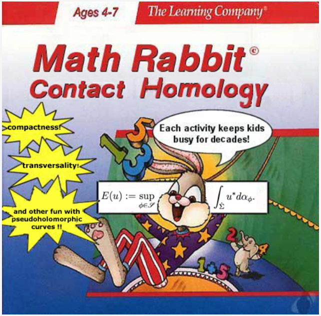
Positive torus knotted Reeb dynamics in the tight 3-sphere

Jo Nelson (Rice University)

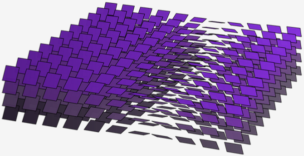
NSF CAREER DMS-2142694



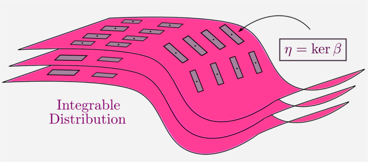
Contact structures

Definition

A **contact structure** is a maximally nonintegrable hyperplane field.



$\xi = \ker(dz - ydx)$



The kernel of a 1-form λ on Y^{2n+1} is a contact structure whenever

- $\lambda \wedge (d\lambda)^n$ is a volume form $\Leftrightarrow$ $d\lambda|_\xi$ is nondegenerate.

Darboux's Theorem

Let λ be a contact form on Y^{2n+1} and $p \in Y$. Then there are coordinates on $U_p \subset Y$ such that $\lambda|_{U_p} = dz - \sum_{i=1}^n y_i dx_i$.

Locally all contact structures look the same!

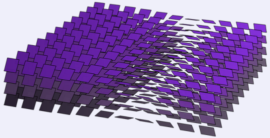
$\rightsquigarrow$ no local invariants like curvature.

Reeb vector fields

Definition

The **Reeb vector field** R on (Y, λ) is uniquely determined by

- $\lambda(R) = 1$
- $d\lambda(R, \cdot) = 0$



$\lambda = dz - ydx, \quad R = \frac{\partial}{\partial z}$

The **Reeb flow** $\varphi_t : Y \to Y$ is defined by $\frac{d}{dt}\varphi_t(x) = R(\varphi_t(x))$.

The Reeb flow preserves the contact form and contact structure.

A closed **Reeb orbit** (modulo reparametrization) satisfies

$$\gamma : \mathbb{R}/T\mathbb{Z} \to Y, \quad \dot{\gamma}(t) = R(\gamma(t)), \tag{1}$$

and is **embedded** whenever (1) is injective.

Reeb orbits on a contact 3-manifold

Given an embedded **Reeb orbit** $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$,
the linearized flow along γ defines a symplectic linear map

$$d\varphi_t : (\xi|_{\gamma(0)}, d\lambda) \to (\xi|_{\gamma(t)}, d\lambda)$$

$d\varphi_T$ is called the **linearized return map**.

If 1 is not an eigenvalue of $d\varphi_T$ then γ is **nondegenerate**. λ is **nondegenerate** if all Reeb orbits associated to λ are nondegenerate.

For $\dim Y = 3$, nondegenerate orbits are either **elliptic** or **hyperbolic** according to whether $d\varphi_T$ has eigenvalues on S^1 or real eigenvalues.

Later, we consider an almost complex structure J on $T(\mathbb{R} \times Y)$:

- J is $\mathbb{R}$-invariant
- $J\xi = \xi$, rotates ξ positively with respect to $d\lambda$
- $J(\partial_s) = R$, where s denotes the $\mathbb{R}$ coordinate

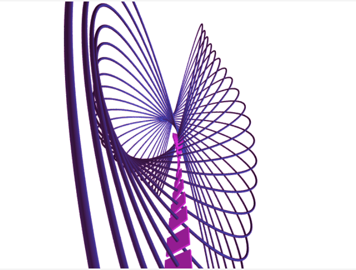
Reeb orbits on S^3

$$S^3 := \{(u,v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}, \lambda = \frac{i}{2}(u d\bar{u} - \bar{u} du + v d\bar{v} - \bar{v} dv).$$

