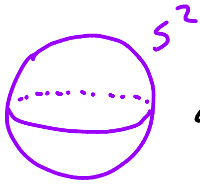
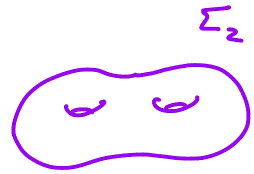
V The Fundamental Group

intuitively the difference between



and

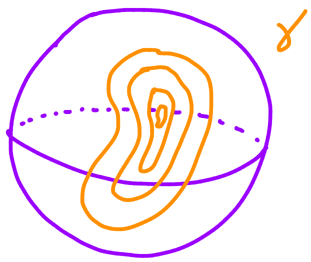
and



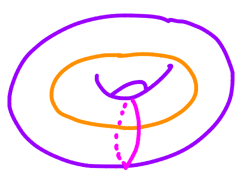
is the "number of holes"

How to make this precise?

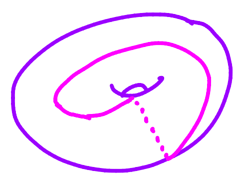
<u>note</u>: if γ is any loop in S^2 then it looks like it can be shrunk to a point



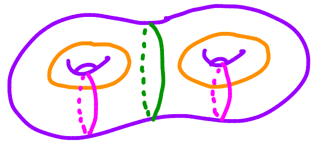
but there are loops in T^2 that can't be shrunk



and others



and looks like even more in Σ_2



we want to make this precise

The idea is to "probe the topology of a space with loops mapped into the space"

<u>Remark</u>: you might want to think about "probing the topology" with things beside loops!

A. Definition of the fundamental group

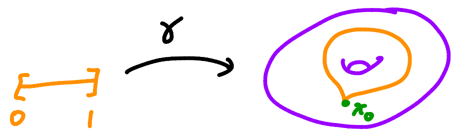
let X be a topological space

fix a point $x_0 \in X$ (call x_0 the base point)

a loop in X based at x_0 is a continuous map

$$\gamma : [0,1] \to X$$

such that $\gamma(0) = \gamma(1) = x_0$



exercise: This is the same as a continuous map

$$\tilde{\gamma} : S^1 \to X \quad \text{with} \quad \gamma((1,0)) = x_0$$

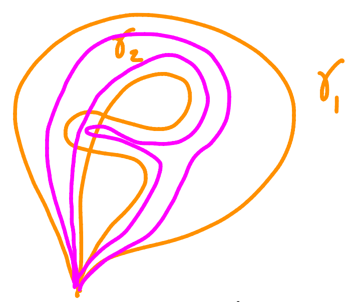
unit circle in $\mathbb{R}^2$

two loops a homotopic, denoted $\gamma_1 \sim \gamma_2$, if there is a continuous map

$$H : \underset{s}{[0,1]} \times \underset{t}{[0,1]} \longrightarrow X$$

such that

1) $H(s,0) = \gamma_1(s)$
2) $H(s,1) = \gamma_2(s)$
3) $H(0,t) = H(1,t) = x_0$



note: H gives a "continuous family of loops from γ_1 to γ_2"

e.g. $H_t(s) = H(t,s)$ is a loop for fixed t

lemma 1:

$$\boxed{\text{homotopy is an equivalence relation on loops based at } x_0}$$

Proof: (reflexive) clearly $\gamma \sim \gamma$

just take $H(s,t) = \gamma(s) \quad \forall\, s$ and t

(symmetric) if $\gamma_1 \sim \gamma_2$ by $H(s,t)$ then let

$$\tilde{H}(s,t) = H(s, 1-t)$$

so $\tilde{H}(s,0) = H(s,1) = \gamma_2(s)$

$\tilde{H}(s,1) = H(s,0) = \gamma_1(s)$

$\tilde{H}(0,t) = \tilde{H}(1,t) = x_0$

and we see $\gamma_2 \sim \gamma_1$

(transitive) if $\gamma_1 \sim \gamma_2$ by $H(s,t)$ and $\gamma_2 \sim \gamma_3$ by $G(s,t)$

then set

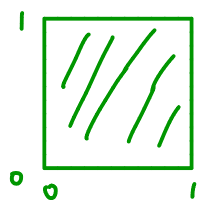
$$\tilde{H}(s,t) = \begin{cases} H(s,2t) & 0 \le t \le 1/2 \\ G(s,2t-1) & 1/2 \le t \le 1 \end{cases}$$

(continuous by Th^{m} II.9)

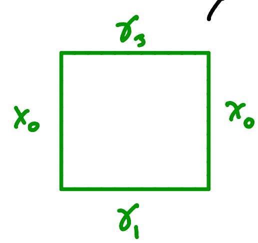
easily check $\tilde{H}$ gives $\gamma_1 \sim \gamma_3$

here is a "picture proof" that gives the idea how to get the formula above

to define a homotopy we need to define a function with domain

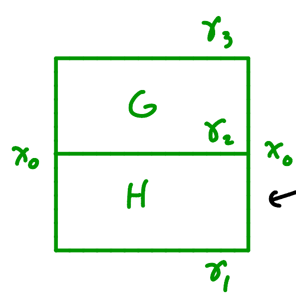


we know on the boundary we need



we just need to say what it is on the interior
we do that by

"this will "simplify" arguments later!"



where this means H with t variable scaled similarly for G set $\boxed{\pi_1(X,x_0) = \{\text{homotopy classes of loops in } X \text{ based at } x_0\}}$

we claim this is a group!
multiplication is just path concatenation

$$(\gamma_1 * \gamma_2)(s) = \begin{cases} \gamma_1(2s) & 0 \le s \le 1/2 \\ \gamma_2(2s-1) & 1/2 \le s \le 1 \end{cases}$$

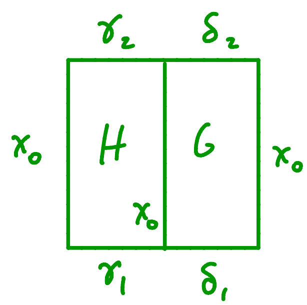
this is well-defined and continuous since

$$\gamma(2(1/2)) = \gamma_1(1) = \gamma_2(0) = \gamma_2(2(1/2)-1)$$

well $*$ is well defined on loops but what about homotopy classes of loops?

