

$$\left| \frac{\partial X_0}{\partial x} \right| = \left( \sum_{i,j=1,2} \left| \frac{\partial X_0^{i}}{\partial x^{j}} \right|^2 \right)^{\frac{1}{2}}.$$

Recall that the classical Liapunov exponent is defined by

$$\mu := \sup_{x \in \Omega} \lim_{t \to \infty} \frac{1}{t} \ln \left| \frac{\partial X_0}{\partial x} \right|(t; x).$$

As we know, $\mu$ is positive when the steady stream function $\psi_0$ has a nondegenerate saddle point. However, we can show that $\mu_{av}$ is zero for almost any steady flow (see Theorem 2). As we shall see, it is exactly this property which enables us to prove nonlinear instability without the restriction on the growth rate. The idea that the pointwise exponential growth of $\left| \frac{\partial X_0}{\partial x} \right|$ can be killed after integration also appeared in our study of nonlinear instability of Vlasov-Poisson system [17]. But in that case, the particle orbit has a very simple structure. Here we have to do much more delicate analysis of geometric property of particle orbits in order to show that $\mu_{av} = 0$ for general plane flows.

The second new idea in this paper is to study nonlinear instability problem in a coupled way. As we mentioned earlier, there are two approaches to prove nonlinear instability. In the vorticity approach ([1]), the linearized operator has good property but the nonlinear term is difficult to handle. In the velocity approach ([26]) the situation is just opposite: easier nonlinear term with more complicated linearized operator. Here we take a novel approach to utilize advantages of both approaches.
Our method is to study the velocity equation (1.1) by the evolution operator of the vorticity equation (1.2). More precisely, denoting by $\omega(t), v(t)$ the evolution of small perturbations of vorticity and velocity according to the Euler equation, then we have

$$
\omega(t) = e^{tM_0} \omega(0) + \int_0^t e^{(t-s)M_0} (-v \cdot \nabla) \omega(s) \, ds = \omega_{\text{lin}} + \omega_{\text{non}}
$$

and from (1.15)

$$
v(t) = \text{curl}^{-1} \left( e^{tM_0} \omega(0) \right) - \int_0^t \text{curl}^{-1} e^{(t-s)M_0} (\nabla \cdot (v \omega))(s) \, ds
= v_{\text{lin}} + v_{\text{non}}.
$$

To study nonlinear instability, we estimate the growth of $\|v(t)\|_{L^p}$ by using the evolution formula (1.16). In [18], we used this approach to prove nonlinear instability in the $L^2$ norm of velocity under the assumption that the growth rate exceeds half of the Liapunov exponent, for general bounded domain and general flows. The idea in [18] is to consider the term $\nabla \cdot (v \omega)$ in (1.16) as a function in $H^{-1}$ and then use the regularizing effect of $\text{curl}^{-1}$ to get back to $L^2$. In this way the nonlinear term essentially becomes $v \omega$ and is easy to be handled by the bootstrap argument as in [1] and [26]. However, to prove the completed result in Theorem 1 with no restriction on the growth rate, we need another new idea.

Our third new idea is to estimate $L^p$ norm of the velocity in a novel way. We have the following estimate for $v_{\text{non}}$ (in (1.16))

$$
\|v_{\text{non}}\|_p \leq C \sup_{u \in \tilde{H}_{p'}} \int_\Omega \int_\Omega u_{\text{non}} \cdot u \, dx \, dx',
$$

where $\tilde{H}_{p'}$ is the admissible velocity space for the Euler equation (1.1), that is

$$
\tilde{H}_{p'} = \left\{ u \in L^{p'}(\Omega) ; \nabla \cdot u = 0, u \cdot n = 0 \text{ on } \partial \Omega \right\},
$$

with $p' = 1/(1 - 1/p)$. Using the stream function $\psi$ of $u$, we get another formula from (1.17), which for a simply connected domain is

$$
\|v_{\text{non}}\|_p \leq C_1 \sup_{\psi \in W^{1,p'}_0} \int_\Omega \int_\Omega \psi \omega_{\text{non}} \, dx \, dx'.
$$

There are several advantages to do estimate based on (1.17). For example in a simply connected domain, to estimate $\|v_{\text{non}}\|_p$ we only need to estimate the integral on the right hand side of (1.18). Thus we can use integration by parts and duality operations to get rid of the derivative in the term $\nabla \cdot (v \omega(s))$ appeared in the expression of $\omega_{\text{non}}$. In this way, the nonlinear becomes essentially $v \omega$ which is easier to estimate by bootstrap. More importantly, it provides us a rather precise way to estimate $\|v_{\text{non}}\|_p$, and finally enables us to use the crucial property that $\mu_{av} = 0$ to get the desired growth estimate for $\|v_{\text{non}}\|_p$. It can be seen from our proof that when the linear growth rate is very small, we have to take $p$ to be very close to 1. But for this case, we can not directly prove nonlinear instability in $L^p$ for $p$ large, even though it is implied by the instability in $L^p$ for $p$ close to 1. This illustrates that the nonlinear instability of Euler equation is rather subtle and we need a flexible way to deal with it.
Many conservative systems in continuum physics and kinetic theory have similar structure as 2D Euler equation. To prove nonlinear instability for those systems, we often encounter similar difficulties as in this problem. The ideas and methods we introduced in this paper are rather general and flexible. Thus it is expected that we could use these new ideas to treat other nonlinear instability problems. In Section 6, we show that the same instability result can be obtained for 2D quasi-geostrophic flows (Charney-Hasegawa-Mima equation). In [17] we use a similar method to get general nonlinear instability result for one dimensional Vlasov-Poisson system, which improved the previous results ([12] [13]) for the case of small amplitude of electric field.

This paper is organized as follows. In Section 2, we study the averaging Liapunov exponent. In Section 3 the regularity of growing modes is proved. In Section 4, we prove the nonlinear instability result for flows defined in a simply connected domain. In Section 5, we prove nonlinear instability for the non-simply connected case, where the boundary conditions are more delicate. Section 6 contains some comments and extensions.

2. Averaging Liapunov exponent

We have the following result about the averaging Liapunov exponent $\mu_{av}$, defined by (1.11).

**Theorem 2.** Consider a steady flow $u_0$ with a $C^2$ stream function $\psi_0$ on a bounded domain $\Omega$ of class $C^2$ and $\psi_0$ has a finite number of critical values. Then we have $\mu_{av} = 0$.

Before proving this theorem, let us briefly explain the main idea about it. As we can see from the definition of $\mu_{av}$, it characterizes the growth rate of

$$\int \int_{\Omega} \left| \frac{\partial X_0}{\partial x} \right|(t) \, dx^1 \, dx^2$$

or

$$\frac{\int \int_{\Omega} \left| \frac{\partial X_0}{\partial x} \right|(t) \, dx^1 \, dx^2}{|\Omega|}$$

which is exactly the average of $\left| \frac{\partial X_0}{\partial x} \right|(t)$ on $\Omega$. The classical Liapunov exponent (1.14) is used to characterize the growth of $\left| \frac{\partial X_0}{\partial x} \right|$. For a steady plane flow $\left| \frac{\partial X_0}{\partial x} \right|$ has exponential growth only on the set of all nondegenerate saddle points of $\psi_0$ and those trajectories connecting them, and beyond this zero measure set $\left| \frac{\partial X_0}{\partial x} \right|$ has only linear growth. Thus we expect the average of $\left| \frac{\partial X_0}{\partial x} \right|(t)$ to not have exponential growth. This is what we prove below.

