Algebraic Topology

Very generally *topology* is the study of spaces on which you can discuss

1) continuity of functions and
2) convergence of sequences

we give general definitions later in the course but two main objects of study in topology are *manifolds* and *CW-complexes*

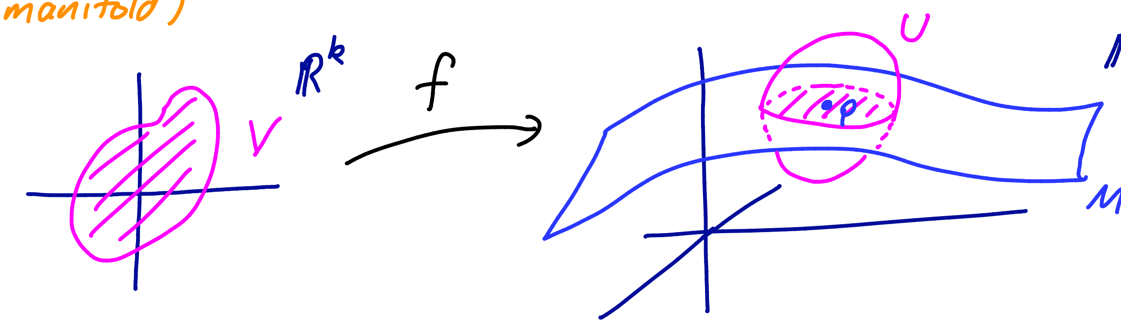
these show up all over math and science as configuration spaces, models of the universe, solution spaces to equations,...

we focus on manifolds for now (requires less background)

a *k-manifold* (or *manifold of dimension k*) M is a subset of $\mathbb{R}^n$ such that for each point $p \in M$ there is

1) an open set U in $\mathbb{R}^n$ containing p
2) an open set V in $\mathbb{R}^k$, and
3) a continuous function $f: V \to U$ such that
   a) f is injective
   b) $\operatorname{im} f = M \cap U$
   c) $(f|_{\operatorname{im} f})^{-1}: M \cap U \to V$ is continuous

(if f is differentiable and $\operatorname{rank}(Df_x) = k$, $\forall x \in V$ then M is a *smooth* manifold)

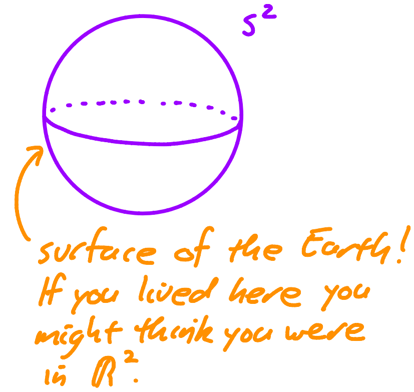


we call f a *coordinate chart* or *local parameterization*

Intuitively M is "locally Euclidean", that is if *you* "lived" in M

then you would probably think you were in $\mathbb{R}^k$

(if you were really small compared to M or had really bad eye sight
eg. surface of Earth not $\mathbb{R}^2$)



examples:

1) $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$

is a 2-manifold

(2-manifolds are also called surfaces)

local parameterizations are of the form

$$(x,y) \longmapsto (x, y, \sqrt{1-x^2-y^2})$$

for $(x,y) \in \{x^2+y^2 \le 1\} \subseteq \mathbb{R}^2$ (need 4 more such charts, what are they?)

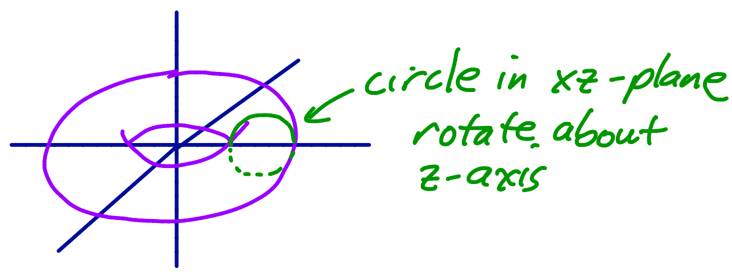
2) $S^k = \{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} : \sum_{i=0}^{k} x_i^2 = 1\}$

is a k-manifold.

