VIII Covering Spaces

A. Covering Spaces

recall, when we computed $\pi_1(S^1)$ we used the map

$$p: \mathbb{R} \to S^1$$
$$t \longmapsto (\cos 2\pi t, \sin 2\pi t)$$

the key facts about p were

<u>path lifting:</u> given $\gamma: [0,1] \to S^1$, then for each $x \in p^{-1}(\gamma(0))$, $\exists$ unique $\tilde{\gamma}_x : [0,1] \to \mathbb{R}$ s.t. $\tilde{\gamma}_x(0) = x$ and $p \circ \tilde{\gamma}_x = \gamma$

<u>homotopy lifting:</u> given a homotopy $H: [0,1] \times [0,1] \to S^1$ then for each $x \in p^{-1}(H(0,0))$, $\exists$ unique $\tilde{H}_x : [0,1] \times [0,1] \to \mathbb{R}$ s.t. $\tilde{H}_x(0,0) = x$ and $p \circ \tilde{H}_x = H$

to prove these properties we used that:

$S^1 = A \cup B$ with A and B open

$p^{-1}(A) = \bigcup_{i=-\infty}^{\infty} A_i$, $p^{-1}(B) = \bigcup_{i=-\infty}^{\infty} B_i$ s.t.

1) A_i, B_i are open in $\mathbb{R}$

2) A_i are disjoint (same for B_i's)

3) $p|_{A_i} : A_i \to A$ and $p|_{B_i} : B_i \to B$ are homeomorphisms

generalizing this we have:

given a topological space X

a <u>covering space</u> of X is a pair $(\tilde{X}, p)$ where $\tilde{X}$ is a topological space and

$$p : \tilde{X} \to X$$

is a continuous map (called a covering map) such that $\forall x \in X$, there is an open set $U \subset X$ containing x

$$\text{s.t. } p^{-1}(U) = \{U_\alpha\}_{\alpha \in I}$$

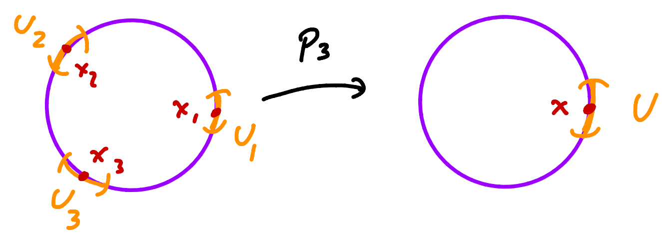
where the U_i are open, pairwise disjoint sets in $\tilde{X}$ and $p|_{U_i} : U_i \to U$ is a homeomorphism

(U is called an evenly covered set)

examples:

1) $p : \mathbb{R} \to S^1$ is clearly a covering map

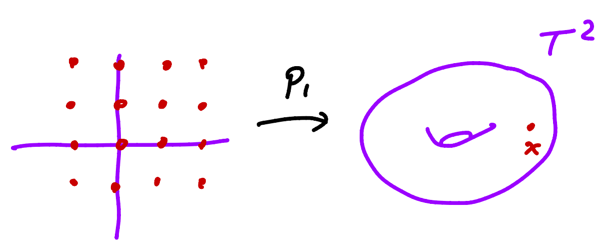
2) $p_n : S^1 \to S^1 : \theta \mapsto n\theta$ can easily be seen to be a cover



3) $p_1 : \mathbb{R}^2 \to T^2 = S^1 \times S^1$ where p is from 1)

$(x,y) \mapsto (p(x), p(y))$

can easily be checked to be a covering map



more generally

exercise: if $p_X : \tilde{X} \to X$ and $p_Y : \tilde{Y} \to Y$ are covering maps, then show $p(x,y) = (p_X(x), p_Y(y))$ is a covering map

$$p : \tilde{X} \times \tilde{Y} \to X \times Y$$

repeating the proofs we gave for $p : \mathbb{R} \to S^1$ (Thm IV.8 and 10) we get

Th$^{\underline{m}}$ 1 (path lifting):

if $p: \tilde{X} \to X$ is a covering map
$\gamma: [0,1] \to X$ is a path and
$x \in p^{-1}(\gamma(0))$
then $\exists$ unique $\tilde{\gamma}_x : [0,1] \to \tilde{X}$ such that
$\tilde{\gamma}_x(0) = x$ and $p \circ \tilde{\gamma}_x = \gamma$

