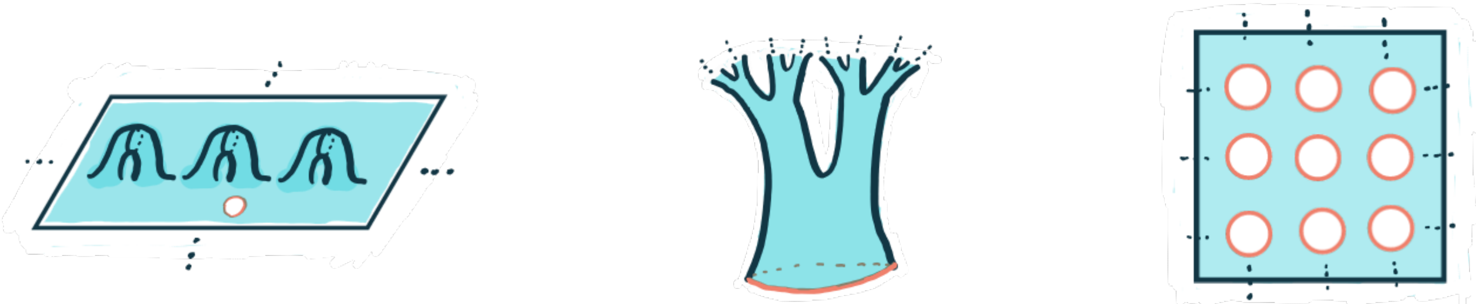
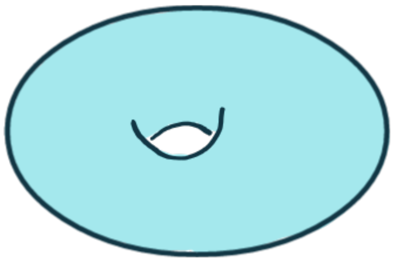
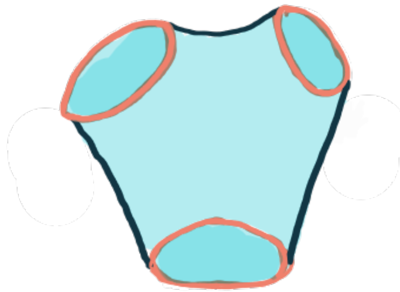
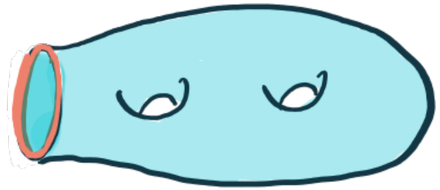
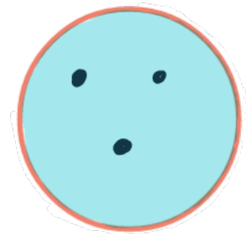
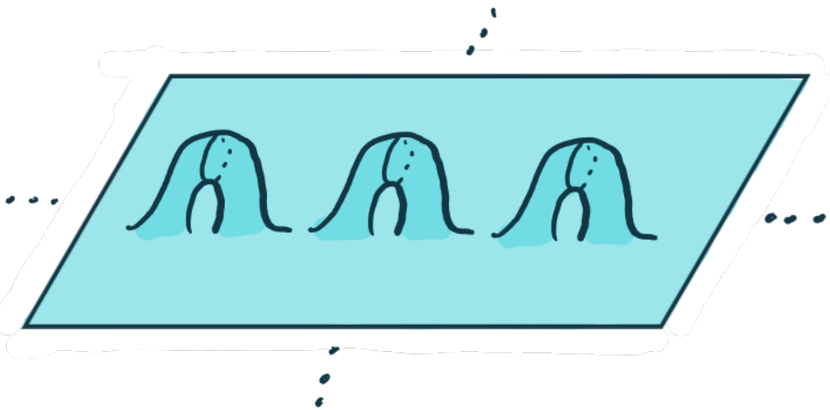
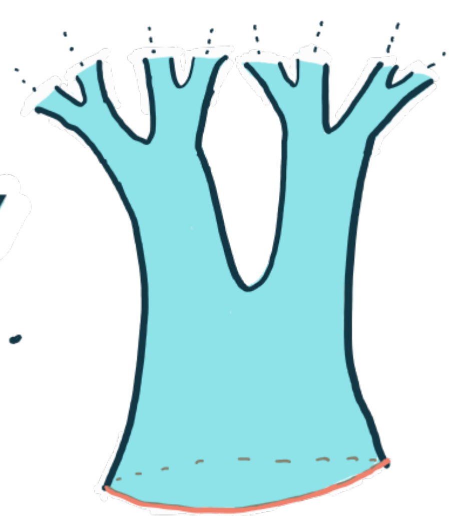
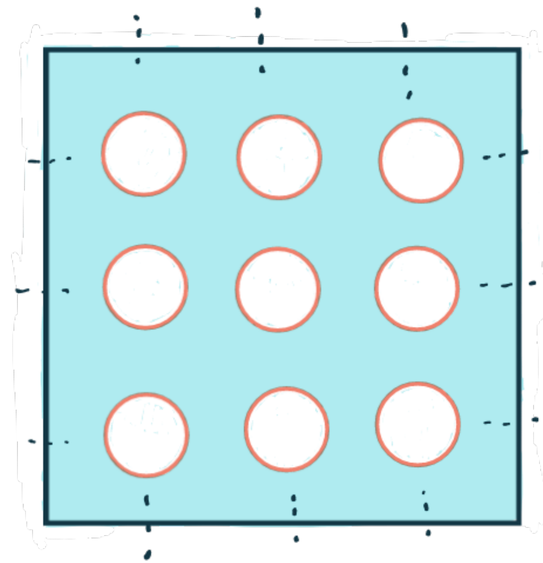
Generalizing the (fractional) Dehn twist coefficient