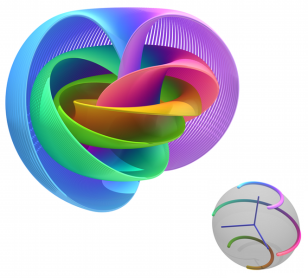
The orbits of the Reeb vector field form the Hopf fibration!

$$R = iu\frac{\partial}{\partial u} - i\bar{u}\frac{\partial}{\partial \bar{u}} + iv\frac{\partial}{\partial v} - i\bar{v}\frac{\partial}{\partial \bar{v}} = (iu, iv).$$

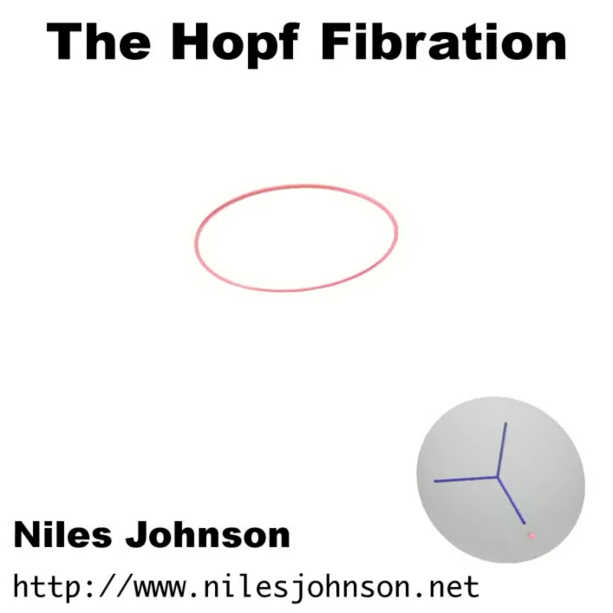
The flow is $\varphi_t(u,v) = (e^{it}u, e^{it}v)$.



Patrick Massot



Niles Johnson, $S^3/S^1 = S^2$


The Hopf Fibration
Niles Johnson
http://www.nilesjohnson.net

Existence of periodic orbits

The Weinstein Conjecture (1978)

Let Y be a closed oriented odd-dimensional manifold with a contact form λ. Then the associated Reeb vector field R_λ has a closed orbit.

- Weinstein (convex hypersurfaces)
- Rabinowitz (star shaped hypersurfaces)
- Star shaped is secretly contact!
- Viterbo, Hofer, Floer, Zehnder ('80's fun)
- Hofer (overtwisted, $\pi_2(Y) \neq 0$, or S^3)
- Taubes (dimension 3)

Tools > 1985: **Floer Theories** and **Gromov's** pseudoholomorphic curves.

Morse theory

Let $f \in C^\infty(M;\mathbb{R})$ be nondegenerate and g be a "reasonable" metric.
$\rightsquigarrow (f,g)$ is **Morse-Smale.**

$CM_* = \mathbb{Z}\langle \text{Crit}(f)\rangle$.
$* = \#\{\text{negative eigenvalues Hess}(f)\}$
∂^{Morse} counts $u \in \mathcal{M}_1(x,y)/\mathbb{R}$, flow lines of $-\nabla f$ between critical points

Theorem (Floer '80s, with technical conditions)

Floer $HF_*(M,\omega,H,J) \cong$ *Morse* $H_*(M,(H,\omega(\cdot,J\cdot))) \cong H_*(M;\mathbb{Q})$

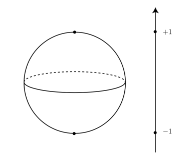
The Arnold Conjecture (Floer '80s...)

Let (M^{2n},ω) be compact symplectic and $H_t = H_{t+1} : M \to \mathbb{R}$ be a smooth time dependent nondegenerate 1-periodic Hamiltonian. Then

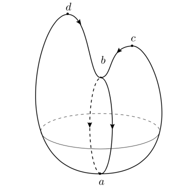
$$\#\{\text{1-periodic orbits of } X_{H_t}\} \geq \sum_{i=0}^{2n} \dim H_i(M;\mathbb{Q})$$

Analytic Necessities:
Transversality (for implicit function theorem $\Rightarrow \mathcal{M}_k(x,y)$ is a manifold)
Compactness (so ∂ is well defined, $\partial^2 = 0$, and invariance holds)