3) Torus: image of the map

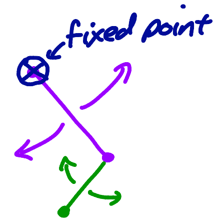
$$f(x,y) = ((3+\cos x)\cos y, (3+\cos x)\sin y, \sin x)$$

is a surface

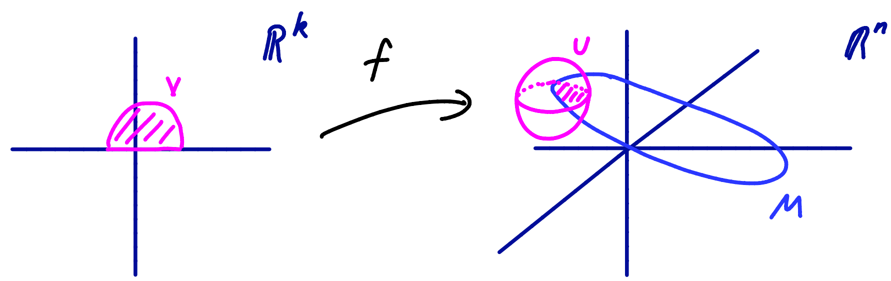


note: T^2 is a configuration space

consider the "double pendulum"



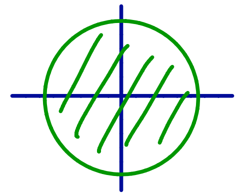
M is called a k-manifold with boundary if we have f, U, V as above except V is an open set in $\mathbb{R}^k_{\ge 0} = \{(x_1, \dots x_k) : x_k \ge 0\}$



examples:

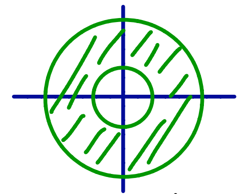
1) $D^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_1^2 + \dots + x_k^2 \le 1\}$

k-disk



2) annulus

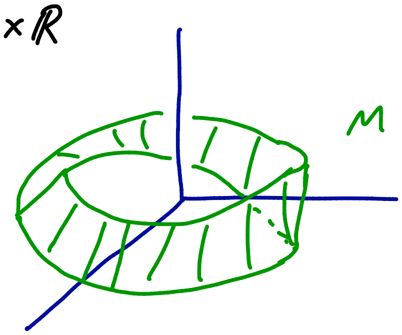
$A = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 2\}$



3) Möbius band

image of $f(x,y) = (\underbrace{2 + x\cos y}_{r}, \underbrace{2y}_{\theta}, \underbrace{x \sin y}_{z})$ cylindrical coordinates

for $(x,y) \in [-1,1] \times \mathbb{R}$



Two manifolds M and N are called homeomorphic if there is a continuous bijection $f: M \to N$ such that $f^{-1}: N \to M$ is also continuous

if two manifolds are homeomorphic then we think of them as being the same.

example:

$S^2 \subset \mathbb{R}^3$ the unit sphere and

$S_r^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\}$ $r > 0$

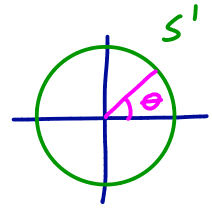
are homeomorphic (what is the map?)

from the topological point of view they are the same (of course "geometrically" they are different, e.g. area)

An embedding of one manifold into another is a continuous injective function $f: M \to N$ that is a homeomorphism onto its image.

examples:

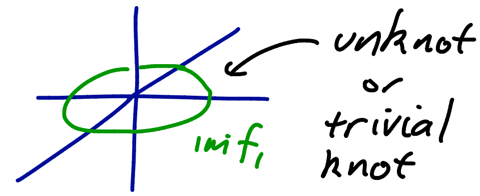
1) S^1 = unit circle in $\mathbb{R}^2$



inclusion $i: S^1 \to \mathbb{R}^2$

is an embedding

2) $f_1: S^1 \to \mathbb{R}^3 : \theta \mapsto (\cos\theta, \sin\theta, 0)$



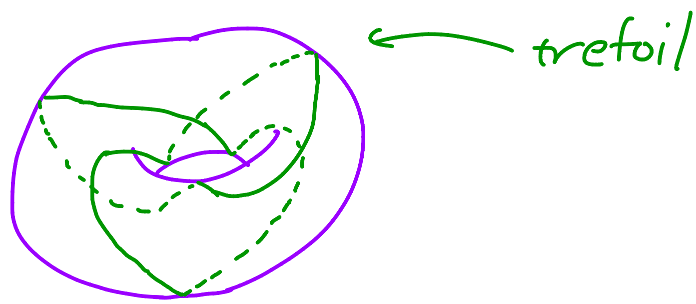
an embedding of S^1 into $\mathbb{R}^3$

is called a <u>knot</u>

think of it as a piece of string with ends glued together

3) $f_2: S^1 \to \mathbb{R}^3$:

$\theta \mapsto (\cos 3\theta(3+\cos 2\theta), \sin 3\theta(3+\cos 2\theta), \sin 2\theta)$



Main Problems: (same as in other areas of math)

1) list or show how to build all manifolds
2) find ways to distinguish manifolds

$\Big\}$ <u>classify</u>

3) study maps between manifolds

usually restrict to special maps

examples: • homeomorphisms and
• embeddings

(again we want to <u>construct</u> them and <u>distinguish</u> them)

(4) study "structures" on manifolds

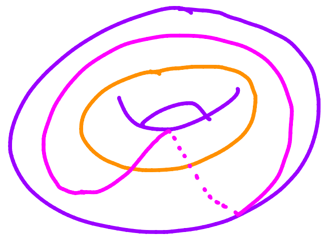
e.g. Riemannian geometry, complex geometry, contact/symplectic geometry, ...)

different course

There are many surprising relations between all these problems

examples:

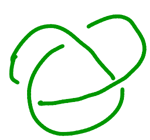
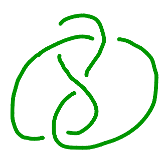
1) use embeddings of curves in surfaces

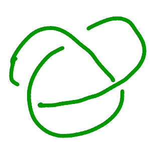


to understand homeomorphisms of surfaces and to distinguish and build surfaces

2) embeddings of S^1 in S^3 (or $\mathbb{R}^3$) can be used to construct 3 and 4-manifolds

the study of such embeddings is called knot theory and is very interesting on its own

e.g. are  and  the "same"?

what about  and  ?

We will study these problems using algebraic techniques

i.e. Algebraic topology (in a very general sense)

The idea is to build a function

$$\left\{ \begin{array}{l} \text{something you} \\ \text{want to study} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} \text{something algebraic that} \\ \text{is hopefully easier to study} \end{array} \right\}$$

e.g. {all manifolds} or $\left\{ \begin{array}{c} \text{all embeddings} \\ S^1 \hookrightarrow \mathbb{R}^3 \end{array} \right\}$ or ...

$\mathbb{Z}$ or

set of groups or

set of vector spaces or

set of polynomials or ...

being a function, if two manifolds/embeddings... are sent to to different algebraic things then they are different!

we call such a function an <u>algebraic invariant</u>

It would be even better if the invariant "reflected" properties of the topological objects

some examples we will study

$$\begin{Bmatrix}\text{topological}\\ \text{spaces}\end{Bmatrix} \xRightarrow[\text{fundamental group}]{} \{\text{groups}\}$$

$$X \longmapsto \pi_1(X)$$

we will see: 1) very good invariant of surfaces and knots

2) studying homeomorphisms of surfaces is essentially the same as studying isomorphisms of the fundamental group

(there are some partial generalizations of this to higher dimensions)

3) can use topology to learn things about groups! (this is called "geometric group theory")

$$\{\text{knots}\} \xRightarrow[\text{Jones}]{\text{Alexander}} \{\text{polynomials}\}$$

$$\{\text{knots}\} \xRightarrow[\text{Khovanov homology}]{} \{\text{vector spaces}\}$$

we (hopefully) will see

1) How the Khovanov vector spaces are related to the Jones polynomial
2) How the fundamental group is related to the Alexander polynomial
3) What these tell us about knots

The main parts of this course will be

I. Intro. to general topology including the classification of surfaces using "surgery theory"

II. Brief intro. to groups and group presentations

III. Fundamental group and homotopy theory

IV. Covering spaces

but before we really get started, let's see some specific examples to illustrate the above themes

we will do this through knot theory, much of the first part of this is <u>very</u> "simple" and could be told to highschool students, but later we will see deap connections to algebraic topology!