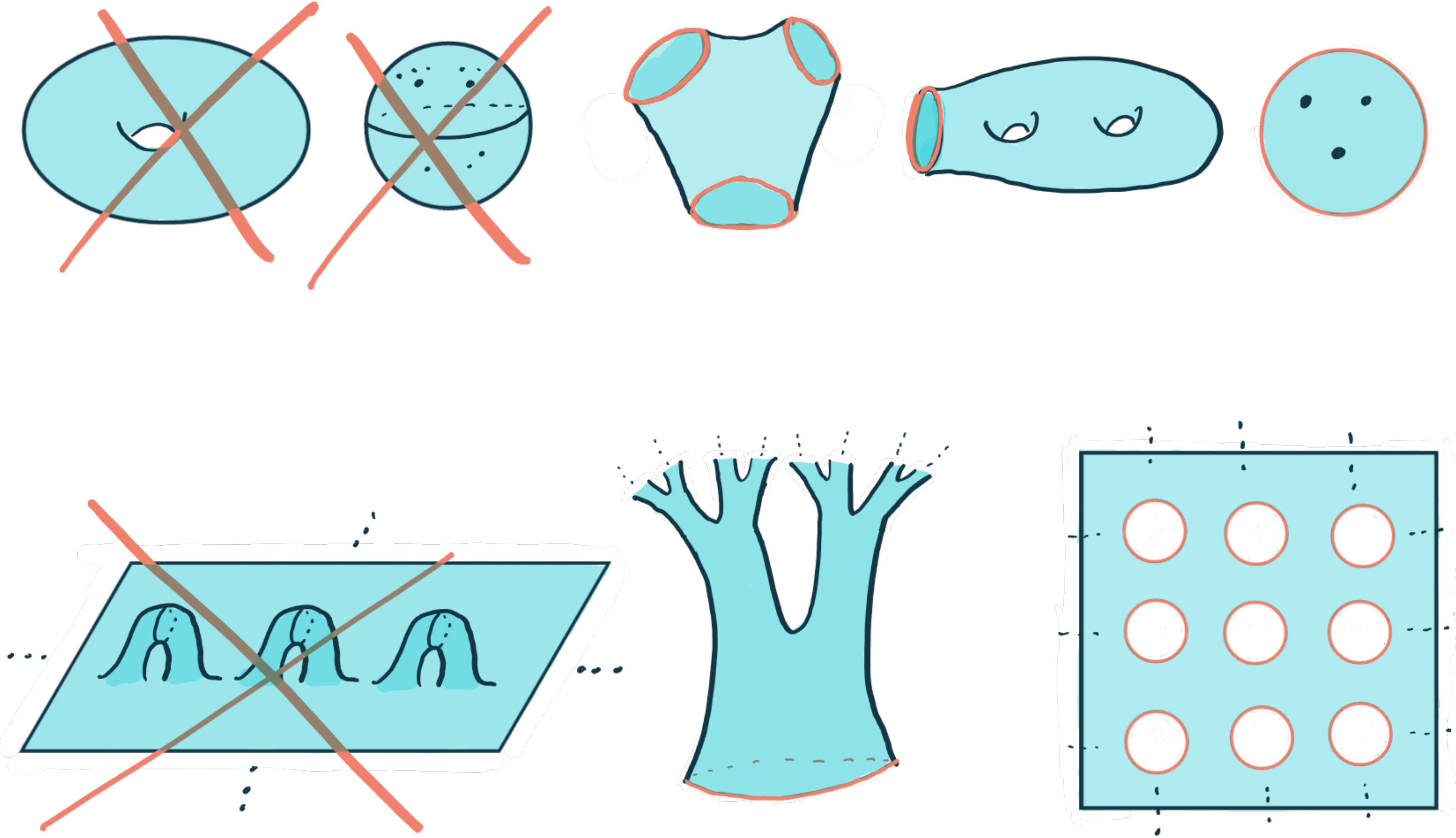
Hannah Turner



joint work in progress with
P. Feller & D. Hubbard



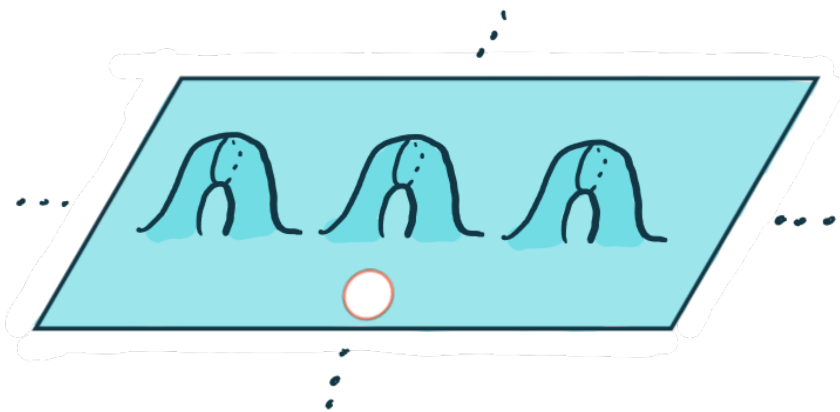









☆ surface S with at least one boundary component