**Proof of Theorem 2.** For the reason we mentioned above, to study the growth of

$$\int \int_{\Omega} \left| \frac{\partial X_0}{\partial x} \right|(t) \, dx^1 \, dx^2,$$

we only need to understand the growth of the integral in the region near nondegenerate saddle points and trajectories connecting them. So we study the structure of this “dangerous” region first. Since nondegenerate saddle points are isolated, their number is finite. We denote them by $b_1, \cdots, b_q$. We take one point, $b_1$, and
denote $C_{cr} = \psi_0 (b_1)$. Now consider the following regions $D^\pm_\varepsilon$ near the level set \( \{ \psi_0 (x) = C_{cr} \} \), defined by

\[
D^-_\varepsilon := \{ x | C_{cr} - \varepsilon < \psi_0 (x) < C_{cr} \}, \quad D^+_\varepsilon := \{ x | C_{cr} < \psi_0 (x) < C_{cr} + \varepsilon \}.
\]

Here we choose $\varepsilon$ small enough so that there are no critical points of $\psi_0$ in $D^\pm_\varepsilon$. This is possible by our assumption that $\psi_0$ has only a finite number of critical values. We take out all the connected components of $D^\pm_\varepsilon$ having $b_1$ in their closure and denote them by $E^1_\varepsilon, \ldots, E^q_\varepsilon$. Notice that $l \leq 4$ since only four trajectories can bifurcate from a nondegenerate saddle point and therefore the neighborhood of $b_1$ can be divided into at most 4 parts. Now we take one of them and denote it by $E_\varepsilon \subset D^\pm_\varepsilon$ (The case $E_\varepsilon \subset D^\pm_\varepsilon$ can be handled in the same way). We denote the connected closed curve $J_\varepsilon = E_\varepsilon \cap \{ \psi_0 (x) = \varepsilon \}$ for each $\varepsilon \in [C_{cr} - \varepsilon, C_{cr}]$. It is obvious that

\[
\int \int_{E_\varepsilon} \left| \frac{\partial X_0}{\partial x} \right| (t) \, dx \, dx^2 \leq \int \int_{E_\varepsilon} \left(\left| \frac{\partial X_0}{\partial x^1} \right|^2 + \left| \frac{\partial X_0}{\partial x^2} \right|^2 \right)^{\frac{1}{2}} \, dx \, dx^2 + \int \int_{E_\varepsilon} \left(\left| \frac{\partial X_0}{\partial x^1} \right|^2 + \left| \frac{\partial X_0}{\partial x^2} \right|^2 \right)^{\frac{1}{2}} \, dx \, dx^2
\]

\[
= I_1 (t) + I_2 (t).
\]

We shall prove that for $\varepsilon$ small enough there exists $C_1, C_2$ such that

\[
I_1 (t), I_2 (t) \leq C_1 t + C_2.
\]

Assuming (2.1), Theorem 2 can be proved. For each nondegenerate saddle point $b_i$ ($1 \leq i \leq q$), denote by $F^i_{\varepsilon_i}$ the union of the set $E_{\varepsilon_i}^1, \ldots, E_{\varepsilon_i}^q$ as defined above. By (2.1), we can take $\varepsilon_i > 0$ such that

\[
\int \int_{F^i_{\varepsilon_i}} \left| \frac{\partial X_0}{\partial x} \right| (t) \, dx \, dx^2 \leq C^i_1 t + C^i_2.
\]

Thus for some constants $C', C'' > 0$ we have

\[
\int \int_{\bigcup_{i=1}^q F^i_{\varepsilon_i}} \left| \frac{\partial X_0}{\partial x} \right| (t) \, dx \, dx^2 \leq C' t + C''.
\]

Notice that $\bigcup_{i=1}^q F^i_{\varepsilon_i}$ is a neighborhood of all nondegenerate saddle points and trajectories connecting them, thus the classical Liapunov exponent is zero on the region $\Omega/ \bigcup_{i=1}^q F^i_{\varepsilon_i}$. So for any $\alpha > 0$, there exists $C_\alpha$ such that

\[
\left| \frac{\partial X_0}{\partial x} \right| (t; x) \leq C_\alpha e^{\alpha t}.
\]

So

\[
\int \int_{\Omega} \left| \frac{\partial X_0}{\partial x} \right| (t) \, dx \, dx^2 \leq \int \int_{\bigcup_{i=1}^q F^i_{\varepsilon_i}} \left| \frac{\partial X_0}{\partial x} \right| (t) \, dx \, dx^2 + \int \int_{\Omega/ \bigcup_{i=1}^q F^i_{\varepsilon_i}} \left| \frac{\partial X_0}{\partial x} \right| (t) \, dx \, dx^2 \leq C' t + C'' + C_\alpha e^{\alpha t} |\Omega|.
\]

Thus by the definition of $\mu_{av}$, we have $\mu_{av} \leq \alpha$. Since $\alpha$ is arbitrary, $\mu_{av} = 0$.

Now we shall prove the key estimate (2.1). We only prove it for $I_1 (t)$ since the proof is the same for $I_2 (t)$. As we will see, the proof of (2.1) strongly depends on the geometric properties of the set $E_\varepsilon$ and the orbits inside it. So we begin with a study
of these properties. First let us study the structure of $E_\varepsilon$. When $\varepsilon$ is very small, we can think of $E_\varepsilon$ as a very narrow “circular” domain bounded by two boundary curves $J_{C_\varepsilon} - \varepsilon$ and $J_{C_\varepsilon} + \varepsilon$ being the inner boundary. Denote $[A_1, A_2]$ to be the projection of $J_{C_\varepsilon}$ to the $x^1$ axis. For each $a \in [A_1, A_2]$ let $I_a = \{x^2 \mid (a, x^2) \in E_\varepsilon\}$. It is easy to see that we can uniquely divide $I_a$ into a finite number of nonoverlapping closed intervals $I_1^a, \ldots, I_m^a$ such that exactly one end point of $I_j^a$ ($1 \leq j \leq m_a$) lies on the curve $J_{C_\varepsilon}$. We have $\sup_{a \in [A_1, A_2]} m_a < \infty$. Actually when $\varepsilon$ is very small, we can see that
\begin{equation}
\sup_{a \in [A_1, A_2]} \sup_{a \in [A_1, A_2]} \# \{x^1 = a \cap J_{C_\varepsilon}\} := M.
\end{equation}

Now we study the orbit of (1.12) starting with a point in $E_\varepsilon$. The trajectory equation can be written as a Hamiltonian system
\begin{equation}
\begin{aligned}
&\dot{X}_0^1 = \partial_2 \psi_0(X_0^1, X_0^2) \\
&\dot{X}_0^2 = -\partial_1 \psi_0(X_0^1, X_0^2).
\end{aligned}
\end{equation}

The energy $\psi_0(x)$ is invariant along the trajectory, and if $e = \psi_0(x)$ is not a critical value of $\psi_0$ then a fluid particle starting with $x$ does periodic motion along its orbit. Since there are no critical points in $E_\varepsilon$, the fluid particle starting with $x \in E_\varepsilon$ and energy $e = \psi_0(x) \in (C_{cr} - \varepsilon, C_{cr})$ circles along the closed curve $J_e$ with a finite period
\begin{equation}
T(e) = \int_{J_e} \frac{1}{|\nabla \psi_0|}.
\end{equation}

Since $\psi_0 \in C^2$ and $|\nabla \psi_0|(x) \neq 0$ on $E_\varepsilon$, the function $T(e)$ defined above is differentiable for $e \in (C_{cr} - \varepsilon, C_{cr})$. When $e$ tends to $C_{cr}$, the particle having energy $e$ will approach the saddle point $b_1$ very closely, and take extremely long time to leave. Thus $T(e) \to \infty$ as $e \to C_{cr}$ and correspondingly $T'(e) \to \infty$ as $e \to C_{cr}$. So we can take $\varepsilon$ small enough such that
\begin{equation}
T'(e) > R > 0 \text{ for } e \in (C_{cr} - \varepsilon, C_{cr}).
\end{equation}

In particular it implies that the period of any particle in $E_\varepsilon$ is no less than $T_1 = T'(C_{cr} - \varepsilon)$.

With the notations introduced above, we have
\begin{equation}
I_1(t) = \int_{E_\varepsilon} \left( \frac{1}{2} \int_{I_a} \left( \frac{\partial X_0^1}{\partial x^2} \right)^2 + \frac{1}{2} \int_{I_a} \left( \frac{\partial X_0^2}{\partial x^2} \right)^2 \right) (t) dx^1 dx^2
\end{equation}
\begin{equation}
= \int_{a \in [A_1, A_2]} \sum_{j=1}^{m_a} \left( \int_{I_j^a} \left( \frac{\partial X_0^1}{\partial x^2} \right)^2 + \frac{1}{2} \int_{I_j^a} \left( \frac{\partial X_0^2}{\partial x^2} \right)^2 \right) \frac{1}{2} dx^2 \right) da.
\end{equation}

Here the key observation is that: For fixed $t$ and $a$, the mapping
\begin{equation}
x^2 \in I_j^a \to (X_0^1(t; a, x^2), X_0^2(t; a, x^2))
\end{equation}
defines a curve $L(j, a, t)$ with the parameter $x^2 \in I_j^a$, while
\begin{equation}
\int_{I_j^a} \left( \frac{\partial X_0^1}{\partial x^2} \right)^2 + \frac{1}{2} \int_{I_j^a} \left( \frac{\partial X_0^2}{\partial x^2} \right)^2 \right) \frac{1}{2} dx^2
\end{equation}
is nothing but the length of this curve which we denote by \( K(j, a, t) \). Thus we have

\[
I_1(t) = \int_{a \in (A_1, A_2)} \sum_{j=1}^{m_\alpha} K(j, a, t) \, da.
\]

So to prove (2.1), it suffices to show that \( K(j, a, t) \) has only linear growth. This can be seen in the following intuitive way. By our construction \( \mathcal{I}_a^j \) has exactly one end point on \( J_{C_c, \varepsilon} \), and the particle starting with this point takes infinite time to get to the saddle point. The particle starting from the other end of \( \mathcal{I}_a^j \) goes around its orbit with the shortest period no less than \( T_1 \), so it can finish at most \([t/T_1] + 1\) periods by time \( t \). Thus the curve \( L(j, a, t) \) consists of at most \([t/T_1] + 1\) circles.

