THE ARC COMPLEX AND CONTACT GEOMETRY: NON-DESTABILIZABLE PLANAR OPEN BOOK DECOMPOSITIONS OF THE TIGHT CONTACT 3-SPHERE

JOHN ETNYRE AND YOULIN LI

ABSTRACT. In this note we introduce the (homologically essential) arc complex of a surface as a tool for studying properties of open book decompositions and contact structures. After characterizing destabilizability in terms of the essential translation distance of the monodromy of an open book we given an application of this result to show that there are planer open books of the standard contact structure on S^3 with 5 (or any number larger than 5) boundary components that do not destabilize. We also show that any planar open book with 4 or fewer boundary components does destabilize.

1. INTRODUCTION

The Giroux correspondence between open book decompositions and contact structures on 3–manifolds has been a key tool in contact geometry and low dimensional topology for quite some time. In studying relations between open book decompositions and contact structures one frequently wants to know if the open book decomposition is in some sense "minimal" or at least "non-destabilizable". The main theme explored in this note is destabilizability of open book decompositions.

The bindings of open book decompositions of S^3 are in fact fibered links. The construction and study of fibered links in S^3 has a long history, see for example [8, 9, 14, 17]. Hedden [11] proved (Baader and Ishikawa [1] later independently re-proved and Baker, Etnyre and Van Horn-Morris generalized [2]) that a fibered link in S^3 supports the unique tight contact structure on S^3 if and only if it is quasipositive. Also in [1], Baader and Ishikawa raised the following question: Does there exist a quasipositive fiber surface, other than a disk, from which we cannot de-plumb a Hopf band? In the terminology of this paper, that is: Does there exist an open book decomposition that supports the standard tight contact structure on S^3, other than the (D^2, Id), that does not destabilize?

In [2], Baker, Etnyre, and Van Horn-Morris and in [19], Wand found examples of open book decompositions that support the standard tight contact structure on S^3, have page genus 2, and cannot be destabilized down to the trivial open book (D^2, Id). Thus there are non-destabilizable examples of genus 2, or 1. (We note the results in this paper imply that if their examples destabilized to a planar open book then they would destabilize to the trivial open book, thus their examples prove the existence of a genus 2 or 1 non-destabilizable open book.)

In this paper, we completely answer this question in the planar case. We first note the existence of non-destabilizable examples with 5 or more boundary components.

Theorem 1.1. *For each integer $n \geq 5$, there is a planar open book decomposition (Σ_n, ϕ_n) which supports the tight contact structure on S^3, has n binding components, and cannot be destabilized.*

However, if the binding number is small enough, then the planar open books can be destabilized.

Theorem 1.2. *Suppose (Σ, ϕ) is a planar open book decomposition which supports the standard tight contact structure on S^3 other than the (D^2, Id), and has at most 4 binding components, then (Σ, ϕ) can be destabilized.*

We leave the following question open.

Question 1.3. *Are there open book decompositions supporting the standard tight contact structure on S^3 with page genus 1 that cannot be destabilized? What about higher genus?*

One of the main tools we use in proving the above theorems is the homologically essential arc complex and essential translation distance. The arc complex was introduced by Harer in [9] to study the mapping class group but was first used to study open books in Saito and Yamamoto's [16] and later by Johnson in [13]. Here we introduce the homologically essential arc complex and show its efficacy in studying stabilization of open books. We hope these new tools will find more applications in contact geometry. One can define the arc complex of a surface with boundary whose vertices are, more or less, isotopy classes of properly embedded arcs and two vertices have an edge if they are represented by disjoint arcs, see Subsection 2.2 for precise definitions. There is also the homologically essential arc complex defined in the same way, but only using homologically essential arcs. A diffeomorphism ϕ of the surface acts on both these complexes and we define the translation distance $\mathrm{dist}(\phi)$ and the essential translation distance $\mathrm{dist}_e(\phi)$ to be the minimal distance ϕ moves vertices in the respective complexes. We make the following observations.

Lemma 1.4. *Let (Σ, ϕ) be an open book decomposition. Then $\mathrm{dist}_e(\phi) = 0$ implies that $M_{(\Sigma,\phi)}$ has an $S^2 \times S^1$ summand.*

We note the examples from [19] imply that the other implication is not true.

Theorem 1.5. *An open book decomposition for a tight contact structure on any manifold without an $S^2 \times S^1$ summand destabilizes if and only if its essential translation distance is 1.*

Generalizing previous conjectures, now known to be false, we can ask the following question.