Th$^{\underline{m}}$ 2 (Homotopy lifting):

if $p: \tilde{X} \to X$ is a covering map,
$H: [0,1] \times [0,1] \to X$ a homotopy, and
$x \in p^{-1}(H(0,0))$
then $\exists$ unique $\tilde{H}_x : [0,1] \times [0,1] \to \tilde{X}$ such that
$\tilde{H}_x(0,0) = x$ and $p \circ \tilde{H}_x = H$

lemma 3:

let $p: \tilde{X} \to X$ be a covering map with X connected
if $\exists$ a point $x_0 \in X$ with $|p^{-1}(x_0)| = k$, then
$|p^{-1}(x)| = k, \ \forall x \in X$

$|p^{-1}(x)|$ is called the <u>degree</u> of the covering space

<u>Proof</u>: let $A = \{x \in X \text{ s.t. } |p^{-1}(x)| = k\}$
$A \neq \emptyset$ since $x_0 \in A$
<u>Claim</u>: A is open
indeed if $x \in A$, then let U be an evenly covered open set containing x
$p^{-1}(U) = \{U_\alpha\}_{\alpha \in I}$

but $U_\alpha \cap p^{-1}(x) = 1$ point $\forall \alpha$

$\therefore \; I = \{1, \dots, k\}$

so $|p^{-1}(y)| = k, \; \forall y \in U$

$\therefore$ A is open

<u>Claim</u>: A is closed

similar argument exercise

since X is connected $A = X$ (lemma II.10)

<u>lemma 4</u>:

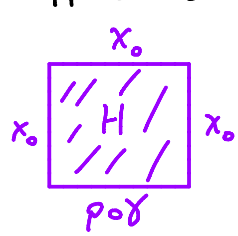
$$\boxed{\begin{array}{l} p: \tilde{X} \to X \text{ a covering map, } \tilde{x}_0 \in \tilde{X}, \; x_0 = p(\tilde{x}_0) \\ \text{then} \\ \qquad p_*: \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0) \\ \text{is injective} \\ \text{Moreover, } [\gamma] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \\ \qquad\qquad \Leftrightarrow \\ \qquad \text{the lift of } \gamma \text{ to a path based} \\ \qquad\quad \text{at } \tilde{x}_0 \text{ is a loop in } \tilde{X} \end{array}}$$

<u>Proof</u>: $[\gamma] \in \pi_1(\tilde{X}, \tilde{x}_0)$

constant path

suppose $p_*([\gamma]) = e$ i.e. $p \circ \gamma \simeq x_0$

so $\exists$ homotopy $H: [0,1] \times [0,1] \to X$

s.t. 

homotopy lifting says $\exists \; \tilde{H}: [0,1] \times [0,1] \to \tilde{X}$

s.t. $\tilde{H}(0,0) = \tilde{x}_0$ and $p \circ \tilde{H} = H$

<u>note:</u> $p \circ \tilde{H}(s,0) = p \circ \gamma$ so $\tilde{H}(s,0)$ is a lift of $p \circ \gamma$ starting at $\tilde{x}_0$, so it is γ

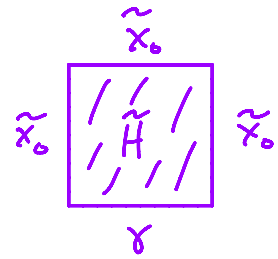
$\therefore\ H(s,0) = \gamma(s)$

also $\tilde{H}(0,t) \in p^{-1}(x_0)$ ← points with discrete topology

so $\tilde{H}(0,t) = \tilde{x}_0 \ \forall t$

similarly $\tilde{H}(1,t) = \tilde{x}_0 \ \forall t$ and $\tilde{H}(s,1) = \tilde{x}_0 \ \forall s$

i.e.



is a homotopy $\gamma \simeq \tilde{x}_0$

$\therefore\ [\gamma] = e$

and p_* is injective

now, if $[\eta] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ then $\exists [\gamma] \in \pi_1(\tilde{X}, \tilde{x}_0)$

s.t. $p_*([\gamma]) = [\eta]$ i.e. $p \circ \gamma \simeq \eta$

let $\tilde{\eta}$ be a lift of η starting at $\tilde{x}_0$

by homotopy lifting $\gamma \simeq \tilde{\eta}$ rel end points

but γ a loop so $\tilde{\eta}$ a loop too

if $[\eta] \notin p_*(\pi_1(\tilde{X}, \tilde{x}_0))$, then the lift $\tilde{\eta}$ of η based at $\tilde{x}_0$ can't be a loop since if it were then $[\tilde{\eta}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ and $[\eta] = p_*([\tilde{\eta}])$ $\otimes$