Here we use the term circle to mean a part of the curve \( L(j, a, t) \) rotating exactly by \( 2\pi \). If \( \varepsilon \) is very small we expect each circle to tightly wind around \( J_{C_c, -\varepsilon} \) and have comparable length with the curve \( J_{C_c, -\varepsilon} \) or \( J_{C_c, \varepsilon} \). So the length of \( L(j, a, t) \) can roughly be controlled by \((t/T_1 + 1)\) times the length of \( J_{C_c, \varepsilon} \). This is the intuition behind the following rigorous proof. First we need to quote a result in [2] to estimate the length of a curve.

**Theorem** (The Cauchy-Crofton formula) Let \( C \) be a regular plane curve with length \( l \). The measure of the set of straight lines (counted with multiplicities) which meet \( C \) is equal to \( 2l \).

The measure for straight lines is defined in the following way ([2]). A straight line \( L \) in the plane is determined by the distance \( p \geq 0 \) from \( L \) to the origin \( O \) and by the angle \( \theta, 0 \leq \theta \leq 2\pi \), which a half-line starting at \( 0 \) and normal to \( L \) makes with the \( x \) axis. In this way, the set of all straight lines in the plane is represented by the set

\[
\mathcal{L} = \{(p, \theta) \mid p \geq 0, 0 \leq \theta \leq 2\pi\}.
\]

Let \( n(p, \theta) \) be the number of intersection points of the straight line \( (p, \theta) \) with \( C \). Then the Cauchy-Crofton formula is

\[
\int \int n(p, \theta) \, dpd\theta = 2l.
\]

Denote \( B \subset \mathcal{L} \) to be the set of all straight lines intersecting \( \bar{E}_c \). Since \( E_c \) is a bounded set, \( B \) is also bounded. Now we divide the curve \( L(j, a, t) \) into circles. To make it precise we divide the interval \( \mathcal{I}_a^j \) into a set of neighboring closed subintervals \( \mathcal{I}_a^{j,r} \) \((0 \leq r \leq R(j, a, t))\) such that particles starting inside \( \mathcal{I}_a^{j,r} \) (here we mean the initial position is \((a, x^2)\) with \( x^2 \) lying inside \( \mathcal{I}_a^{j,r} \)) finishes their periodic motions exactly \( r \) times by time \( t \). Here \( R(j, a, t) \) is the maximal number of periodic motions a particle can finish if starting in \( \mathcal{I}_a^j \). As we explained before,

\[
R(j, a, t) \leq \lfloor t/T_1 \rfloor + 1.
\]

By our construction it is clear that a small interval starting with the end point on \( J_{C_c, \varepsilon} \) is \( \mathcal{I}_a^{j,0} \) and \( \mathcal{I}_a^{j,1} \) is neighboring to \( \mathcal{I}_a^{j,0}, \ldots \). By the definition of \( \mathcal{I}_a^{j,r} \) the curve

\[
L^r(j, a, t) : \mathcal{I}_a^{j,r} \to (X_0^l(t; a, x^2), X_{2r}^l(t; a, x^2))
\]

is exactly the \( r \)-th circle of \( L(j, a, t) \), and we denote its length by \( K^r(j, a, t) \). Denote \( n(p, \theta, a, j, r, t) \) to be the number of intersection points of the curve \( L^r(j, a, t) \) with a straight line \((p, \theta) \in B \). Then by the Cauchy-Crofton formula,

\[
K^r(j, a, t) = \int \int_B n(p, \theta, a, j, r, t) \, dpd\theta.
\]
From (2.6), we have

\[ I_1 (t) = \int_{a \in (A_1, A_2)} \sum_{j=1}^{m_a} \sum_{r=1}^{R(j,a,t)} K_r^* (j,a,t) \, da \]

\[ = \int_{a \in (A_1, A_2)} \sum_{j=1}^{m_a} \sum_{r=1}^{R(j,a,t)} \int_B n (p, \theta, a, j, r, t) \, dp \, d\theta \, da. \]

We shall show that

\[ n (p, \theta, a, j, r, t) \]

for large \( t \), by (2.2) and (2.8). Thus the theorem is proved.

We show (2.9) by a contradiction argument. Suppose otherwise, \( n (p, \theta, a, j, r, t) \) has no upper bound for large \( t \). Since \( B \) is compact and \( m_a \leq M \), we have two cases:

Case 1: There exists some straight line \( I_0 = (p_0, \theta_0) \in B, a_0 \in (A_1, A_2) \) and \( 1 \leq j_0 \leq m_{a_0} \) such that \( n (p_0, \theta_0, a_0, j_0, r, t) \) has no upper bound for \( r, t \).

Case 2: There exists some straight line \( I_0 = (p_0, \theta_0) \in B, 1 \leq j_0 \leq M \), such that \( n (p_0, \theta_0, a, j_0, r, t) \) has no upper bound for \( r, t \) as \( a \) tends to \( A_1 \) or \( A_2 \).

We analysis Case 1 first. There exists a sequence \( t_k \to \infty \) such that

\[ n (p_0, \theta_0, a_0, J_0, r_k, t_k) > k \]

for some \( r_k \leq R (j_0, a_0, t_k) \). Let \( I_{a_0}^{j_0, r_k} = [x_1^0, x_2^0] \). Then \( x_1^k \to x_1^0, x_2^k \to x_2^0 \) for some \( x_1^0, x_2^0 \in I_{a_0}^{j_0, r_k} \). We must have \( x_1^0 = x_2^0 = x^0 \) since otherwise two particles starting from two end points of \( I_{a_0}^{j_0, r_k} \) finish different number of periods when \( t_k \) is very large. We have the following two situations.

(1) \( x^0 \) is not on \( \mathcal{C}_i \). Then for \( k \) large the periods of particles starting in \( I_{a_0}^{j_0, r_k} \) are upper bounded. Thus \( r_k \to \infty \) since by definition \( r_k \) is the number of periods finished by time \( t_k \). When \( k \) is large, for any \( x \in I_{a_0}^{j_0, r_k} \), the particle orbit \( J_{\psi_0 (a_0, x)} \) is almost like \( J_{\psi_0 (a_0, x^0)} \). In particular, the number of intersection points of \( J_{\psi_0 (a_0, x)} \) with the line \( l_0 \) is the same as that of \( J_{\psi_0 (a_0, x^0)} \), which we denote by \( Q \). The function \( \psi_0 (a_0, x) \) is a strictly monotone function in \( I_{a_0}^{j_0, r_k} \) by our construction. Assuming it is increasing, then

\[ e = \psi_0 (a_0, x) \in (\psi_0 (a_0, x_1^k), \psi_0 (a_0, x_2^k)) := E^k. \]

and the correspondence of \( e \) to \( x \) is 1-1 in \( I_{a_0}^{j_0, r_k} \). Denote \( P_i (e) (1 \leq i \leq Q) \) to be the time a particle takes to hit the line \( l_0 \) \( i \)th time within one period of motion, starting from some point in \( I_{a_0}^{j_0, r_k} \) with energy \( e \). For \( e \in (C_{\text{cr}} - \varepsilon, C_{\text{ce}}) \), \( P_i (e) \) is differentiable. It is clear that \( n (p_0, \theta_0, a_0, j_0, r_k, t_k) \) is the total number of solutions to the following \( Q \) equations

\[ t_k = r_k T (e) + P_i (e), 1 \leq i \leq Q \]
for $e \in E^k$. Since $x^0 \notin J_{C_{cr}}$, the interval $E^k$ is away from $C_{cr}$. Thus we can find a lower bound $M_1$ for $P_i'(e)$ ($1 \leq i \leq Q$). Then by (2.5),
\[ r_kT'(e) + P_i'(e) \geq r_kR + M_1 > 0 \]
when $k$ is large since $r_k \to \infty$. So for large $k$, $r_kT(e) + P_i(e)$ is monotone and thus the equation (2.10) has at most one solution for each $i$. This implies that
\[ n(p_0, \theta_0, a_0, j_0, r_k, t_k) \leq Q \]
for $k$ large, which is a contradiction.