suppose $\gamma_1 \sim \gamma_2$ by H and $\delta_1 \sim \delta_2$ by G

then $\gamma_1 * \delta_1 \sim \gamma_2 * \delta_2$ by



that is

$$\tilde{H}(s,t) = \begin{cases} H(2s,t) & 0 \le s \le 1/2 \\ G(2s-1,t) & 1/2 \le s \le 1 \end{cases}$$

so $[\gamma_1]*[\delta_1] = [\gamma_1 * \delta_1]$ is well-defined on $\pi_1(X,x_0)$

Thm 2:

> X a topological space, $x_0 \in X$
> Then $\pi_1(X,x_0)$ is a group under $*$

we call $\pi_1(X,x_0)$ the <u>fundamental group of X (based at x_0)</u>

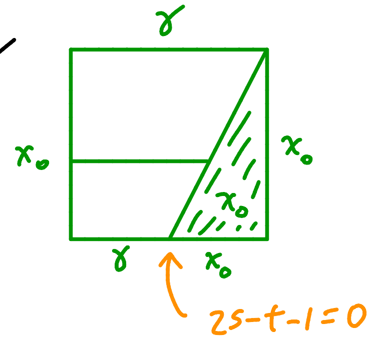
Proof: let $e : [0,1] \to X : s \mapsto x_0$ be the constant loop

<u>Claim</u>: $[e]$ is the identity element

<u>Pf</u>: let $\gamma : [0,1] \to X$ be any loop

then $$\gamma * e : [0,1] \to X : s \mapsto \begin{cases} \gamma(2s) & 0 \le s \le \frac{1}{2} \\ x_0 & \frac{1}{2} \le s \le 1 \end{cases}$$

now



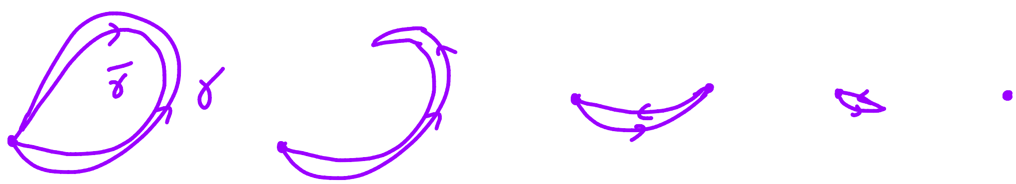
so $H(s,t) = \begin{cases} \gamma\left(\frac{2s}{1+t}\right) & 0 \le s \le \frac{1+t}{2} \\ x_0 & \frac{1+t}{2} \le s \le 1 \end{cases}$

is a homotopy $\gamma * e \sim \gamma$

(similarly $\gamma * e \sim \gamma$)

<u>Claim</u>: $\gamma : [0,1] \to X$ has inverse $\bar{\gamma}(s) = \gamma(1-s)$

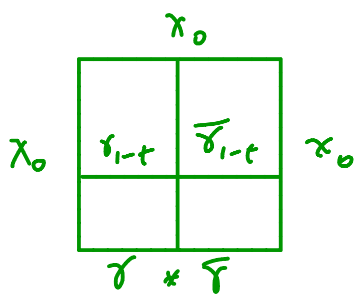
<u>Pf</u>:



Homotopy $\gamma * \bar{\gamma}$ to e

let $\gamma_t = [0,1] \to X : s \mapsto \gamma(ts)$

(only go along γ to $\gamma(t)$)



so

$$H(s,t) = \begin{cases} \gamma((1-t)2s) & 0 \le s \le \frac{1}{2} \\ \gamma(1-t(2-2s)) & \frac{1}{2} \le s \le 1 \end{cases}$$

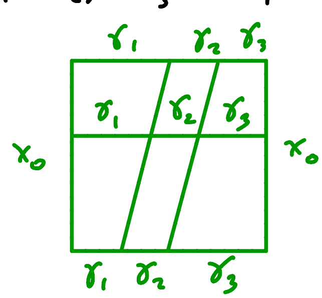
is a homotopy $\gamma * \bar{\gamma}$ to e

similarly $\bar{\gamma} * \gamma \sim e$

<u>Claim</u>: multiplication is associative

<u>Pf</u>: given loops $\gamma_1, \gamma_2, \gamma_3$ need to see

$$(\gamma_1 * \gamma_2) * \gamma_3 \sim \gamma_1 * (\gamma_2 * \gamma_3)$$



<u>exercise</u>: write down homotopy

let $f: X \to Y$ be a continuous map

$x_0 \in X$ any $y_0 = f(x_0)$

given $\gamma: [0,1] \to X$ a loop based at x_0

then $f \circ \gamma: [0,1] \to Y$ is a loop based at y_0

if $\gamma \sim \tilde{\gamma}$ by a homotopy $H(s,t)$ then $f \circ H: [0,1] \times [0,1] \to Y$ is a homotopy $f \circ \gamma$ to $f \circ \tilde{\gamma}$

so for each $[\gamma] \in \pi_1(X, x_0)$ we get a well-defined $[f \circ \gamma] \in \pi_1(Y, y_0)$

we define
$$f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$$
$$[\gamma] \longmapsto [f \circ \gamma]$$

Thm 3: $\boxed{f_* \text{ is a homomorphism}}$

Proof: $[\gamma_1], [\gamma_2] \in \pi_1(X, x_0)$

$$\gamma_1 * \gamma_2 (s) = \begin{cases} \gamma_1(2s) & 0 \le s \le 1/2 \\ \gamma_2(2s-1) & 1/2 \le s \le 1 \end{cases}$$

and
$$(f \circ \gamma_1) * (f \circ \gamma_2) = \begin{cases} f \circ \gamma_1(2s) & 0 \le s \le 1/2 \\ f \circ \gamma_2(2s-1) & 1/2 \le s \le 1 \end{cases}$$

so $f \circ (\gamma_1 * \gamma_2) = (f \circ \gamma_1) * (f \circ \gamma_2)$

i.e. $f_*([\gamma_1] * [\gamma_2]) = f_*([\gamma_1]) * f_*([\gamma_2])$ ▦

exercise: 1) $id_X: X \to X$ the identity, then $(id_X)_*: \pi_1(X, x_0) \to \pi_1(X, x_0)$ the identity