Recollections of spheres



$$C_*(S^2,(f,g)) = \begin{cases} \mathbb{Z}_2 & * = 0,\ 2 \\ 0 & \text{else} \end{cases} \qquad \partial \equiv 0$$



$$C_*(S^2,(f,g)) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & * = 2 \\ \mathbb{Z}_2 & * = 1 \\ \mathbb{Z}_2 & * = 0 \end{cases} \qquad \begin{aligned} \partial c = \partial d &= b \\ \partial b = 2a &= 0 \end{aligned}$$

Theorem (Reeb '46)

If there exists a Morse function on a compact connected M with only two critical points then M is homeomorphic to a sphere.

Theorem (Hutchings-Taubes 2008)

A closed contact 3-manifold admits ≥ 2 embedded Reeb orbits and if there are exactly two then Y is diffeomorphic to S^3 or a lens space.

Embedded contact homology (ECH)

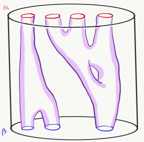
ECH is a gauge theory for (Y^3, λ) and $\Gamma \in H_1(Y;\mathbb{Z})$ due to Hutchings.

$ECC_*(Y, \lambda, \Gamma, J)$ is a $\mathbb{Z}_2$ vector space generated by **Reeb currents** $\alpha = \{(\alpha_i, m_i)\}$:

- α_i is an embedded Reeb orbit, $m_i \in \mathbb{Z}_{>0}$,
- if α_i is hyperbolic, $m_i = 1$,
- $\sum_i m_i[\alpha_i] = \Gamma$.

$*$ is given by the **ECH index**, a topological index defined via c_1, CZ, and relative self-intersection pairing, wrt $Z \in H_2(Y, \alpha, \beta)$.
Get a relative $\mathbb{Z}_d$-grading, d is divisibility of $c_1(\xi) + 2PD(\Gamma)$ in $H^2(Y;\mathbb{Z})$ mod torsion.

$\langle \partial^{ECH} \alpha, \beta \rangle$ counts **currents**, realized by unions of **holomorphic curves**



Partition writhe fun,
index inequality,
(yay for adjunction!)

-Hutchings' 02 Haiku

Dee squared is zero;
obstruction bundle gluing
is complicated.

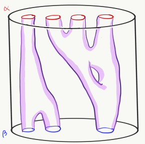
Hutchings-Taubes' 07 & 09 Haiku

Invariance of ECH

$ECC_*(Y, \lambda, \Gamma, J)$ is generated by **Reeb currents** $\alpha = \{(\alpha_i, m_i)\}$ over $\mathbb{Z}_2$

Grading is given by the **ECH index**, a topological index defined via c_1, CZ, and relative self-intersection pairing, wrt $Z \in H_2(Y, \alpha, \beta)$.

$\langle \partial^{ECH} \alpha, \beta \rangle$ counts **currents**, realized by unions of **holomorphic curves**



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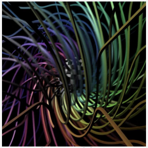


Jason Hise

Theorem (Taubes G&T (2010), no. 5, 2497-3000)

If Y is connected, there is a canonical isomorphism of relatively graded $\mathbb{Z}[U]-$modules

$$ECH_*(Y, \lambda, \Gamma, J) = \widehat{HM}^{-*}(Y, \mathfrak{s}_\xi + \text{PD}(\Gamma))$$

ECH is a topological invariant of Y !

(shift Γ when changing choice of ξ)

Prequantization bundles

Theorem (Boothby-Wang construction '58)

Let (Σ_g, ω) be a Riemann surface such that $\frac{[\omega]}{2\pi}$ admits an integral lift. Let $\mathfrak{p} : Y \to \Sigma_g$ be the principal S^1-bundle with Euler class $e = -\frac{[\omega]}{2\pi}$. Then there is a connection 1-form $-i\lambda$ on Y whose Reeb vector field R is tangent to the S^1-action.

- (Y, λ) is the **prequantization bundle** over (Σ_g, ω).
- The Reeb orbits of R are the S^1-fibers of this bundle.
- $d\lambda = \mathfrak{p}^*\omega$
- $\mathfrak{p}_*\xi = T\Sigma_g$
- The Reeb orbits of R are degenerate.

