Results on exact Lagrangian fillings of Legendrian links

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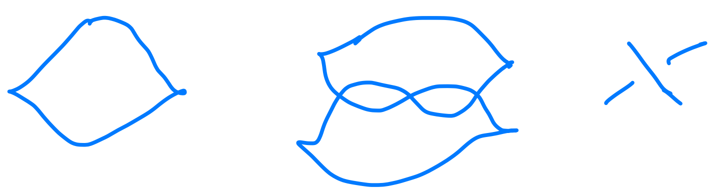
2 projects: (A) joint w/ R. Casals UC. Davis
(B) joint w/ J. Hughes Duke
D. Weng U.C. Davis

let $(\mathbb{R}^3, \xi_{std} = \ker(dz - y\,dx)) = \langle \partial_y, y\partial_x + \partial_x \rangle$
$p = (x_0, y_0, z_0) \in \mathbb{R}^3$

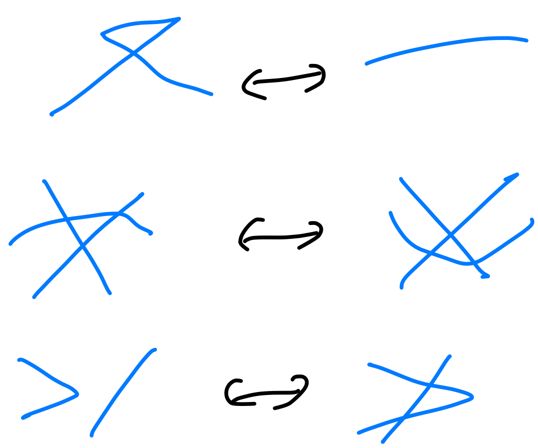
let $\Lambda \subseteq \mathbb{R}^3$ be a smooth link

we say it is Legendrian if $T_p\Lambda \subseteq \xi_p \ \forall p \in \Lambda$

front projection



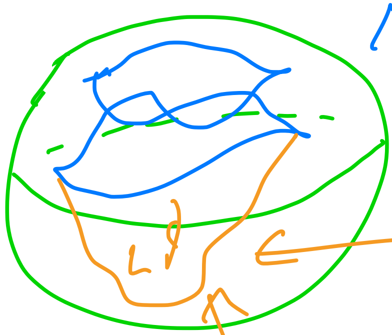
Legendrian Reidemeister



lagrangian projection $\pi_L : \mathbb{R}^3 \to \mathbb{R}^2\langle x,y \rangle$

$\pi_L(\Lambda)$ immersed curves w/ oriented area zero

Consider



Λ

$(B^4, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$

filling of Λ

exact Lagrangian

$$L \subseteq (B^4, \omega_{std} = d(\underbrace{y_1 dx_1}_{\lambda}))$$

Lagrangian surface cf $\omega|_{T_pL} = 0 \ \forall p \in L$

if $\lambda|_L = df$ for $f: L \to \mathbb{R}$

then L is exact Lagrangian

L is an <u>exact lag filling</u> of $\Lambda \subseteq (S^3, \xi_{std})$ if

$L \subseteq (B^4, \omega_{std})$ is an exact Lagrangian

if $L \cap S^3 = \Lambda$

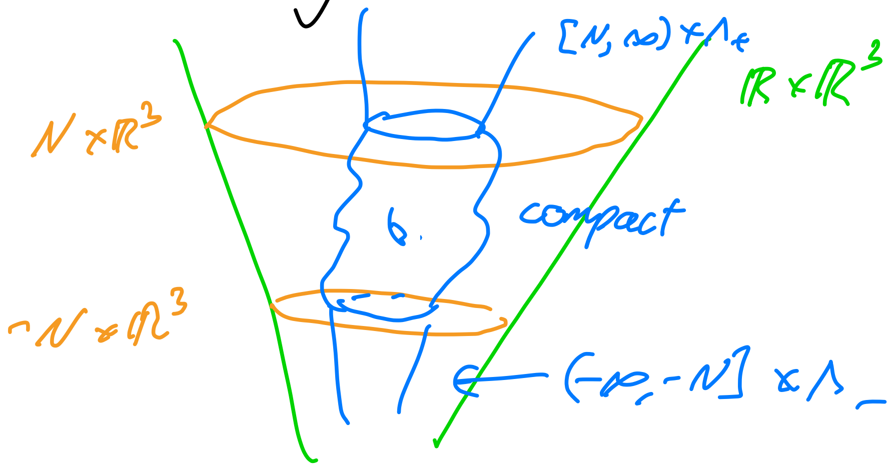
<u>Goal</u>: Study Lag fillings up to exact Lag isotopy fixing the boundary