(2) $x^0$ lies on $J_{C_{cr}}$. We use the same notations as above. Now
\[ (\psi_0(a, x_1^k), \psi_0(a, x_2^k)) := E^k \]
is an interval approaching $C_{cr}$ when $k$ is large. If $P_i(C_{cr}) = \infty$, then $P_i'(e) \to \infty$ as $e \to C_{cr}$. So if $k$ is large, $P_i'(e)$ is positive. Thus $r_kT(e) + P_i(e)$ is again a monotone function on the interval $E^k$ and the equation (2.10) has at most one solution for each $i$. If $P_i(C_{cr}) < \infty$, then there are no critical points of $\psi_0$ on the path from $x^0$ to hit the line $l_0 i$–th time along $J_{C_{cr}}$. So $P_i'(C_{cr})$ is well defined, being a finite number. Thus when $k$ is large, we can find a lower bound $M_1$ for $P_i'(e)$ defined in $E^k$. Since $P_i(e)$ remains bounded in $E^k$ as $k$ is large, for (2.10) to be satisfied we must have $r_k \geq 1$. Since $T'(e) \to \infty$ as $e \to C_{cr}$, we can take $k$ large enough such that $T'(e) > -M_1$ in $E^k$. Then $r_kT(e) + P_i(e)$ is again a monotone function and there is at most one solution to (2.10) for each $i$. So we still have $n(p_0, \theta_0, a_0, j_0, r_k, t_k) \leq Q$ for $k$ large, and we get contradiction again.

Now we study Case 2. The analysis is very similar to above second situation of Case 1. We assume $a \to A_1$. Let $I_{a^0,r}^u = [x_1^{a,0}, x_2^{a,0}]$. Obviously we have $|I_{a}^{a_0}| \to 0$ as $a \to A_1$. So the interval
\[ (\psi_0(a, x_1^{a,0}), \psi_0(a, x_2^{a,0})) := E^{a,r} \]
approaches $C_{cr}$ as $a \to A_1$. Let $Q$ be the number of intersection points of $J_{C_{cr}}$ with the line $l_0$. We define $P_{i,a}(e)$ ($1 \leq i \leq Q$) to be the time which a particle takes to hit the line $l_0 i$–th time within one period of motion, starting from some point in $I_{a^0,r}$ with energy $e$. Then $n(p_0, \theta_0, a_0, j_0, r, t)$ is the total number of solutions to the following $Q$ equations
\[ t = rT(e) + P_{i,a}(e), \quad 1 \leq i \leq Q \]
for $e \in E^{a,r}$. If $P_{i,A_1}(C_{cr}) = \infty$, then $P_{i,a}'(e) \to \infty$ as $e \to C_{cr}$ and $a \to A_1$. So if $a$ is very close to $A_1$, $P_{i,a}'(e)$ is positive on the interval $E^{a,r}$. Thus $rT(e) + P_{i,a}(e)$ is a monotone function on the interval $E^{a,r}$ and the equation (2.10) has at most one solution for any $t$. The discussion of the case $P_{i,A_1}(C_{cr}) < \infty$ is also similar to the corresponding case in the second situation of Case 1. We just need to replace “for $k$ large” there with “for $a$ very close to $A_1$”, so we skip it here. We get the same conclusion that for $a$ very close to $A_1$ and large $t$,
\[ n(p_0, \theta_0, a_0, j_0, r, t) \leq Q. \]
A contradiction. Thus (2.9) is proved and this ends the proof of Theorem 2. □
3. Regularity of growing modes

In this section, we use the property of the averaging Liapunov exponent proved in the last section to show regularity of growing modes.

**Proof of Theorem 1 (i).** The vorticity function \( \omega \) of the growing mode satisfies

\[
\lambda \omega + u_0 \cdot \nabla \omega = -v \cdot \nabla \omega_0.
\]

Integrating above equation along the trajectory (1.12), we have

\[
(3.1) \quad \omega = \int_0^\infty e^{-Re \lambda s} v \cdot \nabla \omega_0 (X_0(s)) \, ds.
\]

Since \( v \in H^1 \), by Sobolev embedding \( v \in L^p \) for any \( 1 \leq p < \infty \). From (3.1) and the fact that the mapping \( x \to X_0(s;x) \) has Jacobian 1, we have

\[
\int_0^\infty e^{-Re \lambda s} \|v\|_p \, ds \leq \|\omega_0\| C \|v\|_p \int_0^\infty e^{-Re \lambda s} \|v\|_p \, ds.
\]

So \( v \in L^p \), and by elliptical regularity theory the stream function \( \phi \in W^{2,p} \). Thus \( v = \nabla^\perp \phi \in W^{1,p} \), \( 1 \leq p < \infty \). Taking \( x \)-derivative in (3.1), we have

\[
(3.2) \quad \partial \omega (x) = \int_0^\infty e^{-Re \lambda s} \left( \partial (v \cdot \nabla \omega_0) (X_0(s)) \cdot \frac{\partial X_0}{\partial x} (s;x) \right) \, ds.
\]

For any \( 1 \leq p < p^* \), we can find \( p_0 \in (1,p^*) \) and \( p_1 > 1 \) such that

\[
\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}.
\]

Then from (3.2) and Hölder inequality

\[
(3.3) \quad \|\partial \omega (x)\|_p \leq C \int_0^\infty e^{-Re \lambda s} \|v\|_{W^{1,p_1}} \left( \left\| \frac{\partial X_0}{\partial x} (s) \right\|_{p_0} \right) \, ds.
\]

Since \( p_0 \in (1,p^*) \) we have

\[
\mu \left( 1 - \frac{1}{p_0} \right) |\mu| \frac{1}{p} = \left( 1 - \frac{1}{p^*} \right) = \left\{ \begin{array}{ll}
\mu & \text{if } \mu \leq Re \lambda \\
Re \lambda & \text{if } \mu > Re \lambda 
\end{array} \right. = \min \{Re \lambda, \mu\}.
\]

So we have

\[
\varepsilon_0 = Re \lambda - \mu \left( 1 - \frac{1}{p_0} \right) > 0.
\]

Choose \( \varepsilon_1, \varepsilon_2 > 0 \) such that \( \varepsilon_1 + \varepsilon_2 < \frac{\varepsilon_0}{2} \). By Theorem 2 and the definition of the classical Liapunov exponent, there exist constants \( C_{\varepsilon_1}, C_{\varepsilon_2} \) such that

\[
\int \int_{\Omega} \left\| \frac{\partial X_0}{\partial x} (t) \right\| dx^1 dx^2 \leq C_{\varepsilon_1} e^{\varepsilon_1 t}, \quad \left\| \frac{\partial X_0}{\partial x} (t) \right\| \leq C_{\varepsilon_2} e^{(\mu + \varepsilon_2) t}.
\]

So

\[
\left\| \left\| \frac{\partial X_0}{\partial x} (s) \right\| \right\|_p \leq \left( C_{\varepsilon_2} e^{(\mu + \varepsilon_2) s} \right)^{p_0 - 1} p_0 \left( \int \int_{\Omega} \left\| \frac{\partial X_0}{\partial x} (t) \right\| dx^1 dx^2 \right)^{\frac{1}{p_0}}
\]

\[
\leq C' \exp \left( \left( \mu \left( 1 - \frac{1}{p_0} \right) + \varepsilon_2 \frac{p_0 - 1}{p_0} + \varepsilon_1 \frac{1}{p_0} \right) s \right)
\]

\[
\leq C' e^{(Re \lambda - \frac{\varepsilon_0}{2}) s}.
\]
by our choice of $\varepsilon_1, \varepsilon_2$. Plugging above estimate into (3.3), we have
\[
\|\partial \omega (x)\|_p \leq CC' \|v\|_{W^{1,p}_1} \int_0^\infty e^{-Re\lambda s} e^{(Re\lambda - \frac{\omega}{2})s} \, ds
\]
\[
= CC' \|v\|_{W^{1,p}_1} \frac{2}{\varepsilon_0}.
\]
Thus $\omega \in W^{1,p}$ and Theorem 1 (ii) is proved. \hfill \Box