<u>exercise:</u> $[\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0))]$ = degree of $(\tilde{X}, p)$

↑ index of subgroup

<u>Hint:</u> Show there is a bijection from right cosets of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ to $p^{-1}(x_0)$

examples:

1) $p : \mathbb{R} \to S^1$

$$p_* : \pi_1(\mathbb{R}) \longrightarrow \pi_1(S^1)$$
$$\{e\} = \pi_1(\mathbb{R}), \quad \pi_1(S^1) \cong \mathbb{Z}$$

no non-trivial loop in S^1 lifts to a loop in $\mathbb{R}$

degree $= \infty = [\mathbb{Z} : \{e\}]$

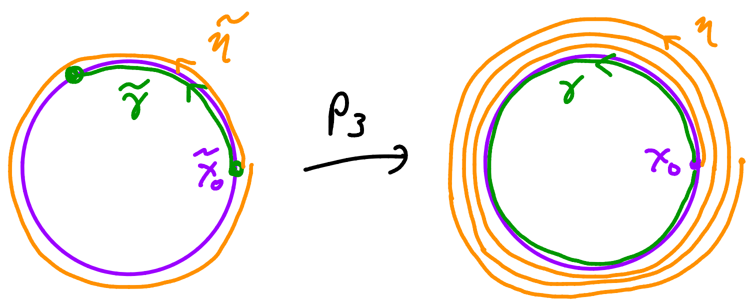
2) $p_n : S^1 \to S^1 : \theta \mapsto n\theta$

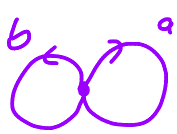
$$(p_n)_* : \pi_1(S^1) \longrightarrow \pi_1(S^1)$$
$$\mathbb{Z} \cong \pi_1(S^1), \quad \pi_1(S^1) \cong \mathbb{Z}$$
$$m \longmapsto nm$$

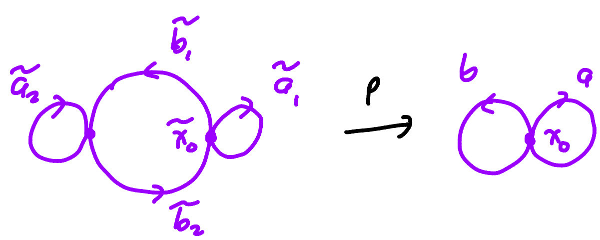
so $\mathrm{im}\,(p_n)_* = n\mathbb{Z}$

degree $= n = [\mathbb{Z} : n\mathbb{Z}]$

loop in S^1 lifts to a loop iff it "goes around" S^1 a multiple of n times



3) $X =$ 

$\tilde{X} =$  $= X$

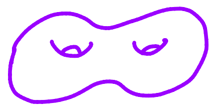
<u>exercise</u>: this is a covering map

<u>note</u>: $\pi_1(\tilde{X}) = F_3$ generated by $\tilde{a}_1,\ \tilde{b}_1 * \tilde{b}_2,\ \tilde{b}_1 * \tilde{a}_2 * \overline{\tilde{b}_1}$