[Eliashberg, Polterovich]  has a unique exact Lag disk filling

$\Lambda_-, \Lambda_+ \subseteq (\mathbb{R}^3, \xi_{std})$ Legendrians

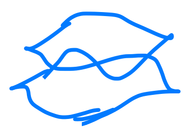
an <u>exact Lagrangian cobordism</u> L from Λ_- to Λ_+

is $L \subseteq (\mathbb{R}\times\mathbb{R}^3, d(e^t(dz - ydx)))$
exact Lag and $\exists\, N \in \mathbb{N}$ st



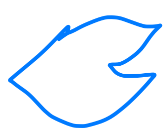
and f is constant on the cylindrical ends

Facts: if orientable, then $g(L) = g_4(\Lambda)$
[Chantraine]



if Λ has an orientable lag filling then Λ
must be quasipositive [Hayden, Sabloff]

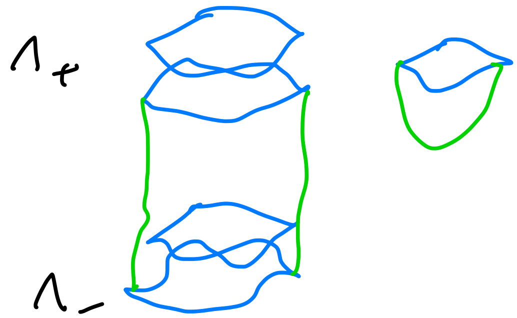
L nonorientable $\Rightarrow$ some topological restrictions
[Chan-Cuder, Phillips
Reno... Sabloff Yao] ?

 has no exact fillings

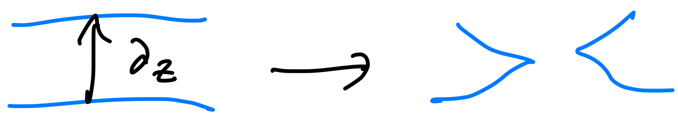
Decomposable Lagrang
(1) trace of a Legendrian isotopy

(2) Minimum cobordisms:

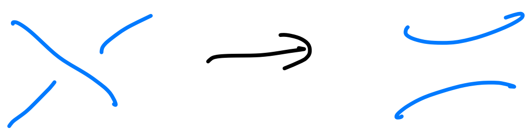
If $\Lambda_+ = \Lambda_- \cup$ [unknot]



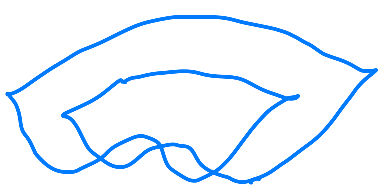
3) Saddle cobordism



front



Lag proj

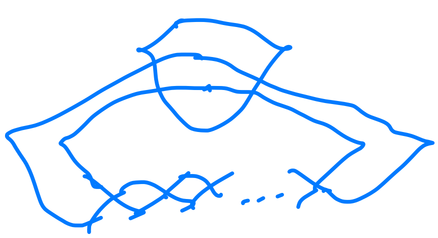


3rd Catalan #

$\Lambda[2,3]$ has C_3 fillings [Pan]

$\Lambda[2,n]$ has C_n

expected this is all



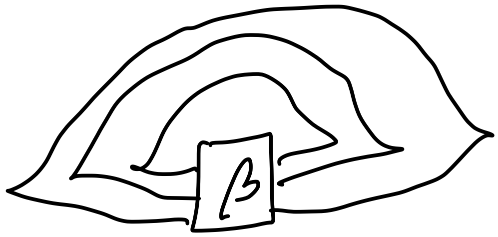
finitely many constructed

Hughes ...?

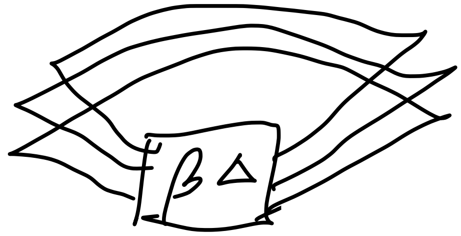
max tb $T(p,q)$ torus links ∞-many fillings

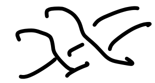
$p,q \geq 4$ [Casals-Gao]