4. Nonlinear instability (simply connected case)

We shall only prove nonlinear instability (Theorem 1 (ii)) for $p_0$ very close to 1. Then nonlinear instability for bigger $p_0$ follows from the inequality
\[
\|v\|_{p_1} \leq |\Omega|^{\frac{1}{p_1}} - \frac{1}{p_2} \|v\|_{p_2}
\]
for any $p_1 < p_2 \leq \infty$. We take the growing mode $e^\lambda t v_\delta (x)$ such that $Re\lambda$ is the largest growth rate and $v_\delta \in H^1 (\Omega)^2$. By the regularity theorem proved in the last section, we have $\omega_\delta = \text{curl} v_\delta \in W^{1,p}_1 \cap L^{p_1}$ for any $1 \leq p < p^*$ and $1 \leq p_1 < \infty$. If $\lambda$ is real, we normalize $\omega_\delta$ in the way
\[
\|\omega_\delta\|_{p_2} + \|\partial \omega_\delta\|_{p_1} = 1
\]
for given $p_1, p_2$ in Theorem 1 (ii), and let $\omega_\delta (0, x) = \omega_0 + \delta \omega_\delta$, $v_\delta (0, x) = u_0 + \delta v_\delta$. If $\lambda$ is not real, notice that
\[
\|e^{t L_0} \text{Im} v_\delta\|_{p_0} \geq c_0 |\text{Re} \lambda|
\]
for some constant $c_0$, where $L_0$ is the linearized velocity operator (1.9). We normalize $\text{Im} \omega_\delta$ in the same way as above and let $\omega_\delta (0, x) = \omega_0 + \delta \text{Im} \omega_\delta$, $v_\delta (0, x) = u_0 + \delta \text{Im} v_\delta$. We consider the case of non real $\lambda$ in the below. Denote $v_\delta (t, x)$ to be the perturbed flow satisfying the Euler equation (1.1) with initial $v_\delta (0, x)$ as defined. Denote by $v (t, x) = v_\delta (t, x) - u_0$ and $\omega (t, x) = \omega_\delta (t, x) - \omega_0$ the perturbations of velocity and vorticity. Then $\omega (t, x)$ satisfies
\[
\partial_t \omega = M_0 \omega - v \cdot \nabla \omega
\]
where $M_0$ is the linearized vorticity operator defined by (1.10). So we have
\[
\omega (t) = e^{t M_0} \omega (0) - \int_0^t e^{(t-s)M_0} (v \cdot \nabla \omega) (s) \, ds
\]
\[
= \omega_L + \omega_N
\]
Thus
\[
v (t) = \text{curl}^{-1} \omega = \text{curl}^{-1} \omega_L + \text{curl}^{-1} \omega_N
\]
\[
= v_L + v_N.
\]
For simply connected domain it is convenient to write $v = \text{curl}^{-1} \omega$ as $v = \nabla^\perp (-\Delta)^{-1} \omega$. Here the stream function $\phi = (-\Delta)^{-1} \omega$ solves the Dirichlet problem
\[-\Delta \phi = \omega, \phi = 0 \text{ on } \partial \Omega.
\]
Denote the admissible velocity space
\[
H_p = \{ u \in L^p (\Omega)^2 \mid \nabla \cdot u = 0, u \cdot n = 0 \text{ on } \partial \Omega \},
\]
and
\[
G_p = \{ w \in L^p (\Omega)^2 \mid w = \nabla q \text{ for some } q \in W^{1,p}_{\text{loc}} (\Omega) \}.
\]
If $\Omega$ is a bounded domain of class $C^2$, then we have the following theorem (see III.1 in [9]):

**Theorem (Helmholtz-Weyl decomposition)** If $\Omega$ is a bounded domain of class $C^2$, then the following Helmholtz-Weyl decomposition holds for $1 < p < \infty$

$$L^p(\Omega) = H^p(\Omega) \oplus G^p(\Omega).$$

The projection operator $P^p : L^p(\Omega) \to H^p(\Omega)$ is a linear bounded operator having $H^p$ as its range and $G^p$ as its null space. We also have $G^p \subset H^p_0$. Combining these facts, we have the following result.

**Lemma 3.** If $\Omega$ is a bounded domain of class $C^2$, $v \in H^p$ with $1 < p < \infty$, then

$$\|v\|_p \leq \|P^p v\| \sup_{u \in H^p} \int_\Omega v \cdot u \, dx^1 \, dx^2. \quad (4.4)$$

Moreover, if $\text{curl} v \in L^p(\Omega)$ then we have

$$\|v\|_p \leq C_1 \sup_{\psi \in W^{1,p}'} \int_\Omega \int_\Omega \psi \text{curl} v \, dx^1 \, dx^2 \quad (4.5)$$

for some constant $C_1$.

**Proof.** By Helmholtz-Weyl decomposition theorem, for any $w \in L^p(\Omega)$ we have $w = u_1 + u_2$, $u_1 \in H^p$, $u_2 \in G^p$ and $\|u_1\|_p = \|P^p w\| \leq \|P^p\| \cdot \|w\|_p$. So

$$\|v\|_p = \sup_{w \in L^p'} \int_\Omega v \cdot w \, dx^1 \, dx^2$$

$$= \sup_{w \in L^p'} \int_\Omega v \cdot u_1 \, dx^1 \, dx^2 \quad (\text{since } G^p \subset H^p_0)$$

$$\leq \sup_{u_1 \in H^p} \int_\Omega v \cdot u_1 \, dx^1 \, dx^2$$

$$= \|P^p v\| \sup_{u \in H^p} \int_\Omega v \cdot u \, dx^1 \, dx^2.$$

To prove (4.5), we denote $\phi \in W^{1,p}_0$ and $\psi \in W^{1,p}_0$ to be the stream functions of $v, u$ appeared in (4.4). Then $v = \nabla^\perp \phi, u = \nabla^\perp \psi$. Since $\|\partial \phi\|_p = \|u\|_p = 1$, by
Poincaré’s inequality $\|\psi\|_{W^{1,p'}} \leq C$. So from (4.4), we have

$$\|v\|_p \leq \|P_{p'}\| \sup_{u \in H^p} \int \int \Omega v \cdot u \, dx^1 \, dx^2 \leq \|P_{p'}\| \sup_{\|\psi\|_{W^{1,p'}} \leq C_1} \int \int \Omega \nabla^1 \phi \cdot \nabla^1 \psi \, dx^1 \, dx^2 = \|P_{p'}\| \sup_{\|\psi\|_{W^{1,p'}} \leq C_1} \int \int \Omega \nabla \phi \cdot \nabla \psi \, dx^1 \, dx^2 = \|P_{p'}\| \sup_{\|\psi\|_{W^{1,p'}} \leq C_1} \int \int \Omega (-\Delta \phi) \psi \, dx^1 \, dx^2 = \|P_{p'}\| C \sup_{\|\psi\|_{W^{1,p'}} = 1} \int \int \Omega \psi \text{curl} \, dx^1 \, dx^2.$$

Lemma 3 gives us a new way to estimate $\|v\|_p$. In particular (4.5) enables us to get the estimate of $\|v\|_p$ from the estimate of the vorticity function. This is very useful in the nonlinear instability proof later.

Now we study some properties of the linearized vorticity operator $M_0$ (defined by (1.10)) and its dual operator $M_0^*$. Denote two operators exists $D_1 = u_0 \cdot \nabla$ and $D_2 = \nabla \omega_0 \cdot \nabla$. Then

$$M_0 = -D_1 + D_2 (-\Delta)^{-1}.$$ 

Since $\omega_0|_{\Lambda_1}$ and $\psi_0|_{\Lambda_1}$ are constant and $\nabla^1 \omega_0 \cdot n = \nabla^1 \psi_0 \cdot n = 0$ on $\partial \Omega$, $D_1, D_2$ are skew-adjoint and

$$M_0^* = D_1 - (-\Delta)^{-1} D_2.$$ 

Because $M_0, M_0^*$ are compact perturbations of $\pm D_1$ which generate isometry groups in any $L^p(\Omega)$ ($1 \leq p < \infty$), their essential spectrum lie in the imaginary axis. By our assumption $\text{Re} \lambda$ is the maximal growth rate of $M_0$ and also $M_0^*$, thus for any $1 \leq p < \infty$ and $\varepsilon > 0$ there exists $C_{\varepsilon,p}$ such that

$$\left\| e^{t M_0^*} \right\|_{L^{p} \rightarrow L^{p}} \leq C_{\varepsilon,p} e^{(\text{Re} \lambda + \varepsilon) t}.$$ 

If we choose $p_0$ close to 1, there exists $p_3 \in (1, p^*)$ such that $\frac{1}{p_0} + \frac{1}{p_3} < 1$. Let

$$p_4 = \frac{p_3 p_0'}{p_3 + p_0'}.$$ 

The following estimate is crucial in the later proof. We again use the property of the averaging Liapunov exponent.