Use a Morse-Smale $H : \Sigma_g \to \mathbb{R}$, which is C^2 close to 1 to perturb λ. The perturbed Reeb vector field for $\lambda_\varepsilon := (1 + \varepsilon \mathfrak{p}^* H)\lambda$

$$R_\varepsilon = \frac{R}{1 + \varepsilon \mathfrak{p}^* H} + \frac{\varepsilon \tilde{X}_H}{(1 + \varepsilon \mathfrak{p}^* H)^2}$$

ECH from Morse H_*

Theorem (Nelson-Weiler '20, $\mathbb{Z}_2$-grading in Farris '11)

Let $(Y, \xi = \ker\lambda)$ be a prequantization bundle over (Σ_g, ω) of negative Euler class e. Then as $\mathbb{Z}_2$-graded $\mathbb{Z}_2$-modules,

$$\bigoplus_{\Gamma \in H_1(Y;\mathbb{Z})} ECH_*(Y, \xi, \Gamma) \cong \Lambda^* H_*(\Sigma_g; \mathbb{Z}_2).$$

There is an explicit upgrade to a (relatively) $\mathbb{Z}$-graded isomorphism.

Corollary (Nelson-Weiler '20)

For $*$ sufficiently large and $g > 0$, the groups $ECH_*(Y, \xi, \Gamma)$ are isomorphic to $\mathbb{Z}_2^{f(g)}$, where $f(g) = 2^{2g-1}$.

1. Critical points of a perfect H form a basis for $H_*(\Sigma_g; \mathbb{Z}_2)$.
   Generators of ECC are $e_-^{m_-} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^{m_+}$ where $m_i = 0, 1$.
   $\rightsquigarrow$ basis for $\Lambda^* H_*(\Sigma_g; \mathbb{Z}_2)$
2. ∂^{ECH} only counts cylinders corresponding to Morse flows on Σ_g;
   $\partial^{ECH}(e_-^{m_-} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^{m_+})$ is sum of ways to apply ∂^{Morse} to h_i or e_+.

More details

Theorem (Nelson-Weiler '20)

Let $(Y, \xi = \ker\lambda)$ *be a prequantization bundle over* (Σ_g, ω) *of negative Euler class* e*. Each* $\Gamma \in H_1(Y;\mathbb{Z})$ *satisfying* $ECH_*(Y,\xi,\Gamma) \neq 0$ *corresponds to a number in* $\{0, \dots, -e-1\}$*,*

$$ECH_*(Y,\xi,\Gamma) \cong \bigoplus_{d\in\mathbb{Z}_{\geq 0}} \Lambda^{\Gamma+(-e)d} H_*(\Sigma_g;\mathbb{Z}_2), \qquad d = \frac{M-N}{|e|}$$

$$|\alpha|_* - |\beta|_* = -e(d_\alpha^2 - d_\beta^2) + (\chi(\Sigma_g) + 2\Gamma + e)(d_\alpha - d_\beta) + |\alpha|_\bullet - |\beta|_\bullet$$

$$I(\alpha,\beta) = \chi(\Sigma_g)d - d^2 e + 2dN + m_+ - m_- - n_+ + n_-$$

$$c_\tau(\alpha,\beta) + Q_\tau(\alpha,\beta) \quad + CZ_\tau^I(\alpha) - CZ_\tau^I(\beta), \; CZ_\tau^I(\gamma) = \sum_i \sum_{k=1}^{\ell_i} CZ_\tau(\gamma_i^k)$$

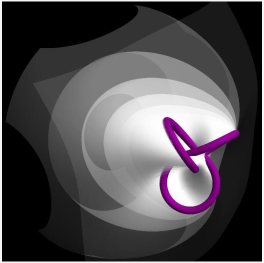
1. There exists $\varepsilon > 0$ so that the generators of $ECC_*^L(Y, \lambda_\varepsilon, J)$ are $e_-^{m_-} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^{m_+}$, e.g. orbits which are fibers over critical points.
2. $\partial^{ECH,L}$ only counts cylinders over Morse flow lines in Σ_g.
3. Finish with a direct limit argument, sending $\varepsilon \to 0$ and $L \to \infty$, by way of the action filtered isomorphism with Seiberg-Witten.