β positive n-braid



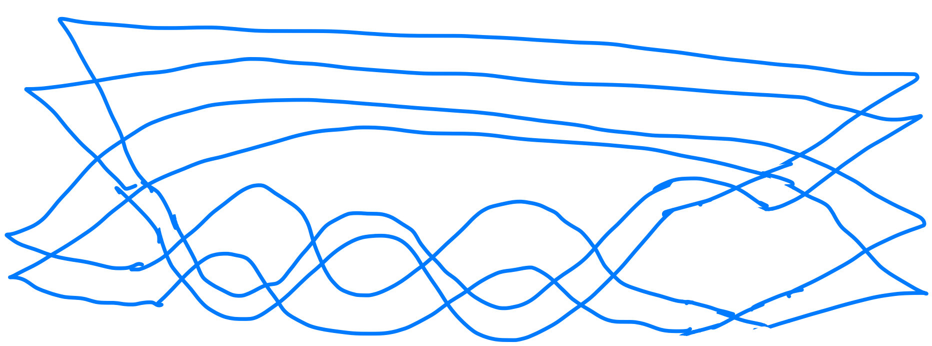
Gao Shen Weng



$\Delta =$ 

Casals-Gorsky^2
Simental-Le-Shen

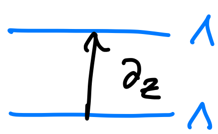
Casals-Ng : if $\exists$ lag cobordism, Λ_- to Λ_+
and Λ_- has ∞'ly many fillings
distinguished by augmentations
$\Rightarrow \Lambda_+$ does too



$\Lambda(\beta_{11})$ has ∞'ly many fillings
genus 1 Casals-Ng

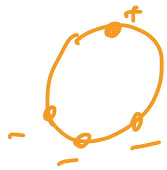
Legendrian dga $(A(\Lambda, R), \partial)$
legendrian invariant

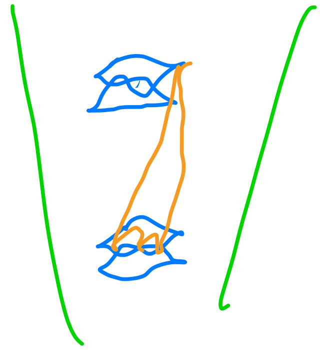
noncommutative differential graded algebra (dga)
generated

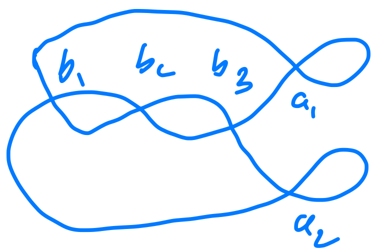


crossings in Lag proj

differential counts rigid J-disks







$$\partial a_1 = b_1 b_2 b_3 + b_1 + b_3 + 1$$
$$\partial a_2 = b_3 b_2 b_1 + b_1 + b_3 + 1$$
$$\partial b_1 = 0$$

Thm [Ekholm Honda Kalman, Karlsson]
if L is Lag cobordism from Λ_- to Λ_+
$\Rightarrow$ chain map
$$\Phi_L : (A(\Lambda_+, R), \partial) \to (A(\Lambda_-, R), \partial)$$
if $\Lambda_- = \emptyset$ we have
$$\varepsilon_L : (A(\Lambda_+, R), \partial) \longrightarrow (R, 0)$$
$$(\varepsilon_L \circ \partial = 0)$$

if L_1, L_2 exact Lag isotopic $\Rightarrow$ ε_{L_1} and ε_{L_2} are chain homotopic

can use $R = \mathbb{Z}_2[H_1(L, \Lambda; \mathbb{Z})]$

$$\varepsilon_L(a) \in \mathbb{Z}_2[H_1(L, \Lambda; \mathbb{Z})]$$

$$\cong \mathbb{Z}_2[s_1^{\pm 1}, \dots, s_n^{\pm 1}]$$

Laurent poly

if $L_1 \cong L_2 \Rightarrow \exists\, \phi \in GL(K, \mathbb{Z})$

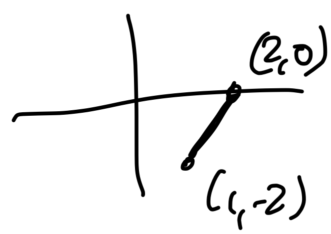
$$\phi(H_1(L_1, \Lambda)) = H_1(L_2, \Lambda)$$

$\phi \circ \varepsilon_{L_1}$ chain homotopic to ε_{L_2}

we construct Lag fillings where Laurent polys are not related by $GL(K, \mathbb{Z})$

Newton polytope

$N(x^2 + y^{-2}x)$



consider

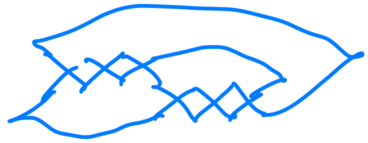
$(\text{volume}, \#(N(f) \cap \mathbb{Z}^k))$

use to distinguish Lag's

Thm: [CS-Casals]

can distinguish examples above

$\exists$ Leg. with infinitely many non-orient fillings



Thm: [CS, Hughes, Weng]

for certain Leg 2-bridge knots Λ & ungraded augmentations of Λ has a ring of functions which algebraically has a cluster structure of type $A_{n_1} \times \cdots \times A_{n_k}$