**Lemma 4.** There exists some constant $C_2$, such that for any $\psi \in W^{1,p_0'}_0$

$$\left\| \partial (e^{t M_0^*} \psi) \right\|_{p_4} \leq C_2 e^{\frac{2}{\text{Re} \lambda}} \|\psi\|_{W^{1,p_0'}}.$$
Proof. Denote \( \tilde{\psi}(t, x) = e^{tM_0^\lambda} \psi \), then \( \tilde{\psi}(t, x) \) satisfies
\[
\partial_t \tilde{\psi} = D_1 \tilde{\psi} - (\Delta)^{-1} D_2 \tilde{\psi},
\]
\[
\tilde{\psi}(0, x) = \psi(x).
\]
So
\[
\tilde{\psi}(t, x) = \psi(X_0(t; x)) - \int_0^t \left( (-\Delta)^{-1} D_2 \tilde{\psi} \right)(s; X_0(t-s; x)) \, ds
\]
and
\[
(4.8) \quad \partial_t \tilde{\psi}(t, x) = \partial_t \psi(X_0(t; x)) \frac{\partial X_0}{\partial x}(t) - \int_0^t \left( \partial (-\Delta)^{-1} D_2 \tilde{\psi} \right)(s; X_0(t-s)) \frac{\partial X_0}{\partial x}(t-s) \, ds.
\]
By (4.6), there exists \( C' \) such that
\[
\left\| \tilde{\psi}(t) \right\|_{p_0} \leq C' \epsilon \frac{3}{2} \text{Re} \lambda \left\| \psi \right\|_{p_0}.
\]
Let
\[
\tilde{\phi}(t) = (\Delta)^{-1} D_2 \tilde{\psi}(t),
\]
then
\[
(4.9) \quad \tilde{v}(t) = \nabla^\perp \tilde{\phi}(t) \in H_{p_1}'.
\]
By (4.5) of Lemma 3, we have
\[
\left\| \partial_t \tilde{\phi}(t) \right\|_{p_0} = \left\| \tilde{v}(t) \right\|_{p_1'}
\]
\[
\leq C_1 \sup_{\psi \in W_0^{1,p_0}} \int_{\Omega} \int_{\Omega} \psi \text{curl} \tilde{v}(t) \, dx \, dx^2
\]
\[
= C_1 \sup_{\psi \in W_0^{1,p_0}} \int_{\Omega} \int_{\Omega} \psi D_2 \tilde{\psi}(t) \, dx \, dx^2
\]
\[
= C_1 \sup_{\psi \in W_0^{1,p_0}} \int_{\Omega} \int_{\Omega} \tilde{\psi}(t) D_2 \psi \, dx \, dx^2
\]
\[
\leq C_1 \left\| \omega_0 \right\|_{C_1} \left\| \tilde{\psi}(t) \right\|_{p_0}
\]
\[
\leq C_1 C' \left\| \omega_0 \right\|_{C_1} \epsilon \frac{3}{2} \text{Re} \lambda \left\| \psi \right\|_{p_0}.
\]
As in the regularity proof in Section 3, we denote \( \varepsilon_0 = \text{Re} \lambda - \mu (1 - \frac{1}{p_3}) > 0 \) and have
\[
\left\| \frac{\partial X_0}{\partial x}(s) \right\|_{p_3} \leq C'' e^{(\text{Re} \lambda - \frac{2}{p_3}) s}
\]
for some constant $C''$. Combining above, we have
\[
\left\| \partial \tilde{\psi} (t) \right\|_{p_4} \leq \left\| \partial \psi (X_0 (t; x)) \frac{\partial X_0}{\partial x} (t) \right\|_{p_4} \\
+ \int_0^t \left\| \left( \partial (-\Delta)^{-1} D_2 \tilde{\psi} \right) (s; X_0 (t - s)) \frac{\partial X_0}{\partial x} (t - s) \right\|_{p_4} ds \\
\leq \left\| \partial \psi \right\|_{p_0} \left\| \frac{\partial X_0}{\partial x} (t) \right\|_{p_3} + \int_0^t \left\| \partial \tilde{\psi} (s) \right\|_{p_0} \left\| \frac{\partial X_0}{\partial x} (t - s) \right\|_{p_3} ds \text{ (by Hölder)} \\
\leq C'' e^{(\Re \lambda - \frac{2\pi}{\lambda})t} \left\| \psi \right\|_{W^{1,p_0}} \\
+ \int_0^t C_1 C'' \left\| \omega_0 \right\|_{C^1} e^{\frac{3}{2} \Re \lambda s} \left\| \psi \right\|_{p_0} C'' e^{(\Re \lambda - \frac{2\pi}{\lambda})(t - s)} ds \\
\leq C_2 e^{\frac{3}{2} \Re \lambda t} \left\| \psi \right\|_{W^{1,p_0}}
\]
for some constant $C_2$.

Denote $p_5 = 2p_4$ and $c_1 = 2 \| v \|_{p_0}$. If $p_0$ is very close to 1, we have $p_5 > p_0$. We need the following bootstrap Lemma as in [1] and [26].

**Lemma 5.** If $\| v (t) \|_{p_0} \leq c_1 \delta e^{\Re \lambda t}$ for $0 \leq t \leq T$, then there exists some constant $C_3$ such that
\[
\| \omega (t) \|_{p_5}, \| v (t) \|_{p_5} \leq C_3 \delta e^{\Re \lambda t}
\]
for $0 \leq t \leq T$.

**Proof.** We only need to prove the estimate for $\| \omega (t) \|_{p_5}$, since the operator
\[
\text{curl}^{-1} : \omega \rightarrow v
\]
is bounded for any $p > 1$. We rewrite the equation (4.2) for $\omega$ as
\[
(4.10) \quad \partial_t \omega + (u_0 + v) \cdot \nabla \omega = -v \cdot \nabla \omega.
\]
Denoting $\tilde{X} (s; x)$ to be the flow generated by perturbed velocity field $-(u_0 + v)$ and integrating (4.10) along it, we get
\[
(4.11) \quad \omega (t, x) = \omega (0, \tilde{X} (t; x)) - \int_0^t v \cdot \nabla \omega_0 \left( s; \tilde{X} (t - s; x) \right) ds.
\]
Taking $L^{p_5}$ norm of (4.11) and using the fact that the Jacobian of the mapping $x \rightarrow \tilde{X} (s; x)$ is 1, we get
\[
\| \omega (t) \|_{p_5} \leq \| \omega (0) \|_{p_5} + \| \omega_0 \|_{C^1} \int_0^t \| v (s) \|_{p_5} ds.
\]
Since the embedding $W^{1,p_5} (\Omega)$ to $L^{p_5} (\Omega)$ is compact, for any $\alpha > 0$ there exists $C_\alpha$ such that
\[
\| v \|_{p_5} \leq \alpha \| v \|_{W^{1,p_5}} + C_\alpha \| v \|_{p_0} \leq \alpha C'' \| \omega \|_{p_5} + C_\alpha \| v \|_{p_0}.
\]
So
\[
\| \omega (t) \|_{p_5} \leq \delta \| \text{Im} \omega_0 \|_{p_5} + \alpha C'' \| \omega_0 \|_{C^1} \int_0^t \| \omega (s) \|_{p_5} ds + C_\alpha \| \omega_0 \|_{C^1} \int_0^t \delta \| \omega (s) \|_{p_5} ds + C_1 C_\alpha \| \omega_0 \|_{C^1} \frac{1}{\Re \lambda} \delta e^{\Re \lambda t}.
\]
We choose $\alpha$ small enough such that

$$\alpha C'' \|\omega_0\|_{C_1} < \text{Re} \lambda.$$  

Then the conclusion follows from Gronwall type inequality. $\square$

Now we can prove the nonlinear instability.