☆ homeo$^+$ / diffeo$^+$ φ on S $/\sim$ $:=$ MCG(S)

- fix ∂S pointwise
- permute punctures P
- $\sim$ up to isotopy rel ∂ and rel P

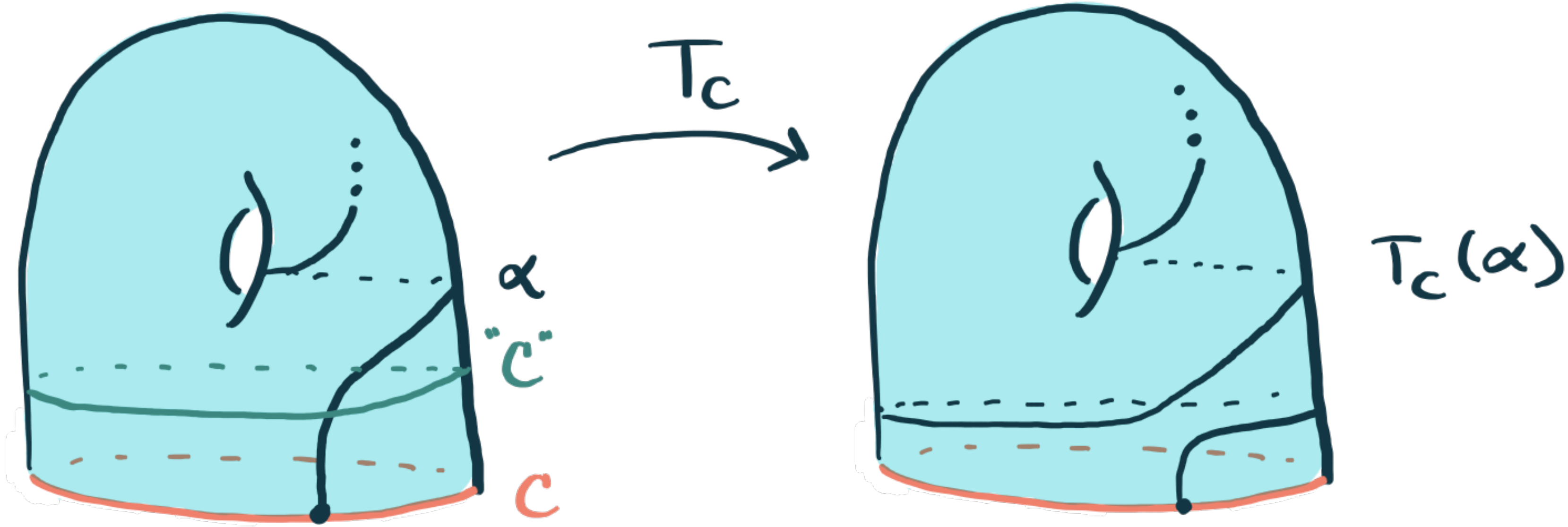


the fractional Dehn twist coefficient

FDTC(φ, c)