so $\operatorname{image}(p_*) = \langle a, b^2, bab^{-1} \rangle = G$

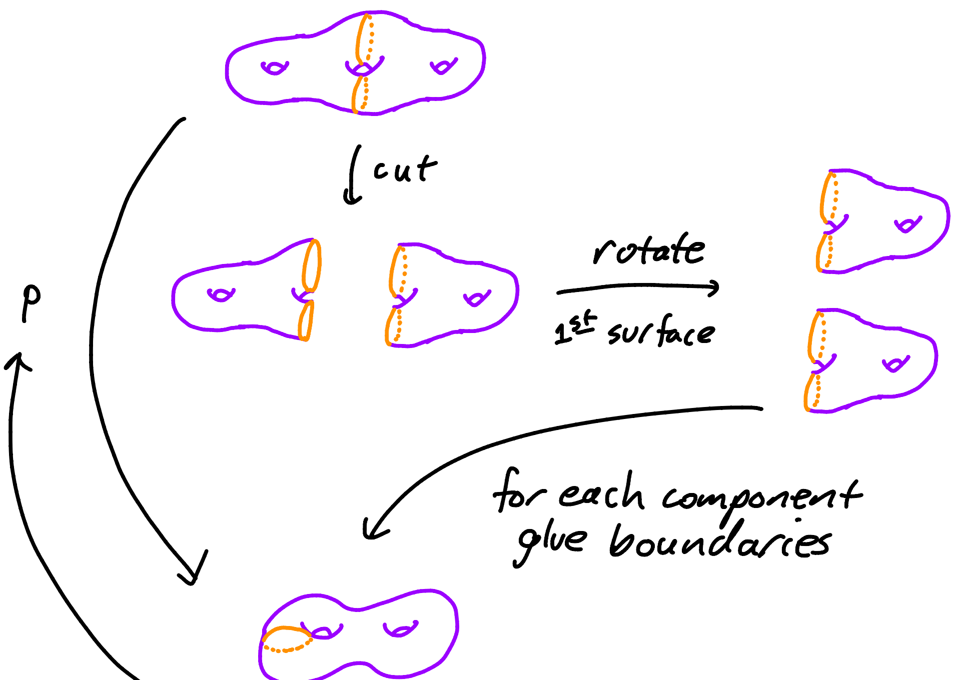
G has index 2 in $\pi_1(X, x_0) \cong F_2$!

note rank went up !

4) consider Σ_2 

let's find a degree 2 cover (there are actually a lot)

consider



define p to be

exercise:

1) Show this is a 2-fold covering map $\Sigma_3 \to \Sigma_2$

2) Work out $\operatorname{im}(p_*)$

3) Experiment constructing other covers of other surfaces

e.g. $\Sigma_n \to \Sigma_2$ by an $n-1$ fold cover for $n \geq 2$

let $p: \tilde{X} \to X$ be a covering map with $p(\tilde{x}_0) = x_0$

$f: Y \to X$ be a continuous map such that $f(y_0) = x_0$

a lift of f to $\tilde{X}$ is a continuous map $\tilde{f}: Y \to \tilde{X}$

s.t. $\tilde{f}(y_0) = \tilde{x}_0$ and $p \circ \tilde{f} = f$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Thm 5 (lifting criterion):

$p: \tilde{X} \to X$ a covering map, $p(\tilde{x}_0) = x_0$

$f: Y \to X$ a continuous map s.t. $f(y_0) = x_0$

assume Y is path connected and locally path connected

Then $\exists$ a lift $\tilde{f}: Y \to X$ of f

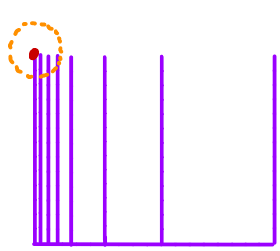
$$\Longleftrightarrow$$

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

if $\tilde{f}$ exists it is unique

a space is locally path connected if for every point x and open set U containing it, there is an open set V such that $x \in V \subset U$ and V is path connected

example:



$(\{1/n\} \times [0,1]) \cup (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$

path connected but not locally path connected

note: all manifolds are locally path connected

Proof: ($\Rightarrow$) if $\tilde{f}$ exists, then clearly

$$f_*(\pi_1(Y, y_0)) = p_* \circ \tilde{f}_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

($\Leftarrow$) need to construct $\tilde{f}: Y \to \tilde{X}$

given $y \in Y$, let $\gamma_y : [0,1] \to Y$ be a path s.t.

$$\gamma_y(0) = y_0, \ \gamma_y(1) = y$$ (use path connected)

$f \circ \gamma_y$ is a path in X from $x_0 = f(y_0)$ to $f(y)$

lift $f \circ \gamma_y$ to a path $\tilde{\gamma}_y$ in $\tilde{X}$ starting at $\tilde{x}_0$