Open book decomposition of (S^3, ξ_{std}) along $T(p, q)$

Definition

An **open book decomposition** of Y^3 is a pair (B, π) where,

- B is an oriented link in Y, aka the **binding**;
- $\pi : Y \setminus B \to S^1$ is a **fibration** of the complement of B such that $\pi^{-1}(\theta) = \mathring{\Sigma}_\theta,\ \partial\Sigma_\theta = B$ for all $\theta \in S^1,\ \Sigma \cong \Sigma_\theta$ is the **page**.
- The **monodromy** ϕ is the self diffeo of the page.



(Henry Blanchette)

The right handed torus knot is the binding of an open book decomposition of (S^3, ξ_{std})

$$T(p, q) = \left\{(z_1, z_2) \in S^3 \mid z_1^p + z_2^q = 0\right\},$$

with the Milnor fibration projection map

$$\pi : S^3 \setminus T(p, q) \to S^1, \quad (z_1, z_2) \mapsto \tfrac{z_1^p + z_2^q}{|z_1^p + z_2^q|}.$$

The page Σ is a surface of genus $\frac{(p-1)(q-1)}{2}$.
The monodromy ϕ is pq-periodic.

http://people.reed.edu/~ormsbyk/projectproject/posts/milnor-fibrations.html

Reeb current generators

The open book of (S^3, ξ_{std}) along $T(p,q)$ is strictly contactomorphic to

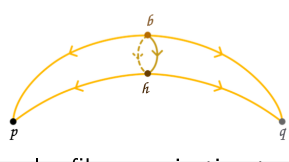
- certain Seifert fiber spaces with $e = -\frac{1}{pq}$ (Lisca-Matic, Colin-Honda)
- the S^1-orbibundle $\mathfrak{p} : S^3 \to \mathbb{CP}^1_{p,q}$ (Kegel-Lange, Dan CG–Mazzuchelli)

$$\lambda_{p,q} = \frac{\lambda_0}{p|z_1|^2+q|z_2|^2} \quad \rightsquigarrow \quad \text{Reeb VF is tangent to the fibers}$$

- Perturb using orbifold Morse function $H_{p,q}$ on $\mathbb{CP}^1_{p,q}$

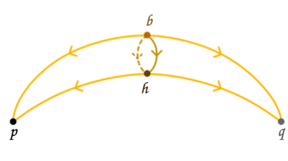
$$\lambda_{p,q,\varepsilon} := (1+\varepsilon \mathfrak{p}^* H_{p,q})\lambda_{p,q}$$

$$R_{p,q,\varepsilon} = \frac{R}{1+\varepsilon\mathfrak{p}^* H_{p,q}} + \frac{\varepsilon \tilde{X}_{H_{p,q}}}{(1+\varepsilon\mathfrak{p}^* H_{p,q})^2}$$



- **p** and **q** are the singular fibers projecting to minima at the orbifold points of isotropy $\mathbb{Z}/p$ and $\mathbb{Z}/q$. (elliptic)
- The binding **b** is a regular fiber projecting to max. (elliptic)
- **h** is a regular fiber projecting to saddle. (positive hyperbolic)

The chain complex



- **p** and **q** are the singular fibers projecting to minima at the orbifold points of isotropy $\mathbb{Z}/p$ and $\mathbb{Z}/q$. (elliptic)
- binding **b** is a regular fiber projecting to max. (elliptic)
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Two nontrivial cylinders with boundary $\mathbf{h} - \mathbf{p}^p$ and $\mathbf{h} - \mathbf{q}^q$.
Two cylinders with boundary $\mathbf{b} - \mathbf{h}$, which cancel.