2) if $f: X \to Y$ and $g: Y \to Z$, then $(g \circ f)_* = g_* \circ f_*$

two maps $f, g: X \to Y$ are called homotopic if $\exists$ a continuous function
$$H: X \times [0,1] \to Y$$
such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$

note: $H_t: X \to Y: x \mapsto H(x,t)$ is a continuous family of maps interpolating between f and g

so maps are homotopic if there is a "continuous deformation" between them

we say f and g are homotopic rel. base point if all H_t take x_0 to y_0

exercise: if $f \simeq g$ rel. base point, then $f_* = g_*$

two spaces X and Y are homotopy equivalent if there are continuous functions

$$f: X \to Y \text{ and } g: Y \to X$$

such that

$f \circ g: Y \to Y$ is homotopic to the identity on Y, and

$g \circ f: X \to X$ " " " " X

f is called a homotopy equivalence and g its homotopy inverse

denote this by $X \simeq Y$ or $X \simeq_f Y$

if the homotopies in the definition preserve the base point, then we say X and Y are based homotopy equivalent

lemma 4:

If $f: X \to Y$ is a based homotopy equivalence, then

$$f_*: \pi_1(X, x_0) \to \pi_1(Y, \underset{\overset{\shortparallel}{y_0}}{f(x_0)})$$

is an isomorphism

Proof: let g be the homotopy inverse of f

so $f \circ g \sim id_Y \quad \therefore \quad f_* \circ g_* = (f \circ g)_* = (id_Y)_*: \pi_1(Y, y_0) \to \pi_1(Y, y_0)$

$$\begin{tikzcd} \pi_1(Y, y_0) \arrow[r, "g_*"] \arrow[rr, bend right, "g_* \circ f_* = (id_Y)_* = id_{\pi_1(Y, y_0)}"'] & \pi_1(X, x_0) \arrow[r, "f_*"] & \pi_1(Y, y_0) \end{tikzcd}$$

bijective

and we f_* is surjective (and g_* injective)

similarly $f_* \circ g_* = id_{\pi_1(X, x_0)}$ so f_* injective

i.e. f_* an isomorphism ⊞

examples:

1) $X = D^n \quad Y = \{x_0\} \quad x_0 = \text{origin in } D^n$

Claim: $Y \simeq Y$ (based)

Proof: $f: X \to Y: x \to x_0$

$g: Y \to X: x_0 \to x_0$

so $f \circ g : Y \to Y$ is identity on Y

$g \circ f : X \to X : x \mapsto x_0$

and $F_t(x) = tx$ is a homotopy from $g \circ f$ to id_X ✓

so $\pi_1(D^n, x_0) \cong \pi_1(\{x_0\}, x_0)$

<u>note</u>: there is <u>exactly</u> one function

$$[0,1] \longrightarrow \{x_0\}$$

so $\pi_1(\{x_0\}, x_0) = \{e\}$ the trivial group

first computation!

$\therefore \pi_1(D^n, x_0) = \{e\}$

2) if $f : X \to Y$ is a homeomorphism

then it is a homotopy equivalence, since $f \circ f^{-1} = id_Y$, $f^{-1} \circ f = id_X$

$\therefore$ <u>lemma 5</u>:

homeomorphic spaces are (based) homotopy equivalent (with correct choice of base points) and hence have the same fundamental group

<u>note</u>: homotopy equivalent $\not\Rightarrow$ homeomorphic (e.g. D^n and point)

3) $A = S^1 \times [0,1]$ and $B = S^1$

<u>Claim</u>: $A \simeq S^1$

<u>Proof</u>: $f : S^1 \times [0,1] \to S^1 : (x,y) \longmapsto x$

$g : S^1 \longrightarrow S^1 \times [0,1] : x \longmapsto (x, 0)$

$f \circ g : S^1 \to S^1 : x \longmapsto x$ so $f \circ g = id_{S^1}$

$g \circ f : S^1 \times [0,1] \to S^1 \times [0,1] : (x,y) \longmapsto (x, 0)$

<u>note</u>: $F_t(x,t) = (x, ty)$ is a homotopy $g \circ f$ to id_A ✓

so $\pi_1(S^1 \times [0,1], x_0) \cong \pi_1(S^1, x_0)$

we compute π_1 of S^1 soon.

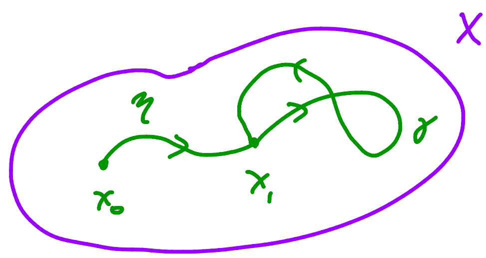
How does π_1 depend on x_0?

let $x_0, x_1 \in X$ and η be a path x_0 to x_1

given a loop $\gamma : [0,1] \to X$ based at x_1, then

$$\eta * \gamma * \bar{\eta}\,(s) = \begin{cases} \eta(3t) & 0 \le t \le 1/3 \\ \gamma(3t-1) & 1/3 \le t \le 2/3 \\ \eta(1-(3t-2)) & 2/3 \le t \le 1 \end{cases}$$

is a loop based at x_0



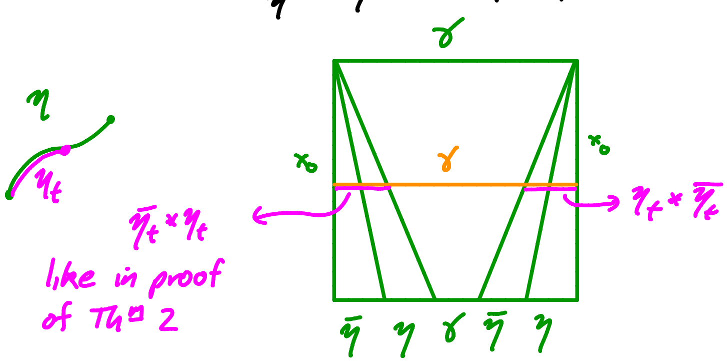
exercise: the map $\Phi_\eta : \pi_1(X, x_1) \to \pi_1(X, x_0)$, $[\gamma] \longmapsto [\eta * \gamma * \bar{\eta}]$ is a well-defined homomorphism.