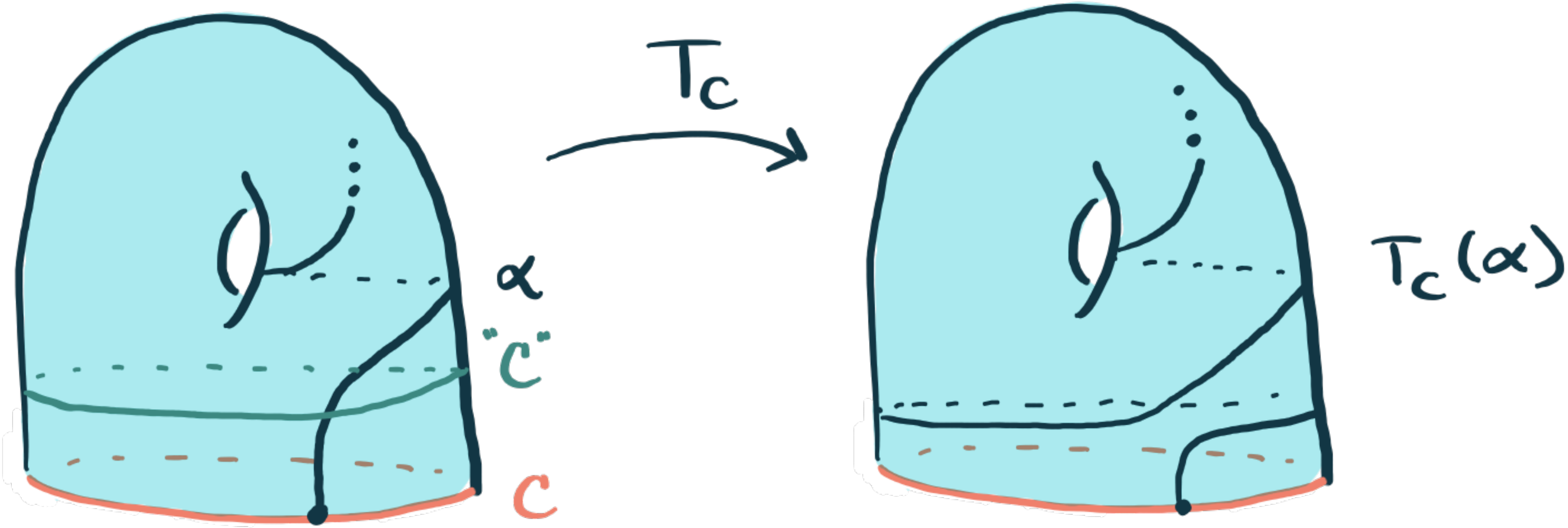
Heuristic: measures twisting of φ near c.

Ex:



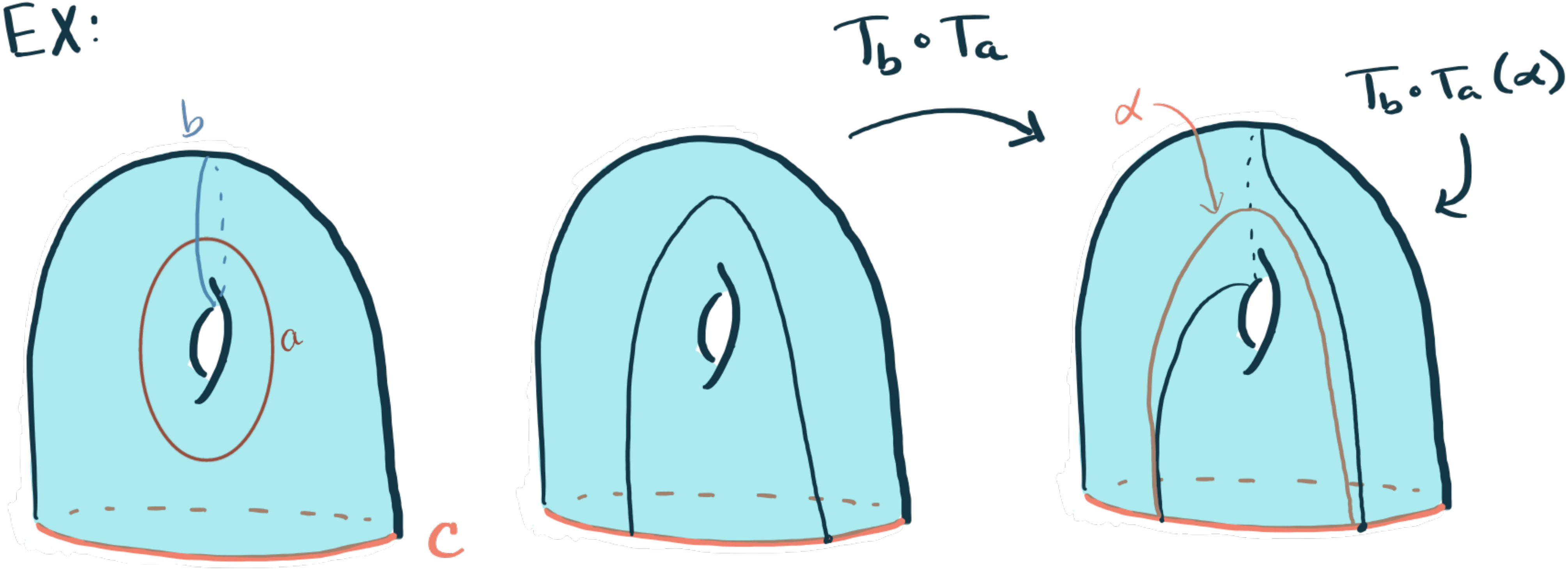
the fractional Dehn twist coefficient

Ex:

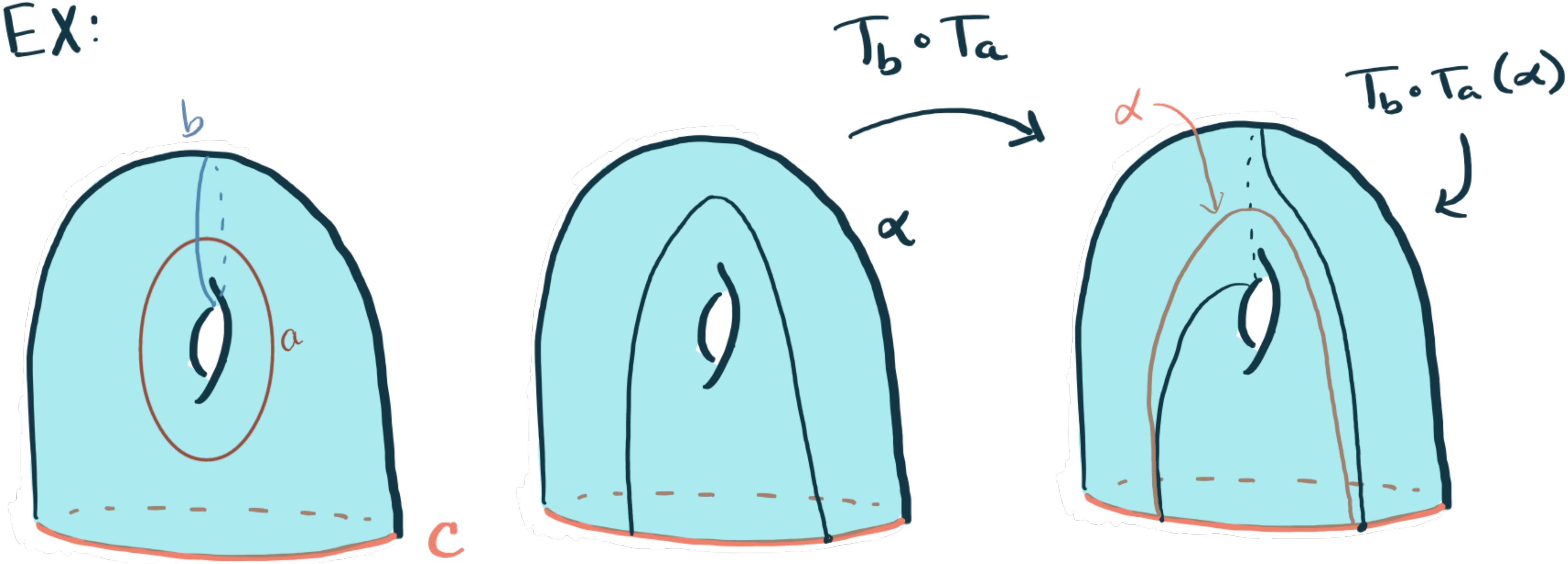


$$\mathrm{FDTC}(T_c, c) = 1$$

EX:



EX:



• $(T_b \circ T_a)^6 \cong T_c$

$$\boxed{\mathrm{FDTC}(\varphi^n) = n\,\mathrm{FDTC}(\varphi)}$$

$$6\,\mathrm{FDTC}(T_b \circ T_a) = \mathrm{FDTC}(T_c) = 1$$

Why study FDTC ?

① It's an invariant of a mapping class

② 3-manifolds

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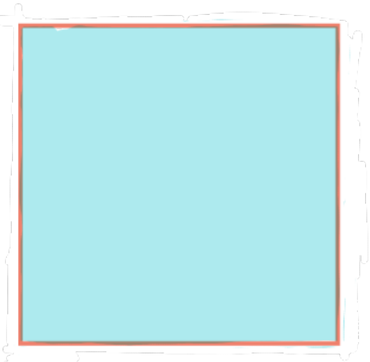
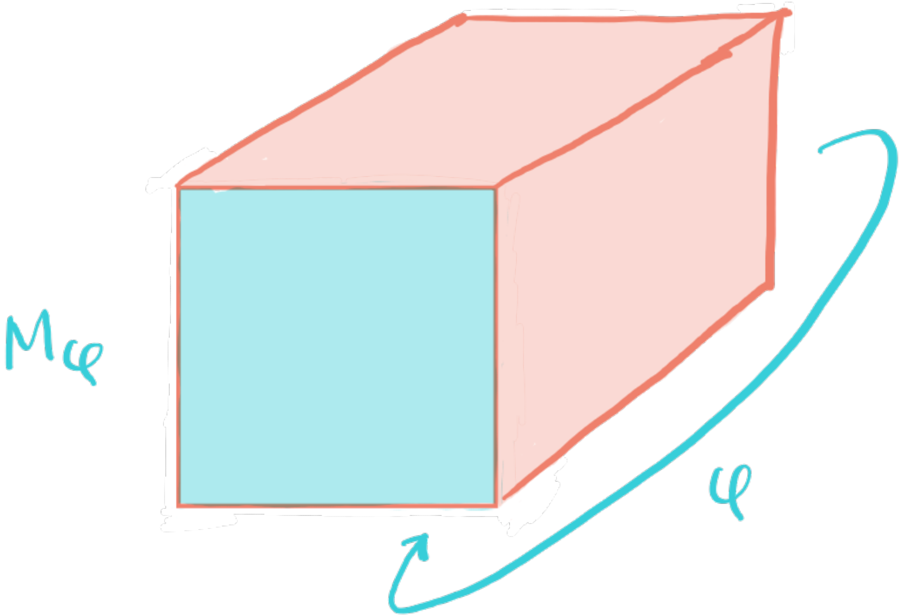
S 