define: $\tilde{f}(y) = \tilde{\gamma}_y(1)$

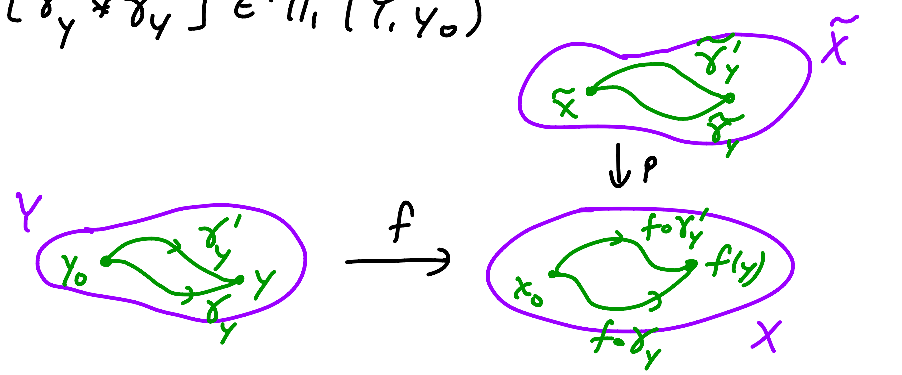
if $\tilde{f}$ is well-defined, the clearly $p \circ \tilde{f}(y) = f(y)$

so $\tilde{f}$ is a lift of f

to see $\tilde{f}$ is well-defined, let γ_y' be another path from y_0 to y

note: $\gamma_y * \overline{\gamma}_y$ is a loop in Y based at y_0

so $[\gamma_y * \overline{\gamma}_y] \in \pi_1(Y, y_0)$



and $f_*[\gamma_y * \overline{\gamma}_y] = [(f \circ \gamma_y) * (\overline{f \circ \gamma_y'})] \in \pi_1(X, x_0)$

by assumption $[(f \circ \gamma_y) * \overline{(f \circ \gamma_y')}] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

so by lemma 4 $(f \circ \gamma_y) * \overline{(f \circ \gamma_y')}$ lifts to a loop

in $\tilde{X}$ based at x_0: $\widetilde{(f \circ \gamma_y) * \overline{(f \circ \gamma_y')}}$

can easily check by uniqueness of lifts that

this loop is $\widetilde{(f \circ \gamma_y)} * \widetilde{\overline{(f \circ \gamma_y')}}$ lift starting at $\tilde{\gamma}_y(1)$

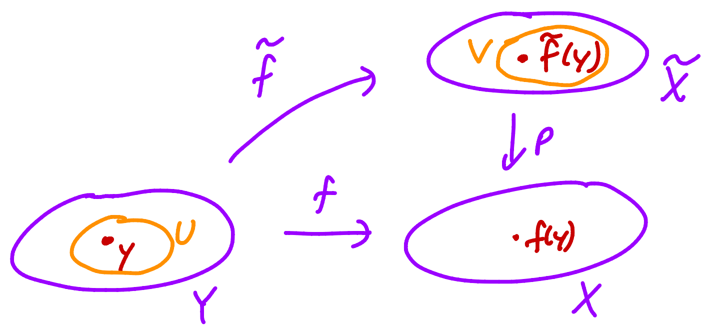
so $\tilde{f}$ is well-defined

the last thing we need to do is see $\tilde{f}$ is continuous.

this is more involved (and uses local connectivity)

you can find a proof in Hatcher, but the idea is:

given $y \in Y$, $\exists$ an open set $U \subset Y$ containing y and open set V in $\tilde{X}$ containing $\tilde{f}(y)$ such that $\tilde{f}|_U = \underbrace{p|_V^{-1}}_{\text{continuous}} \circ f$



<u>Fact</u>: given a surface Σ_g of genus g

if $g > 0$, then $\exists$ a covering map

$$p : \mathbb{R}^2 \longrightarrow \Sigma_g$$

(for $g > 1$, this uses "hyperbolic geometry")

<u>Th$^{\underline{m}}$6</u>:

$$\boxed{\begin{array}{l}\text{If } g \geq 1 \text{ and } n \geq 2, \text{ then any} \\ \qquad f : S^n \to \Sigma_g \\ \text{is homotopic to the constant map!}\end{array}}$$

Recall, this was used in the proof of Th$^{\underline{m}}$ VI.6

<u>Proof</u>: given f, clearly $f_*(\pi_1(S^n)) = \{e\} \subset p_*(\pi_1(\mathbb{R}^2))$ covering map above

so f lifts to a map $\tilde{f} : S^n \to \mathbb{R}^2$ by Th^{m}5

let $\tilde{H} : S^n \times [0,1] \to \mathbb{R}^2$
$(p, t) \longmapsto t\tilde{f}(p)$

$\tilde{H}(p,0) = \text{constant}$

$\tilde{H}(p,1) = \tilde{f}$

set $H = p \circ \tilde{H} : [0,1] \times [0,1] \to \Sigma_g$

this is a homotopy from the constant map to f

We saw that for every covering $p : \tilde{X} \to X$, there is a subgroup $G = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ of $\pi_1(X, x_0)$

For most spaces, there is a converse!