**Proof of Theorem 1 (ii).** Denote $T$ to be the maximal time such that

$$\|v(t)\|_{p_0} \leq c_1 \delta e^{\text{Re}\lambda t}.$$  

Since $c_1 = 2 \|v_g\|_{p_0}$ and $\|v(0)\|_{p_0} = \delta \|\text{Im}v_g\|_{p_0} < c_1$, $T > 0$. Now we estimate $\|v_N\|_{p_0}$. By (4.5), we have

$$\|v_N(t)\|_{p_0} \leq C_1 \sup_{\psi \in W_0^{1,p_0'}} \int \int_{\Omega} \psi \omega_N(t) \, dx \, dx.$$

For any $\psi \in W_0^{1,p_0}$ with $\|\psi\|_{W_0^{1,p_0'}} = 1$ we have

$$\int \int_{\Omega} \psi \omega_N(t) \, dx \, dx = -\int \int_{\Omega} \int_0^t \psi \, e^{(t-s)M_0} (v \cdot \nabla \omega)(s) \, ds \, dx \, dx$$

$$= -\int_0^t \int \int_{\Omega} \psi e^{(t-s)M_0} (\nabla \cdot (v \omega))(s) \, dx \, dx \, ds$$

$$= -\int_0^t \int \int_{\Omega} (\nabla \cdot (v \omega))(s) e^{(t-s)M_0^*} \psi \, dx \, dx \, ds$$

$$= \int_0^t \int \int_{\Omega} v \omega(s) \cdot \partial e^{(t-s)M_0^*} \psi \, dx \, dx \, ds - \int_0^t \int \left( \frac{\omega e^{(t-s)M_0^*} \psi v(s) \cdot n}{\partial \Omega} \right) \, ds$$

Thus for $0 \leq t \leq T$, we have

$$\left| \int \int_{\Omega} \psi \omega_N(t) \, dx \, dx \right| \leq \int_0^t \|v(s)\|_{p_0} \|\omega(s)\|_{p_0} \|\partial e^{(t-s)M_0^*} \psi\|_{p_4} \, ds \text{ (by Hölder)}$$

$$\leq \int_0^t (C_3 \delta e^{\text{Re}\lambda s})^2 C_2 e^{2 \text{Re}\lambda (t-s)} \, ds \text{ (by Lemmas 4 and 5)}$$

$$\leq C_4 (\delta e^{\text{Re}\lambda s})^2.$$  

So finally we get

$$\|v_N(t)\|_{p_0} \leq C_1 C_4 (\delta e^{\text{Re}\lambda s})^2 \text{ for } t \in [0,T]$$

Let $T^*$ be such that $\delta e^{\text{Re}\lambda} = \theta$, where

$$\theta = \frac{\min \{c_0, c_1\}}{4C_1 C_4} \text{ (c_0 is defined in (4.1))}.$$
We show that $T^* \leq T$. Suppose otherwise, we have $T^* > T$. Then at time $T$, we have

$$\|v(T)\|_{p_0} \leq \|v_L(T)\|_{p_0} + \|v_N(T)\|_{p_0}$$

$$\leq \|e^{T\delta g}\|_{p_0} + C_1 C_4 \left(\delta e^{R \lambda T}\right)^2$$

$$\leq \left(\|v_g\|_{p_0} + C_1 C_4 \theta\right) \delta e^{R \lambda T}$$

$$< \frac{3}{4} C_1 C_4 e^{R \lambda T}$$

which is a contradiction to the definition of $T$. So at time $T^*$, we must have

$$\|v(T^*)\|_{p_0} \geq \|v_L(T^*)\|_{p_0} - \|v_N(T^*)\|_{p_0}$$

$$\geq c_0 \delta e^{R \lambda T^*} - C_1 C_4 \left(\delta e^{R \lambda T^*}\right)^2 \text{ (by (4.1))}$$

$$= c_0 \theta - C_1 C_4 \theta^2$$

$$\geq \frac{3}{4} \theta.$$  

We let $\theta_0 = \frac{3}{4} \theta$ and the proof of Theorem 1 (ii) is finished for simply connected case. \qed

5. **Nonlinear instability (non simply connected case)**

The proof basically follows the same line as in the simply connected case. We only point out some differences, mostly about boundary conditions. We first discuss the operator $\text{curl}^{-1}$ appeared in the vorticity approach. For a non simply connected domain $\Omega$, to determine velocity $v = \text{curl}^{-1}\omega$ from vorticity $\omega$ uniquely, we need to specify circulations

$$\oint_{\Lambda_i} u \cdot dl$$

on each boundary component $\Lambda_i$. For the instability study, it turns out that we need to set

$$\oint_{\Lambda_i} u \cdot dl = 0$$

for each $\Lambda_i$. This is due to the fact that each circulation is invariant for both nonlinear equation (1.1) and linearized equation (1.8). The conservation for nonlinear equation is well known. The conservation for the linearized equation also follows from a direct computation. Since it is crucial in our instability study, we give the proof here. Let $u(t) = (u^1, u^2)$ be a solution to (1.8) and $\omega = \text{curl}u$, then

$$\frac{d}{dt} \oint_{\Lambda_i} u \cdot dl = \oint_{\Lambda_i} u_t \cdot dl$$

$$= - \oint_{\Lambda_i} \left( u_0 \cdot \nabla u + u \cdot \nabla u_0 + \nabla p \right) \cdot dl$$

$$= \oint_{\Lambda_i} \left( - (u_0 + u) \cdot \nabla (u_0 + u) + u_0 \cdot \nabla u_0 + u \cdot \nabla u \right) \cdot dl$$

$$= \oint_{\Lambda_i} (I_1 + I_2 + I_3) \cdot dl.$$
We have

$$\oint_{\Lambda_i} I_3 \cdot dl = \oint_{\Lambda_i} \left( \omega \left( \frac{u^2}{-u^1} \right) + \frac{1}{2} \nabla |u|^2 \right) \cdot dl = \oint_{\Lambda_i} \omega u \cdot n = 0$$

and the other terms can be treated in the same way, thus

$$\frac{d}{dt} \oint_{\Lambda_i} u \cdot dl = 0.$$

From this invariance property it is readily seen that for an exponentially growing mode $e^{\lambda t} v_g (x)$ we must have

$$\oint_{\Lambda_i} v_g \cdot dl = 0$$

for each $\Lambda_i$. Since we take our initial perturbation $v(0, x) = \delta v_g$ or $\delta \text{Im} v_g$, we have zero circulations initially. Denote

$$v(t, x) = e^{t L_0} v(0) + v_N$$

to be the solution of the nonlinear equation

$$\partial_t v = L_0 v - v \cdot \nabla v.$$

Then we have

$$\oint_{\Lambda_i} v_N(t) \cdot dl = 0$$

for each $\Lambda_i$. Thus in our instability study, $\text{curl}^{-1}$ is defined by zero circulation condition. We define the stream function $\phi = \left( -\tilde{\Delta} \right)^{-1} \omega$ to be the unique solution of the following problem

$$-\Delta \phi = \omega$$

$$\phi|_{\Lambda_i} = \psi_i, \quad \oint_{\Lambda_i} \frac{\partial \phi}{\partial n} = 0,$$

$$\int_\Omega \int_\Omega \phi dx_1 dx_2 = 0,$$

where $\psi_i$ are unspecified constants. Then

$$\text{curl}^{-1} \omega = \nabla^\perp \phi = \nabla^\perp \left( -\tilde{\Delta} \right)^{-1} \omega.$$

Now

$$M_0 = -D_1 + D_2 \left( -\tilde{\Delta} \right)^{-1}$$

and we have

$$M_0^* = D_1 - \left( -\tilde{\Delta} \right)^{-1} D_2.$$
This follows from the fact that \((-\Delta)^{-1}\) is self-dual, which is checked in the following. Let \(\phi = (-\Delta)^{-1}\omega, \phi^* = (-\Delta)^{-1}\omega^*\), then
\[
\int \int_{\Omega} (-\Delta)^{-1}\omega \omega^* dx_1 dx_2 = \int \int_{\Omega} \phi (-\Delta \phi^*) dx_1 dx_2
\]
\[
= \int \int_{\Omega} \nabla \phi \cdot \nabla \phi^* dx_1 dx_2 - \sum_i \int_{\Lambda_i} \phi \frac{\partial \phi^*}{\partial n}
\]
\[
= \int \int_{\Omega} \nabla \phi \cdot \nabla \phi^* dx_1 dx_2 - \sum_i \int_{\Lambda_i} \phi \frac{\partial \phi^*}{\partial n} \quad \text{since} \quad \phi|_{\Lambda_i} \int_{\Lambda_i} \frac{\partial \phi^*}{\partial n} = 0
\]
\[
= \int \int_{\Omega} (-\Delta)^{-1}\omega \omega^* dx_1 dx_2.
\]
From the above computations, we can see how the zero circulation condition is used. Denote
\[
\tilde{W}^{1,p}_0 = \left\{ \psi \in W^{1,p} \left| \int \int_{\Omega} \psi dx_1 dx_2 = 0 \right\}
\]
and
\[
H^0_p = \left\{ u \in L^p(\Omega)^2 | \nabla \cdot u = 0, u \cdot n = 0 \text{ on } \partial \Omega \text{ and } \int_{\Lambda_i} u \cdot dl = 0 \text{ for each } \Lambda_i \right\}.
\]
Then we still have (4.4) but (4.5) should be changed to the following estimate:
If \(v \in H^0_p\) and \(\text{curl} v \in L^p(\Omega)\) then we have
\[
\sup_{\psi \in \tilde{W}^{1,p}_0} \frac{1}{\|\psi\|_{\tilde{W}^{1,p}_0} = 1} \int \int_{\Omega} \psi \text{curl} v dx_1 dx_2
\]
for some constant \(C_1\). The proof of (5.1) is similar to that of (4.5). So we skip it here. We indicate that in the proof the zero circulation condition is required in the process of integration by parts and Poincaré’s inequality still holds in \(\tilde{W}^{1,p}_0\).
Since \(v_N(t)\) has zero circulations as we have showed, we can use (5.1) to estimate \(\|v_N(t)\|_{p_0}\) just as in the simply connected case. In should be noted that to derive the similar estimate (4.7) for the non-simply connected case, we use the fact that
\[
\tilde{v}(t) = \nabla \perp (-\Delta)^{-1} D2 \tilde{\psi}(s)
\]
has zero circulations. Then we can estimate \(\tilde{v}(t)\) by (5.1) and still get the same growth estimate. The rest of the proof is the same.