Why study FDTC ?

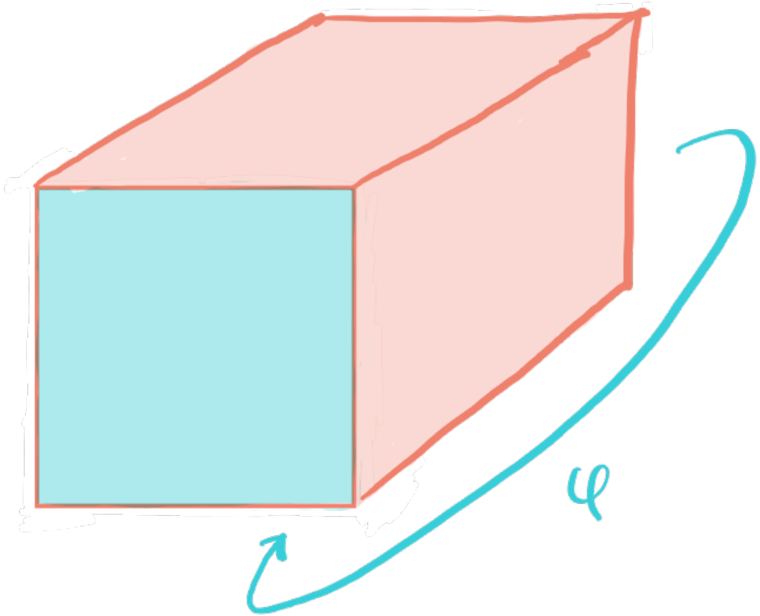


$M_\varphi \overset{\text{fill}}{\rightsquigarrow} M^3 = (S, \varphi)$

Fact (Alexander) Any closed orientable 3-mfd admits an open book decomposition.

Why study FDTC ?

M_φ



$$M_\varphi \overset{\text{fill}}{\rightsquigarrow} M^3 = (S, \varphi)$$

Fact (Alexander) Any closed orientable 3-mfd admits ~~an~~ ∞-many open book decompositions

Question: Can we extract info about M^3 from one OBD? Via FDTC?

☆ (S, φ) has an assoc. FDTC

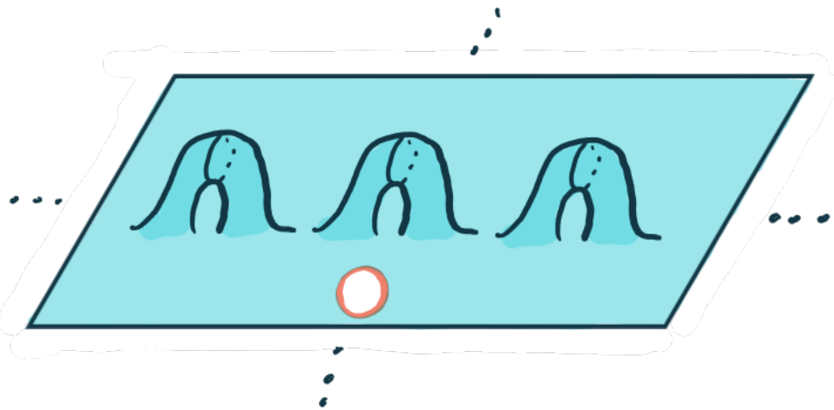
- essential laminations (Gabai)
- taut foliations (Roberts)
- contact structures (Honda-Kazez-Matic, Colin-Honda, ...)
- geometry (Ito-Kawamuro)

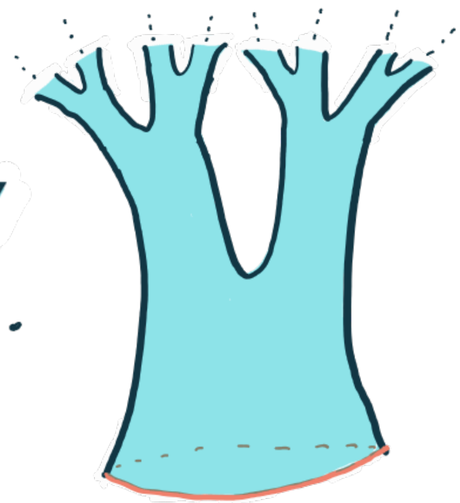
Question: What about when S is infinite-type?