Th^m 6: $\boxed{\Phi_\eta : \pi_1(X, x_1) \to \pi_1(X, x_0) \text{ is an isomorphism}}$

Remark: So isomorphism class of $\pi_1(X, x_0)$ does <u>not</u> depend on choice of x_0 in a path component of X

Proof: <u>note</u> $\Phi_{\bar{\eta}}$ is the inverse of Φ_η

$$\Phi_{\bar{\eta}}(\Phi_\eta(\gamma)) = \bar{\eta} * \eta * \gamma * \bar{\eta} * \eta \sim \gamma$$



so $\Phi_{\bar{\eta}} \circ \Phi_\eta [\gamma] = [\gamma]$

similarly $\Phi_\eta \circ \Phi_{\bar{\eta}} = id$ ∎

Thm 7:

> If $f: X \to Y$ is a homotopy equivalence (not nec. based homotopic) then $f_*: \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism

Proof: let $g: Y \to X$ be a homotopy inverse

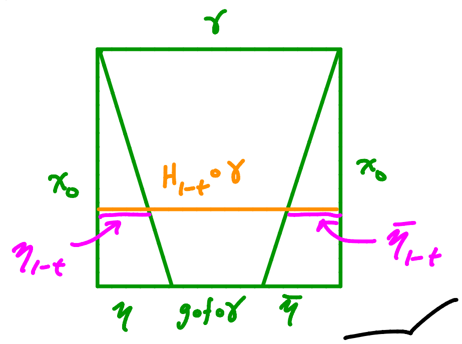
so $g \circ f \simeq id_X$ by a homotopy H_t $\quad (H_0 = id_X,\ H_1 = g\circ f)$

let $\eta(t) = H_t(x_0)$

Claim: $\Phi_\eta \circ (g \circ f)_* = id_{\pi_1(X, x_0)}$

Proof: given $[\gamma] \in \pi_1(X, x_0)$

we need $\Phi_\eta \circ (g\circ f) \circ \gamma \sim \gamma$, which we have by



so $(g \circ f)_*$ is an isomorphism

$\therefore$ g_* is surjective and f_* injective

similarly $(f \circ g)_*$ an isomorphism

$\therefore$ g_* is injective and f_* surjective

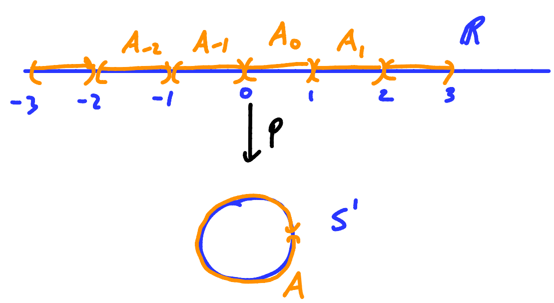
so f_* an isomorphism ▦

B. Fundamental group of S^1

It is surprisingly involved to compute $\pi_1(S^1, x_0)$

but the method is very important!

let $p: \mathbb{R} \to S^1 : x \longmapsto (\cos 2\pi x, \sin 2\pi x)$



set $A = S^1 - \{(1,0)\}$

$$p^{-1}(A) = \bigcup_{i \in \mathbb{Z}} \underbrace{(i, i+1)}_{A_i}$$

note: $p|_{A_i}: A_i \to A$ is a homeomorphism (clearly invertable, check inv. is continuous) similarly for $B = S^1 - \{(-1,0)\}$

then $p^{-1}(B) = \bigcup_{i \in \mathbb{Z}} \underbrace{(i-1/2,\, i+1/2)}_{B_i}$

and $p|_{B_i} : B_i \to B$ a homeomorphism

Obvious but important observation:

if $f: X \to S^1$ has image in A, then after choosing an integer i $\exists$ unique map $\tilde{f}: X \to A_i \subset \mathbb{R}$

such that $p \circ \tilde{f} = f$

i.e. set $\tilde{f} = (p|_{A_i})^{-1} \circ f$

similarly for $f(X) \subset B$.

now given a loop $\gamma: [0,1] \to S^1$ based at $(1,0)$ we want to "lift" it to $\mathbb{R}$

that is we want a map $\tilde{\gamma}: [0,1] \to \mathbb{R}$ such that $p \circ \tilde{\gamma} = \gamma$

if image $\gamma \subset A$ or B then easy!

<u>note</u>: $\{A, B\}$ is an open cover of S^1

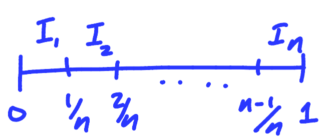
so $\{\gamma^{-1}(A), \gamma^{-1}(B)\}$ is an open cover of $[0,1]$

$[0,1]$ is compact metric space, so $\exists$ a Lebesgue number $\delta > 0$ for cover

(i.e. any set with diameter $< \delta$ is in $\gamma^{-1}(A)$ or $\gamma^{-1}(B)$)

choose n such that $\frac{1}{n} < \delta$

let $I_i : [\frac{i-1}{n}, \frac{i}{n}]$



<u>note</u>: 1) diam $I_i < \delta$ so $I_i \subset \gamma^{-1}(A)$ or $\gamma^{-1}(B)$

2) $\gamma(0) = (1,0)$ so $\gamma(I_1) \subset B$

so we can lift $\gamma|_{I_1}$ to $B_0 \subset \mathbb{R}$

i.e. $\tilde{\gamma}_1 = (p|_{B_0})^{-1} \circ \gamma|_{I_1}$

<u>note</u>: $\tilde{\gamma}_1(0) = 0$

now $\gamma(I_2)\subset A$ or B so we can lift $\gamma|_{I_2}$ to $\mathbb{R}$

we choose lift $\tilde{f}_2$ so that $\tilde{\gamma}_1(1/n)=\tilde{\gamma}_2(1/n)$