6. Comments and extensions

6.1. Nonlinear instability of the 2D quasi-geostrophic flows. Consider a bounded domain \(\Omega\) with \(C^2\) boundary \(\partial \Omega = \bigcup \Lambda_i\). The governing equation for two-dimensional quasi-geostrophic barotropic flow on the \(\beta\)-plane can be written in the
following form ([20], [21])

\[
\frac{\partial P}{\partial t} - \partial_2 \psi \partial_1 P + \partial_1 \psi \partial_2 P = 0
\]

with boundary conditions

\[\psi|_{\Lambda_i} = \Psi_i, \ (\Psi_i \text{ depends on time only})\]

\[\oint_{\Lambda_i} \frac{\partial \psi}{\partial n} = \Gamma_i \ (\Gamma_i \text{ is constant}).\]

Here the stream function \(\psi\) is related to the horizontal velocity components \(u^1\) and \(u^2\) by \((u^1, u^2) = \nabla^\bot \psi\), and the potential vorticity

\[P = -\Delta \psi + F \psi + f (x^1, x^2),\]

represents the combined influence of topography and the Coriolis force to the potential vorticity with \(F\) being a positive constant. This equation is also called the Charney-Hasegawa-Mima equation ([23]), widely used in geophysical fluid dynamics and the study of a weakly magnetized nonuniform plasma. Consider a stationary solution \((\psi_0, P_0)\) to (6.1). We can obtain the same instability result as in Theorem 1. That is, linear instability always implies nonlinear instability for the general steady flows in a bounded domain.

The proof is almost the same. We only indicate one major difference below. Consider the simply connected case first. We have the following new estimate:

\[\|v\|_p \leq C \sup_{\psi \in W^{1,p'}} \int \int_{\Omega} \psi q dx^1 dx^2\]

Proof of (6.2): By (4.5), we have

\[\|v\|_p \leq C_1 \sup_{\psi \in W^{1,p'}} \int \int_{\Omega} \psi (-\Delta \phi) dx^1 dx^2\]

\[= C_2 \sup_{\psi \in W^{1,p'}} \int \int_{\Omega} \psi \left(1 - F (-\Delta + F)^{-1}\right) q dx^1 dx^2\]

\[= C_2 \sup_{\psi \in W^{1,p'}} \int \int_{\Omega} q \left(1 - F (-\Delta + F)^{-1}\right) \psi dx^1 dx^2\]

\[= C_3 \left(1 + F \left((-\Delta + F)^{-1}\right)\right) \sup_{\psi' \in W^{1,p'}} \int \int_{\Omega} q \psi' dx^1 dx^2.\]

For the non simply connected case, we can prove the same estimate for \(v \in H^0_p\) (with zero circulations) as what we did in Section 5. Let \(v(t) = v_L + v_N\) and \(q(t) = q_L + q_N\) be the evolution of perturbations of velocity and potential vorticity.
according to (6.1). Then by (6.2), we have
\[ \|v_N\|_p \leq C \sup_{\psi \in W^{1,p'}} \int_\Omega \psi q_N dx^1 dx^2. \]
The rest of the proof is the same.

6.2. Remarks on the choice of norm for nonlinear instability. The choice of norm is important for the instability study of an infinite dimensional dynamical system. For the 2D Euler equation, this is a particularly subtle issue. In [16] we constructed a steady flow stable in the $L^2$ norm of vorticity, while linearly unstable in the $L^2$ norm of velocity due to highly oscillating perturbations around a nondegenerate hyperbolic point of the steady flow (see [14] [6]). However such perturbations have very large vorticity norm and can not appear if we require initial perturbations to have small vorticity norm. In the below we illustrate, from another point of view, that it is indeed necessary to impose the vorticity norm constrain for the instability study. Our purpose of stability study is to pick out some stable objects for which we might use as possible asymptotic structures in the study of long time behavior. In two-dimensional turbulence ([25]), it is observed that coherent large scale structures appear with “noise” of very small scales. To understand these coherent structures, we need to study stability and instability in the large scale. If a steady state is unstable in the large scale, then we could not expect it to play a role in the study of the long time behavior of the system. So we are more concerned with instability in the large scale, while not in the very small scale.

In the following we show that nonlinear instability showed in Theorem 1 is indeed in the large scale sense. For simplicity, we assume $\Omega$ is simply connected. Denote $0 < \lambda_1 < \lambda_2 < \cdots \lambda_k \cdots$ to be all the eigenvalues of $-\Delta$ and $\phi_1, \phi_2, \cdots, \phi_k, \cdots$ the corresponding normalized eigenfunctions. We take $p_0 = p_1 = p_2 = 2$ in Theorem 1 and denote by $v(t,x) = v_0(t,x) - u_0 = \nabla^\perp \psi(t,x), \omega(t,x) = \omega_0(t,x) - \omega_0$ the perturbations of velocity and vorticity. We have the expansion
\[ \psi(t,x) = \sum_{k=1}^\infty a_k(t) \phi_k \]
and thus
\[ \omega(t,x) = -\Delta \psi = \sum_{k=1}^\infty a_k(t) \lambda_k^2 \phi_k. \]
So
\[ \|\omega(t)\|_2^2 = \sum_{k=1}^\infty |a_k|^2 \lambda_k^4 \]
and
\[ \|v(t)\|_2^2 = \int_\Omega \omega dx^1 dx^2 = \sum_{k=1}^\infty |a_k|^2 \lambda_k^2. \]
For small $k$, $|a_k|$ denotes the amplitude of the large scale perturbation. As we can see from the proof of Lemma 5 and Theorem 1, there exists $C' > 0$ such that $\|\omega(t)\|_2 \leq C' \delta e^{Re \lambda T^*}$ for $0 \leq t \leq T^*$, where $T^*$ is such that $\delta e^{Re \lambda T^*} = \frac{4\alpha}{\lambda}$. So
\[ \| \omega (T^*) \|_2 \leq C' \frac{4\theta_0}{3}. \]

Since \( \lambda_k \to \infty \), there exists \( K_0 \) such that

\[
\sum_{k=K_0}^{\infty} |a_k (T^*)|^2 \frac{\lambda_k^2}{\lambda_{K_0}^2} < \frac{1}{\lambda_{K_0}^2} \| \omega (T^*) \|_2 \leq \frac{1}{\lambda_{K_0}^2} \left( C' \frac{4\theta_0}{3} \right)^2 \leq \frac{1}{2} \theta_0^2.
\]

Since \( \| v(T^*) \|_2 \geq \theta_0 \), we have

\[
\sum_{k=1}^{K_0} |a_k (T^*)|^2 \lambda_k^2 = \| v(T^*) \|_2^2 - \sum_{k=K_0}^{\infty} |a_k (T^*)|^2 \lambda_k^2 \geq \frac{1}{2} \theta_0^2.
\]

So one of \( |a_1 (T^*)|, |a_2 (T^*)|, \ldots, |a_{K_0} (T^*)| \) must be bigger than a fixed number and the deviation from the steady state is in the large scale. However if we impose no constrain on vorticity function, the instability can happen due to the concentration of energy on very small scales and the large scale instability could not be detected.

ACKNOWLEDGMENTS

In particular I thank Laisang Young for stimulating discussions about the proof of Theorem 2. I thank Walter Strauss and Yan Guo for their useful comments.

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