Intersection theory & more:
∂^{ECH} only counts unions of cylinders,
which are lifts of orbifold Morse flow lines:

$$\langle \partial \mathbf{h}\alpha, \mathbf{q}^q\gamma \rangle = \langle \partial \mathbf{h}\alpha, \mathbf{p}^{\mathbf{p}}\alpha \rangle = 1, \qquad \langle \partial \mathbf{b}\gamma, \mathbf{h}\gamma \rangle = 0.$$

$$\lim_{\varepsilon \to 0} ECH_*^{L(\varepsilon)}(S^3, \lambda_{p,q,\varepsilon}, J) = ECH_*(S^3, \xi_{std}) = \begin{cases} \mathbb{Z}/2 & \text{if } * \in 2\mathbb{Z}_{\geq 0} \\ 0 & \text{else,} \end{cases}$$

Knot filtered ECH

- Realizes the relationship between action and linking of orbits.
- kECH is a topological spectral invariant (Hutchings '16)
- $\mathcal{F}_b(b^B\alpha) := B\ \mathrm{rot}(b) + \ell(\alpha, b)$
- $\mathcal{F}_{\mathbf{b}}(\mathbf{b}^B\mathbf{h}^H\mathbf{q}^Q\mathbf{p}^P) =$ *weighted algebraic multiplicity of fibers* $+B\delta_L$

$$= pq\int_{\mathbf{b}^B\mathbf{h}^H\mathbf{q}^Q\mathbf{p}^P}\lambda_{p,q} \quad + B\delta_L$$

The *degree* of any generator of $ECH_{2k}^{L(\varepsilon)}(S^3, \lambda_{p,q,\varepsilon})$ is $N_k(p,q)$,

$c_k(S^3, \lambda_{p,q}) = N_k(\frac{1}{p}, \frac{1}{q})$, the inf of the action any generator with $I = 2k$

Theorem (Nelson-Weiler '23)

Let $\mathbf{b}$ *be the standard transverse positive* $T(p,q)$ *in* (S^3, ξ_{std}) *with rotation number* pq *(aka maximal self-linking number, invoke Etnyre '99). Then*

$$ECH_{2k}^{\mathcal{F}_{\mathbf{b}}\leq K}(S^3, \xi_{std}, \mathbf{b}, pq) = \begin{cases} \mathbb{Z}/2 & K \geq N_k(p,q) = \{mp + nq \mid m, n \in \mathbb{Z}_{\geq 0}\}_k \\ 0 & \text{otherwise} \end{cases}$$

and $ECH_*^{\mathcal{F}_{\mathbf{b}}\leq K} = 0$ *in all other gradings* $*$.

Applications

Theorem (Nelson-Weiler '23)

Let **b** *be the standard transverse positive* $T(p,q)$ *in* (S^3, ξ_{std}) *with rotation number* pq *(aka maximal self-linking number, invoke Etnyre '99). Then*

$$ECH_{2k}^{\mathcal{F}_{\mathbf{b}} \leq K}(S^3, \xi_{std}, \mathbf{b}, pq) = \begin{cases} \mathbb{Z}/2 & K \geq N_k(p,q) = \{mp + nq \mid m, n \in \mathbb{Z}_{\geq 0}\}_k \\ 0 & \text{otherwise} \end{cases}$$

and $ECH_*^{\mathcal{F}_{\mathbf{b}} \leq K} = 0$ *in all other gradings* $*$.

Corollary

Let $pq \geq p'q'$. *If there is a symplectic cobordism from* $T(p,q)$ *to* $T(p',q')$ *in* $\mathbb{R} \times S^3$ *then* $N_k(p,q) \geq N_k(p',q')$ *for all* k.

Corollary

kECH + ECH Weyl Law $\Rightarrow$ *quantitative existence of Reeb orbits.*

Quantitative bounds on arbitrary Reeb currents

Corollary (NW '23 + ECH Weyl law by Cristofaro-Gardiner–Hutchings–Ramos '15)

Let λ *be a contact form on* (S^3, ξ_{std}) *whose Reeb VF admits the positive* $T(p,q)$ *torus knot as an elliptic Reeb orbit with symplectic action 1 and rotation number* $pq + \Delta$, *where* Δ *is a positive irrational number. If* $Vol(\lambda) < \frac{pq}{(pq+\Delta)^2}$ *then*

$$\inf \left\{ \frac{\text{action}(\gamma)}{\text{linking of } \gamma \text{ with } T(p,q)} \right\} \leq \sqrt{\frac{\text{Vol}(\lambda)}{pq}}.$$

This result implies existence of periodic orbits and mean action bounds in terms of the Calabi invariant for surface dynamics.