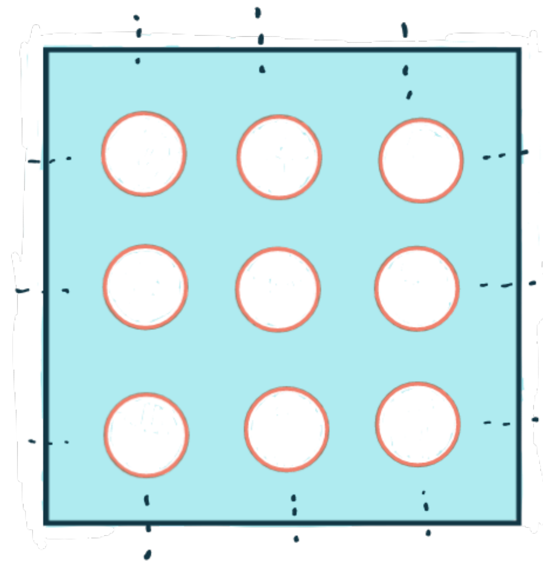
Why study FDTC ?

① It's an invariant of a mapping class

② 3-manifolds (???)

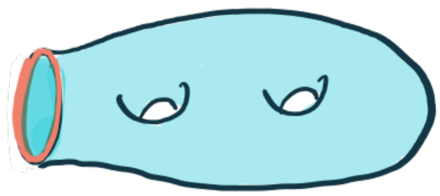






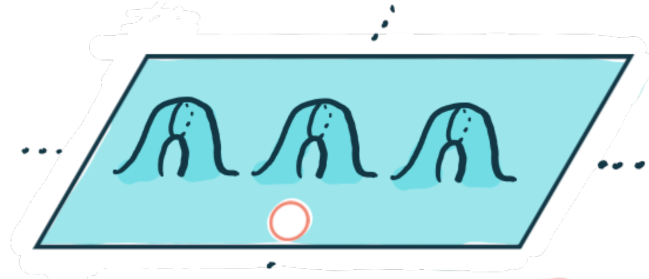
Braids

- FDTC: $B_n \to \mathbb{Q}$
- If $\mathrm{FDTC}(\beta) = p/q$ then $1 \leq q \leq n$ (*) (Malyutin)
- Any p/q satisfying (*) is achieved as $\mathrm{FDTC}(\beta)$ for some $\beta \in B_n$



Finite-type surfaces

- FDTC: $\mathrm{MCG}(S) \to \mathbb{Q}$
- If $\mathrm{FDTC}(\beta) = p/q$ then $1 \leq q \leq 4g+2$ (*) (Ito-Kawamuro)
- Thm (Hubbard-T) If $\varphi \in \mathrm{MCG}(\ \ )$, then $\mathrm{FDTC}(\varphi) \neq \frac{1}{4g+1}$
- Q: What is image of FDTC?



∞-type surfaces

- FDTC: $\mathrm{MCG}(S) \to \mathbb{R}$ (not $\mathbb{Q}$)
- Thm (Feller-Hubbard-T) There is a surface X with $\mathrm{FDTC}: \mathrm{MCG}(X) \to \mathbb{R}$ is surjective.
- Q: What is the image of FDTC in general?

Properties of the FDTC

① $\mathrm{FDTC}(T_c) = 1$

② $\mathrm{FDTC}(\varphi^n) = n\,\mathrm{FDTC}(\varphi)$ (homogeneity)

③ $|\mathrm{FDTC}(\varphi\psi) - \mathrm{FDTC}(\varphi) - \mathrm{FDTC}(\psi)| \leq C$ (quasimorphism)

④ if φ moves some arc α to the right
then $\mathrm{FDTC}(\varphi) \geq 0$ (positivity)

Thm (Feller - Hubbard - T) Let X be a surface with at least one compact boundary component C. There exists a unique map $\omega_c : \mathrm{MCG}(X) \to \mathbb{R}$ satisfying ① - ④.