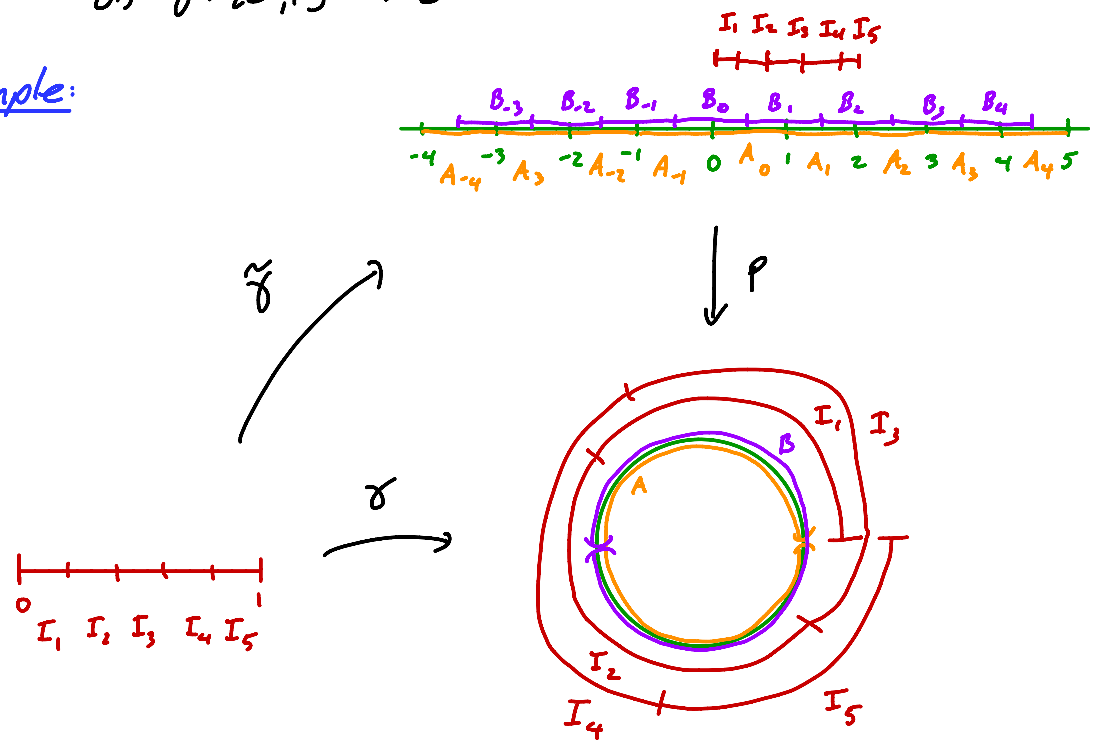
we inductively lift all the $\gamma|_{I_k}$ to get $\tilde{\gamma}_1,\dots,\tilde{\gamma}_n$

since these lifts all agree at the end points, we get a continuous lift

$$\tilde{\gamma}:[0,1]\to\mathbb{R}$$

of $\gamma:[0,1]\to S^1$

example:



note: $\tilde{\gamma}(1)\in p^{-1}((1,0))=\mathbb{Z}$

we have proven

Thm 8 (path lifting):

if $\gamma:[0,1]\to S^1$ is a path based at $(1,0)$, then for each $n\in\mathbb{Z}$

$\exists$ a unique map $\tilde{\gamma}_n:[0,1]\to\mathbb{R}$ such that

$$\tilde{\gamma}_n(0)=n \quad \text{and}$$

$$p\circ\tilde{\gamma}_n=\gamma$$

more generally, if $\gamma:[0,1]\to S^1$ is an unbased loop, then there is a unique lift $\tilde{\gamma}$ once a point in $p^{-1}(\gamma(0))$ is chosen we can define a map

$$\phi : \pi_1(S^1, (1,0)) \longrightarrow \mathbb{Z}$$
$$[\gamma] \longmapsto \tilde{\gamma}_0(1)$$

lift of γ with $\tilde{\gamma}_0(0) = 0$

Th$^{\underline{m}}$ 9:

ϕ is well-defined and an isomorphism

so $\quad \pi_1(S^1, (1,0)) \cong \mathbb{Z}$

to prove this we need

Th$^{\underline{m}}$ 10 (Homotopy lifting):

Given a continuous map $H : [0,1] \times [0,1] \to S^1$

$\exists$ a continuous map $\tilde{H} : [0,1] \times [0,1] \to \mathbb{R}$

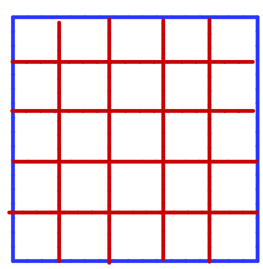
such that $p \circ \tilde{H} = H$

Moreover, $\tilde{H}$ is unique once we have chosen a point $\tilde{x}_0 \in p^{-1}(H(0,0))$ and require $\tilde{H}(0,0) = \tilde{x}_0$

Proof: just like proof of path lifting

let δ be Lebesgue number for $\{H^{-1}(A), H^{-1}(B)\}$

and pick n s.t. $\frac{\sqrt{2}}{n} < \delta$ then consider



$\frac{1}{n} \times \frac{1}{n}$ squares

$H|_{\text{each square}}$ in A or B so can be lifted

exercise: Write out the details

Proof of Th$^{\underline{m}}$ 9:

if $\gamma \simeq \delta$ as based loops in $(S^1, (1,0))$

let H be the homotopy

let $\tilde{H}$ be the lift of H such that $\tilde{H}(0,0)=0$

note: 1) $p\circ\tilde{H}_0(s) = H_0(s) = \gamma(s)$ so $\tilde{H}_0(s)$ is a lift of γ, starting at 0

so $\tilde{\gamma}_0 = \tilde{H}_0$

2) $\tilde{H}|_{\{0\}\times[0,1]} : [0,1] \to p^{-1}(H(\{0\}\times[0,1])) = p^{-1}((1,0)) = \mathbb{Z}$ ← discrete topology

so $\tilde{H}(0,t)$ is constant (i.e. path components are points)

since $\tilde{H}(0,0)=0$, we see $\tilde{H}(0,t)=0$

3) $p\circ\tilde{H}_1(s) = H_1(s) = \delta(s)$ so $\tilde{H}_1$ is a lift of δ starting at 0

$\therefore \tilde{\delta}_0 = \tilde{H}_1$

$\therefore \tilde{\gamma}(1) = \tilde{H}(1,0) = \tilde{H}(1,1) = \tilde{\delta}(1)$

same agrt. as 2)

so ϕ is well-defined

ϕ onto: let $f_n : [0,1] \to \mathbb{R} : x \longmapsto nx$

note: $\gamma_n = p\circ f_n$ is a loop in S^1 based at $(1,0)$ that lifts to f_n

so $\phi([\gamma_n]) = n$

$\therefore \phi$ onto

ϕ homomorphism:

let $[\gamma_1], [\gamma_2] \in \pi_1(S^1, (1,0))$

let $\tilde{\gamma}_i$ be the lift of γ_i based at 0

set $n = \tilde{\gamma}_1(1)$ and $m = \tilde{\gamma}_2(1)$

define: $\tilde{\tilde{\gamma}}_2(s) = n + \tilde{\gamma}_2(s)$

note: $p\circ\tilde{\tilde{\gamma}}_2 = (\cos(2\pi(n+\tilde{\gamma}_2)), \sin(2\pi(n+\tilde{\gamma}_2)))$
$= (\cos(2\pi\tilde{\gamma}_2), \sin(2\pi\tilde{\gamma}_2)) = p\circ\tilde{\gamma}_2 = \gamma_2$