Generalizes Hutchings '16 for $\mathbb{D}$ maps; Weiler '18 for $\mathbb{A}$ maps.

Results for C^∞ generic Hamiltonians by Pirnapasov-Prasad '22.

Applications to surface dynamics and Calabi

Study symplectomorphisms $\psi : (\mathring{\Sigma}_g, d\eta) \circlearrowleft$, $\partial\mathring{\Sigma}_g = T(p,q)$ such that

ψ is freely isotopic to the right handed pq-periodic rep of Mod $\left(\mathring{\Sigma}_g\right)$ and

isotopic rel $\partial\mathring{\Sigma}_g$ to this rep twisted positively near ∂ by $-\frac{1}{pq} < d \leq 0$,

The **action function** f of ψ wrt η measures the ψ distortion of curves; it's defined by $df = \psi^*\eta - \eta$ and $f = \frac{1}{pq} + d$.

The **Calabi invariant** of ψ is the average of the action function:

$$\mathrm{Cal}_\eta(\psi) := \int_{\mathring{\Sigma}_g} f d\eta$$

Theorem (NW '23)

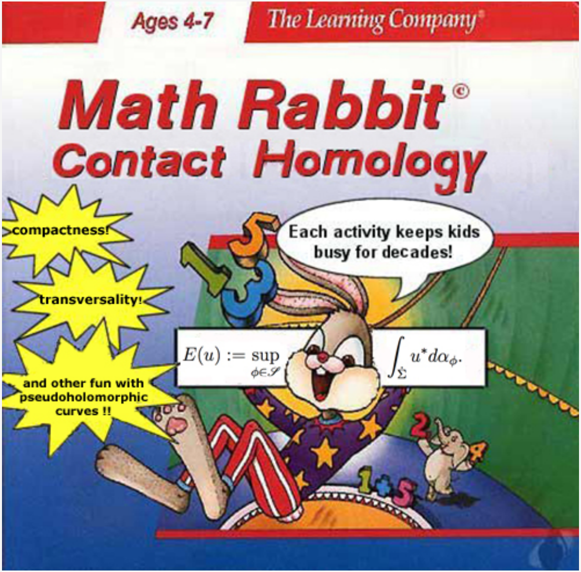
Given any such ψ, if $f > 0$ and $Cal(\psi) < pq \cdot \theta_0^2$, where $\theta_0 = \frac{1}{pq} + d$, then

$$inf \left\{ \frac{Action(\gamma)}{Period(\gamma)} \,\middle|\, \gamma \text{ is a periodic orbit of } \psi \right\} < \sqrt{\frac{\mathrm{Cal}(\psi)}{pq}}.$$

A **periodic orbit** of ψ is a tuple of points $(x_1, ..x_\ell)$ s.t. $\psi(x_i) = x_{i+1}$ mod ℓ.

$$\mathrm{Action}(\gamma) := \sum_{i=1}^{\ell} f(\gamma_i), \quad \mathrm{Period}(\gamma) := \ell.$$

Thanks!


Ages 4-7
The Learning Company
Math Rabbit©
Contact Homology
compactness!
transversality!
and other fun with pseudoholomorphic curves !!
Each activity keeps kids busy for decades!
$E(u) := \sup_{\phi \in \mathscr{S}} \int_{\dot{\Sigma}} u^* d\alpha_\phi.$

Visualizations

Hopf fibration: https://nilesjohnson.net/hopf.html

Spinors exhibit a sign-reversal that depends on the homotopy class of the continuous rotation of the coordinate system between some initial and final configuration in contrast to vectors and other tensors. https://en.wikipedia.org/wiki/Spinor

In the limit, a piece of solid continuous space can rotate in place like this without tearing or intersecting itself.
(A more extreme example of the **belt trick**.)
https://www.youtube.com/watch?v=LLw3BaliDUQ

Milnor fibrations of torus knots (& **open book decompositions**)

http://people.reed.edu/~ormsbyk/projectproject/posts/milnor-fibrations.html

https://www.unf.edu/~ddreibel/research/milnor/milnor.html

https://sketchesoftopology.wordpress.com/2012/08/24/bowman/