Properties of the FDTC

① $\mathrm{FDTC}(T_c) = 1$

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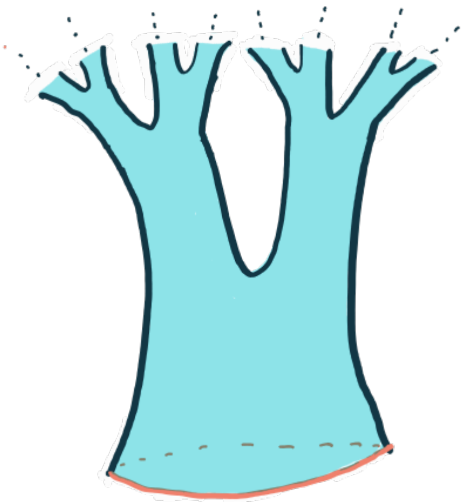
③ $|\mathrm{FDTC}(\varphi\psi) - \mathrm{FDTC}(\varphi) - \mathrm{FDTC}(\psi)| \leq C$ (quasimorphism)

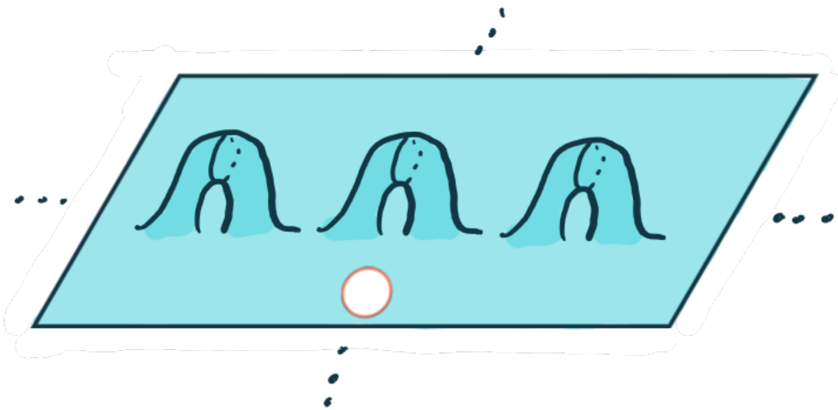
④ if $\varphi \geqslant 0$ ~~moves some arc α to the right~~
then $\mathrm{FDTC}(\varphi) \geqslant 0$ (positivity)

Thm (Feller-Hubbard-T) Let ~~X~~ $(G, \leq^*)$ be a ~~surface~~ group with ~~at least one compact boundary component~~ a central element T C. There exists a unique map $\omega_c : MCG(X) \to \mathbb{R}$ satisfying ①–④.

Thm (Feller-Hubbard-T) There exists a surface X so that for any $r \in \mathbb{R}$, there is $\varphi_r \in MCG(X)$ with $FDTC(\varphi_r) = r$.

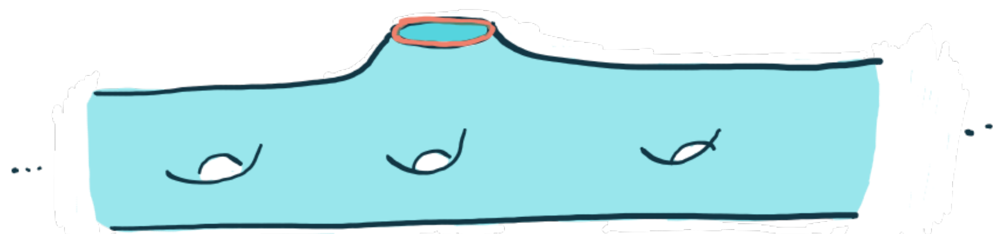


$X = \mathbb{D}^2 - \{\text{Cantor set}\}$



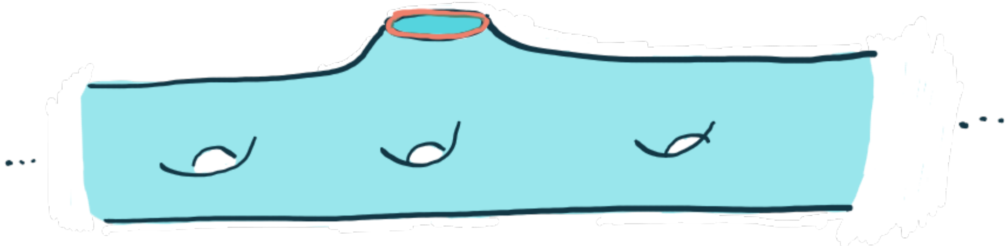
X = Lochness monster − disk

Q: How "bad" does X have to be?



<u>Thm</u> (Feller - Hubbard - T) There exists a surface X so that for any $r \in \mathbb{R}$, there is $\varphi_r \in MCG(X)$ with $FDTC(\varphi_r) = r$.

Q: How "bad" does X have to be?



Q: How "bad" does φ_r have to be?

- φ compactly supported $\Rightarrow FDTC(\varphi) \in \mathbb{Q}$
- the maps we construct are not (generally) pure

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Q: How "bad" does φ_r have to be?

- φ compactly supported $\Rightarrow$ $\mathrm{FDTC}(\varphi) \in \mathbb{Q}$
- the maps we construct are not (generally) pure

Q: Does the FDTC say anything about non-compact 3-mfds?

Thanks for listening!