so $\tilde{\tilde{\gamma}}_2$ is a lift of γ_2 s.t. $\tilde{\tilde{\gamma}}_2(0) = n$

clearly $\tilde{\gamma}_1 * \tilde{\tilde{\gamma}}_2$ is a lift of $\gamma_1 * \gamma_2$ based at 0
and $\tilde{\gamma}_1 * \tilde{\tilde{\gamma}}_2(1) = n+m$

so $\phi([\gamma_1]*[\gamma_2]) = \phi([\gamma_1]) + \phi([\gamma_2])$

ϕ injective:

we check $\ker \phi = \{e\}$

if $[\gamma] \in \ker \phi$, then the lift $\tilde{\gamma}$ of γ based at 0

has $\tilde{\gamma}(1) = 0$

that is $\tilde{\gamma}(s)$ a loop in $\mathbb{R}$ based at 0

set $\tilde{H}(s,t) = t\,\tilde{\gamma}(s)$

note: 1) $\tilde{H}(s,0) = 0$

2) $\tilde{H}(s,1) = \tilde{\gamma}(s)$

3) $\tilde{H}(0,t) = \tilde{H}(1,t) = 0$

$\Big\}$ $\tilde{H}$ a homotopy $\tilde{\gamma}$ to constant loop

let $H = p \circ \tilde{H}$

H is a homotopy of γ to the constant loop e

C. Applications

given a map $f : S^1 \to S^1$

let $\bar{f} : [0,1] \to S^1$ be the map $f \circ q = \bar{f}$

where

$$\begin{tikzcd} {[0,1]} \arrow[d, "q"'] \arrow[dr, bend left, "\bar{f}"] & \\ S^1 \arrow[r, "f"'] & S^1 \end{tikzcd}$$

$q(t) = (\cos 2\pi t, \sin 2\pi t)$

Thm 8 says there is a unique lift $\tilde{f} : [0,1] \to \mathbb{R}$ of $\bar{f}$ once we choose lift of $\bar{f}(0)$

we now define the degree of f to be the number

$$\deg f = \tilde{f}(1) - \tilde{f}(0)$$

note: if $\hat{f}$ is another such lift then $\hat{f}(s) = \tilde{f}(s) + k$ for some k

so $\hat{f}(1) - \hat{f}(0) = \tilde{f}(1) + k - (\tilde{f}(0) + k) = \tilde{f}(1) - \tilde{f}(0)$

and the degree is well-defined

Th^{m} 11:

$$\boxed{\begin{array}{c} f : S^1 \to S^1 \text{ and } g : S^1 \to S^1 \text{ are homotopic} \\ \Longleftrightarrow \\ \deg f = \deg g \end{array}}$$

Proof: ($\Rightarrow$) let $F : S^1 \times [0,1] \to S^1$ be the homotopy

let $\bar{F} : [0,1] \times [0,1] \to S^1$ be the map such that

$$F(q(s), t) = \bar{F}(s,t)$$

$$\begin{tikzcd} {[0,1] \times [0,1]} \arrow[d, "q \times id"'] \arrow[dr, bend left, "\bar{F}"] & \\ S^1 \times [0,1] \arrow[r, "F"] & S^1 \end{tikzcd}$$

let $\tilde{f}, \tilde{g}$ be lifts of $\bar{f}$ and $\bar{g}$ as above

by Th^{m} 10 $\exists!$ lift of $\bar{F}$ to $\tilde{F} : [0,1] \times [0,1] \to \mathbb{R}$ with $\tilde{F}(0,0) = \tilde{f}(0)$

by uniqueness of path lifting we know

$\tilde{F}(s,0) = \tilde{f}(s)$ since $\tilde{F}(s,0)$ is a lift of $\bar{f}$

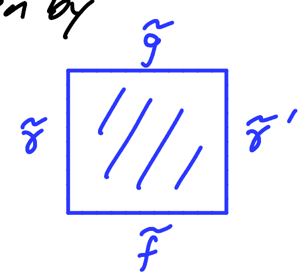
let $\gamma : [0,1] \to S^1 : t \longmapsto \bar{F}(0,t) = \bar{F}(1,t)$

set $\tilde{\gamma} : [0,1] \to \mathbb{R}$ lift of γ s.t. $\tilde{\gamma}(0) = \tilde{f}(0)$

$\tilde{\gamma}' : [0,1] \to \mathbb{R}$ lift of γ s.t. $\tilde{\gamma}'(0) = \tilde{F}(1)$

we can assume $\tilde{g}(0) = \tilde{\gamma}(1)$

note we have $\tilde{F}$ given by



so $\deg f = \tilde{F}(1,0) - \tilde{F}(0,0) = \tilde{\gamma}'(0) - \tilde{\gamma}(0)$

$\deg g = \tilde{F}(1,1) - \tilde{F}(0,1) = \tilde{\gamma}'(1) - \tilde{\gamma}(1)$

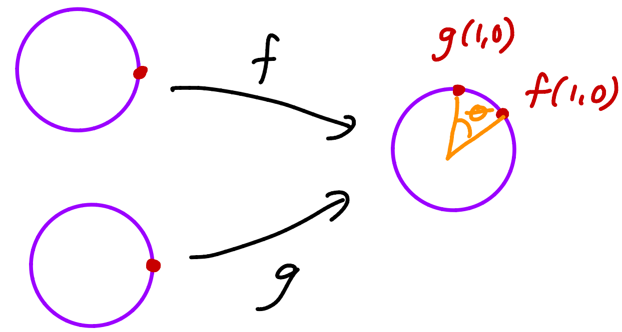
but $\tilde{\gamma}'(t) = \tilde{\gamma}(t) + k$ some k

and $\deg f = \deg g$ ✓

($\Leftarrow$) assume $\deg f = \deg g$

let θ be the angle between $f(1,0)$ and $g(1,0)$

let $R_t : S^1 \to S^1$ be rotation through angle t



set $H(s,t) = R_{t\theta} \circ f(s)$

so $H(s,0) = f(s)$

$H(s,1) = R_\theta \circ f(s)$ i.e. $H((1,0),1) = R_\theta \circ f((1,0)) = g((1,0))$

so after homotopy we can assume $f((1,0)) = g((1,0))$

(from ($\Rightarrow$) we know $\deg f$ unchanged under htpy)

let $\bar{f}, \bar{g} : [0,1] \to S^1$ be as above (note $\bar{f}(0) = \bar{g}(0)$ by above)

let $\tilde{f}, \tilde{g} : [0,1] \to S^1$ be lifts of $\bar{f}, \bar{g}$, respectively

s.t. $\tilde{f}(0) = \tilde{g}(0)$

now $\deg f = \deg g \Rightarrow \tilde{f}(1) = \tilde{g}(1)$

set $\tilde{H}(s,t) = t\tilde{f}(s) + (1-t)\tilde{g}(s)$

note: $\tilde{H}(0,t) = t\tilde{f}(0) + (1-t)\tilde{g}(0) = \tilde{f}(0)$

$\tilde{H}(1,t) = t\tilde{f}(1) + (1-t)\tilde{g}(1) = \tilde{f}(1)$

so $p \circ \tilde{H}_t : [0,1] \to S^1$ decends to a map $H_t : S^1 \to S^1$ $\forall t$