Fact:

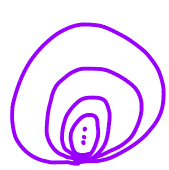
> let X be path connected
> locally path connected
> semi-locally simply connected
>
> Then $\forall\, G < \pi_1(X, x_0)$ there is a covering space $p : \tilde{X} \to X$ such that $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = G$

a space X is *semi-locally simply connected* if $\forall x \in X$, $\exists$ an open set $U \subset X$ such that $x \in U$ and

$$\imath_* : \pi_1(U, x) \to \pi_1(X, x_0)$$

is the trivial map, where $\imath : U \to X$ is inclusion

Fact: mainfolds and CW complexes are semi-locally simply connected.

example:  is not

We will not prove this, but the idea for $G=\{e\}<\pi_1(X)$ is

let $\tilde{X} = \{\text{paths in } X \text{ starting at } x_0\}/\sim$

here $\gamma\sim\eta$ if they are homotopic rel end points

set $p:\tilde{X}\to X : [\gamma]\mapsto \gamma(1)$

you can put a topology on $\tilde{X}$ so this is the desired covering space

B. Subgroups

we use covering spaces to show

Th^m 7 (Nielsen-Schreier):

> any subgroup of a free group is free

We need some lemmas

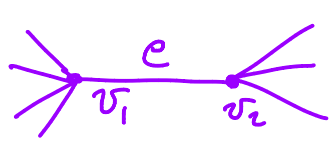
lemma 8:

> let X be a graph, then $\pi_1(X)$ is free

Proof: we can assume X is connected

if X has only one vertex, then X is a wedge of circles so from Section VI we know $\pi_1(X)$ free group

if X has more than one vertex, then there is an edge e in X connecting distinct vertices



CW Fact: if X is a CW complex, and A is a contractible subcomplex, then $X/A \simeq X$

exercise: try to prove this in above situation

so $X/e \simeq X$ and $\pi_1(X) \cong \pi_1(X/e)$,

but X/e is a graph with one less vertex

thus we can inductively find a graph Y with one vertex that is homotopy equivalent to X

$\therefore$ done $\blacksquare$

lemma 9: If X is a graph and $p: \tilde{X} \to X$ is a covering space then $\tilde{X}$ is a graph

more generally, coverings of CW complexes are CW complexes

Sketch of Proof:

$p^{-1}(X^{(0)})$ is a discrete set of points in $\tilde{X}$

0-skeleton

this will be $\tilde{X}^{(0)}$

each edge e of X is a path so it lifts to $\tilde{X}$

the union of all lifts of all edges will be the edges of $\tilde{X}$

to make this rigorous we need to see how to "attach" edges to the verticies

but hopefully this is intuitively clear $\blacksquare$

Proof of Thm 7:

given a free group F_n on n generators

let W_n = wedge of n-circles

so $\pi_1(W_n) \cong F_n$

given any $G < F_n \cong \pi_1(W_n)$, $\exists$ a covering space

$$p : \tilde{X} \to X$$

by injectivity

such that $\pi_1(\tilde{X}) \cong p_*(\pi_1(\tilde{X})) = G$

now lemma 9 says $\tilde{X}$ is a graph

and thus by lemma 8, $\pi_1(\tilde{X})$ is a free group

$\therefore$ G is a free group ∎

lots of other things you can prove about groups using topology, eg.

Thm 10 (Kurosh Subgroup Thm)

let H be a subgroup of a free product $A * B$

Then $H = (*_\lambda H_\lambda) * F$ where

H_λ is a conjugate of a subgroup of A or B

and F is a free group

Cor 11:

G indecomposable if $G = A * B \Rightarrow A$ or B trivial group

an indecomposable subgroup of a free product is isomorphic to $\mathbb{Z}$ or contained in a conjugate of a factor

Cor 12:

$$\boxed{\text{If two non-trivial elements of a free product commute, then they are either powers of a single element or are both contained in a conjugate of a factor}}$$

Cor 13:

$$\boxed{\text{If two elements of a free group commute, then they are powers of a single element}}$$

Cor 14:

$$\boxed{\text{The center of a non-trivial free product is trivial}}$$