H_t give homotopy of f to g ▨

exercise: 1) the constant map $f : S^1 \to S^1$ has degree 0

2) $f_* : \pi_1(S^1, (1,0)) \longrightarrow \pi_1(S^1, (1,0))$ is multiplication by $\deg f$

i.e. $\mathbb{Z} \to \mathbb{Z}$

$[\gamma] \longmapsto (\deg f)[\gamma]$

need to homotop f to preserve base pt!

Corollary 12:

two maps $f, g : S^1 \to S^1$ are homotopic

$\Leftrightarrow$

$f_* = g_* : \pi_1(S^1, (1,0)) \to \pi_1(S^1, (1,0))$

In particular, $f: S^1 \to S^1$ is homotopically trivial $\Leftrightarrow$ it induces trivial map on $\pi_1(S^1, (1,0))$

Proof: imediate from exercises $\blacksquare$

Remark: so maps on S^1 are completely determined by π_1!

Lemma 13:

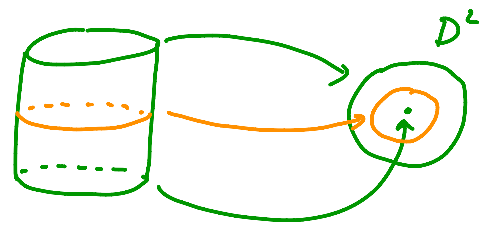
a map $f: S^1 \to S^1$ extends to a map $F: D^2 \to S^1$

$\Leftrightarrow$

$\deg f = 0$

Proof: ($\Rightarrow$) let $P: [0,1] \times S^1 \to D^2$

$(r, \theta) \longmapsto (r, \theta)$ polar coords



given $F: D^2 \to S^1$ such that $F|_{\partial D^2} = f$

set $H(s,t) = F \circ P(s,t)$

this is a homotopy from

$H(s,0) = F \circ P(s,0) = F(0,s) = pt$ origin

to

$H(s,1) = F \circ P(s,1) = F|_{\partial D^2} = f(s)$

so $f \simeq$ constant $\therefore \deg f = 0$ ✓

($\Leftarrow$) if $\deg f = 0$, then $\exists$ a homotopy $H: S^1 \times [0,1] \to S^1$

s.t. $H(s,1) = f(s)$ and $H(s,0) = pt$

so we get an induced map $F: D^2 \to S^1$ that extends f

$$\begin{tikzcd} S^1 \times [0,1] \arrow[r, "H"] \arrow[d, "\text{quotient map}"'] & S^1 \\ D^2 \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

$\blacksquare$

exercise:

think of S^1 as the unit circle in $\mathbb{C}$

let $f_n: S^1 \to S^1 : z \mapsto z^n$

show $\deg(f_n) = n$

$$\begin{tikzcd}
{[0,1]} \arrow[r, "t \mapsto nt"] \arrow[d, "q"] & \mathbb{R} \arrow[d, "p"] \\
S^1 \arrow[r, "z^n"] & S^1
\end{tikzcd}$$

Thm 14 (Fundamental Thm of Algebra):

> any non-constant complex polynomial $P(z)$ has a root
> i.e. z_0 such that $P(z_0) = 0$

Remark: Amazing! We are using algebraic topology to prove basic facts about polynomials!

Proof: let $P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0 \qquad n \geq 1$

assume $P(z)$ has <u>no root</u>

let $M = \max\{|a_0|, \dots, |a_{n-1}|\}$ and choose $k \geq \max\{1, 2nM\}$

note: $P(z) = z^n\Big(1 + \underbrace{a_{n-1}\tfrac{1}{z} + \dots + a_1\tfrac{1}{z^{n-1}} + a_0\tfrac{1}{z^n}}_{b(z)}\Big)$

so if $|z| = k$, then

$$\begin{aligned}
|b(z)| &\leq |a_{n-1}|\frac{1}{|z|} + \dots + |a_0|\frac{1}{|z|^n} \\
&\leq M\Big(\frac{1}{k} + \dots + \frac{1}{k^n}\Big) \leq M\frac{n}{k} \\
&\leq M\frac{n}{2nM} = \frac{1}{2}
\end{aligned}$$

let $f: S^1 \to S^1 : z \mapsto \dfrac{P(kz)}{|P(kz)|}$ $\quad$ well-defined since <u>never zero</u> by assumption

this extends to

$$F: D^2 \to S^1 : z \mapsto \frac{P(kz)}{|P(kz)|}$$

so $\deg f = 0$ by lemma 14

but let $P_t(z) = z^n(1+tb(z))$

from above $P_t(z) \neq 0$ for $|z| = k$

so $f_t : S^1 \to S^1 : z \mapsto \dfrac{P_t(kz)}{|P_t(kz)|}$

is a homotopy from f to $f_1(z) = \dfrac{(kz)^n}{|kz|^n} = \dfrac{k^n z^n}{k^n \underbrace{|z|^n}_{=1}} = z^n$

$\deg f_1 = n \neq 0$ $\cancel{\&}$ $f \simeq f_1$ by Thm 12

$\therefore$ $P(z)$ has a root!

lemma 15:

> If $f : S^1 \to S^1$ is continuous and $f(-x) = -f(x)$ $\forall x$
> then $\deg(f)$ is odd

Proof: given such an $f : S^1 \to S^1$

let $\bar{f} : [0,1] \to S^1$ be as above (i.e. $f \circ q = \bar{f}$)

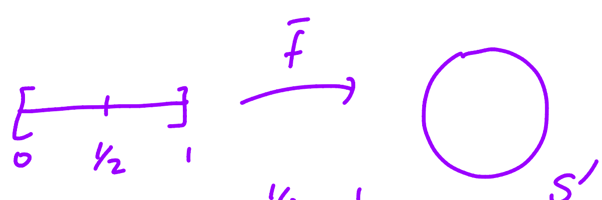
let $a = \bar{f}(0)$ and $p^{-1}(a) = \{\tilde{a}_i\}$ where $p : \mathbb{R} \to S^1$ and
$\tilde{a}_i = \tilde{a}_0 + i$

note $\bar{f}(1/2) = f((-1,0)) = -f((1,0)) = -a$ and

$p^{-1}(-a) = \{\tilde{b}_i\}$ where $\tilde{b}_i = \tilde{a}_i + \frac{1}{2}$

let $f_1 = \bar{f}|_{[0,1/2]}$

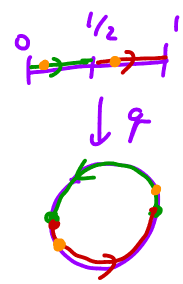
$f_2 = \bar{f}|_{[1/2,1]}$



since $f(x) = f(-(-x)) = -f(-x)$

and

$q(x - 1/2) = -q(x)$



we have $f_2(x) = \bar{f}(x) = f(q(x))$

$= -f(-q(x)) = -f(q(x-1/2)) = -\bar{f}(x - 1/2) = -f_1(x - 1/2)$

so if $\tilde{f}_1$ is a lift of f_1 starting at $\tilde{a}_0$ then $\tilde{f}_1(1/2) = \tilde{b}_i$ some i

and $\tilde{f}_1(x - 1/2) + 1/2$ is a lift of $f_2 : [1/2, 1] \to S^1$ starting at $\tilde{a}_0 + 1/2 = \tilde{b}_0$

just like for q above $p(x - 1/2) = -p(x)$

so $\tilde{f}_2(x) = \tilde{f}_1(x - 1/2) + q + 1/2$ is a lift of f_2 starting at $\tilde{f}_1(1/2) = \tilde{b}_2$

now $\tilde{f}_2(1) = \tilde{f}_1(1/2) + q + 1/2 = \tilde{b}_2 + q + 1/2 = \tilde{a}_1 + q + 1 = \tilde{a}_0 + 2i + 1$

note $\tilde{f}_1 * \tilde{f}_2$ is a lift of f

so $\deg(f) = \tilde{f}_1 * \tilde{f}_2(1) - \tilde{f}_1 * \tilde{f}_2(0) = \tilde{a}_0 + 2q + 1 - \tilde{a}_0 = 2i + 1$ ▦

Thm 16 (Borsuk-Ulam I):

> There does not exist a continuous map
> $$f : S^2 \to S^1$$
> sending antipodal points to antipodal points

Proof: If $f : S^2 \to S^1$ is such a map

then let $S^1 \subset S^2$ be the equator

$$f|_{S^1} : S^1 \to S^1 \quad \text{satisfies} \quad f(-x) = -f(x)$$

so $\deg f|_{S^1}$ is odd by lemma 15

but $f|_{S^1}$ extends over northern hemisphere

so $\deg f|_{S^1} = 0$ by lemma 13 $\otimes$ ▦

Thm 17 (Borsuk-Ulam II):

> Any continuous map $f : S^2 \to \mathbb{R}^2$ must send a pair of antipodal points to the same point

Proof: given any continuous $f : S^2 \to \mathbb{R}^2$

assume $f(x) \neq f(-x) \; \forall x \in S^2$

then consider $g : S^2 \to S^1 : x \longmapsto \dfrac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$

exercise: g is continuous

clearly $g(-x) = -g(x)$ $\otimes$ Thm 16 ▦

Remark: Thm implies that at any point in time there are antipodal points on the earth with the same temperature and humidity!
(or pick your favorite continuously varying quantities)

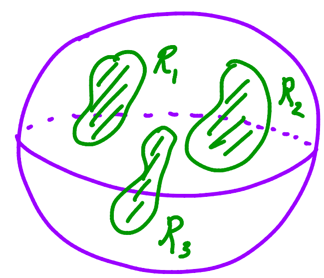
Th^{m} 18 (Ham sandwich th^{m}):

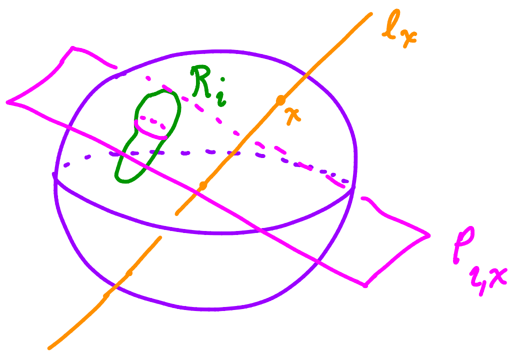
> let R_1, R_2, R_3 be three connected open regions in $\mathbb{R}^3$ each of which is bounded and of finite volume
> Then $\exists$ a plane which cuts them in half by volume

Proof: let $S^2 \subset \mathbb{R}^3$ be a large sphere about origin containing all R_i

given $x \in S^2$, let ℓ_x be the line through x and origin

for each i, $\exists$ plane $P_{i,x}$ perpendicular to ℓ_x that cuts R_i in half





let $d_i(x)$ = distance of $P_{i,x}$ from origin
(where $d_i(x) > 0$ if $P_{i,x}$ on same side of origin as x)

exercise: Show $d_i(x)$ are continuous functions $d_i : S^2 \to \mathbb{R}$

Hint: Equation of planes perpendicular to ℓ_x continuously vary with x

Volume of regions of R_i cut by plane continuously vary with eq^{n} of plane

clearly $d_i(-x) = -d_i(x)$

consider $f : S^2 \to \mathbb{R}^2 : x \mapsto (d_1(x) - d_2(x), d_1(x) - d_3(x))$

Th^{m} 17 $\Rightarrow \exists\, x$ such that $f(x) = f(-x)$

so $d_1(x) - d_2(x) = d_1(-x) - d_2(-x) = -d_1(x) + d_2(x)$

$\therefore\ 2d_1(x) = 2d_2(x) \Rightarrow d_1(x) = d_2(x)$

similarly $d_3(x) = d_1(x) = d_2(x)$

so $\exists$ plane $\perp$ to ℓ_x that cuts R_1, R_2, R_3 in half! ∎