Question 1.6. *Is there some integer n larger than 0 such that if an open book decomposition is right veering and its monodromy has essential translation distance greater than n, then the supported contact structure is tight? The Heegaard-Floer contact invariant is non-zero?*

We end with one last question.

Question 1.7. *Is there a relation between fractional Dehn twist coefficients of a monodromy map and its (essential) translation distance?*

Acknowledgements: We thank Meredith Casey and Dan Margalit for useful conversations during the preparation of this paper. In addition we thank the referee for valuable comments improving the exposition. The first author thanks the Simons Center for their hospitality during the completion of the work in this paper. The first author was partially supported by NSF grant DMS-0804820 and DMS-1309073. The second author was partially supported by NSFC grant 11001171 and the China Scholarship Council grant 201208310626. This work was carried out while the second author was visiting Georgia Institute of Technology and he would like to thank for their hospitality.

2. Background and preliminary notions

In this section we begin by recalling the definition of an open book decomposition, its associated contact structure and other relevant notions. In the following subsection we define the arc complex and the homologically essential arc complex, define a notion of distance in these complexes and in the last subsection give a simple way one can try to bound the distance.

2.1. Open book decompositions and contact structures.

Recall that given a surface Σ with boundary and a diffeomorphism $\phi\colon\ \Sigma \to \Sigma$ that fixes the boundary, we can construct a 3–manifold $M_{(\Sigma,\phi)}$ by collapsing each circle $x \times [0,1]/\sim$ in the boundary of the mapping torus

$$T_\phi : \Sigma \times [0,1]/(x,1) \sim (\phi(x), 0)$$

to a point. We call the image of $\Sigma \times \{t\}$ a *page* of the open book and the boundary of this surface is called the *binding* of the open book. The diffeomorphism ϕ is called the *monodromy* of the open book. If M is a 3–manifold diffeomorphic to $M_{(\Sigma,\phi)}$ then we say that (Σ, ϕ) is an *open book decomposition* for M. See [5] for more details. We will use the terminology "planar open book" to refer to an open book whose page is a surface of genus zero (that is a surface that can be embedded in the plane).

A *positive stabilization* of an open book decomposition (Σ, ϕ) of a manifold M is the open book decomposition (Σ', ϕ') obtained as follows: let Σ' be Σ with a 1–handle attached and let α be a curve in Σ' that (transversely) intersects the co-core of the new 1–handle exactly once. Set $\phi' = \phi \circ \tau_\alpha$, where τ_α is the right handed Dehn twist along α. The *negative stabilization* of (Σ, ϕ) is defined in the same way except one set $\phi' = \phi \circ \tau_\alpha^{-1}$. One may easily check that $M_{(\Sigma',\phi')}$ is diffeomorphic to $M_{(\Sigma,\phi)}$ for both the positive and negative stabilization.

It is well known [7, 18] that to an open book decomposition of M there is a naturally associated contact structure $\xi_{(\Sigma,\phi)}$ and we say that the open book (Σ, ϕ) *supports* this contact structure. Moreover positive stabilization does not change the contact structure whereas negative stabilization does.

In [12] the notion of an open book decomposition being right veering was defined as a way of trying to better understand whether or not the associated contact structure was tight or overtwisted. We recall the definition here. Let γ and γ' be two arcs properly embedded in an oriented surface Σ such that they have a common endpoint x. Isotope the curves (rel boundary) so that they intersect minimally. We say that γ is *to the right* of γ' at x if the tangent vector to γ at x followed by the tangent vector of γ' at x forms an oriented basis for $T_x\Sigma$, we also say that γ' is to the left of γ at x. A diffeomorphism ϕ of Σ that fixes the boundary is called *right veering* if for every arc γ that is properly embedded in Σ, the image of γ is either isotopic to γ or is to the right of γ at each endpoint. The fundamental observation of Honda, Kazez and Matić in [12] was that if ξ is a tight contact structure then all of the open books that support it will be right veering.

2.2. **The arc complex.** Given a surface Σ with non-empty boundary. Choose a distinguished point on each boundary component. We will consider properly embedded arcs with endpoints on a subset of these distinguished points. We call a properly embedded arc *essential* if it is not isotopic into the boundary of Σ.

Important convention: When discussing embedded arcs we only require that they are embedded on their interior. That is both the endpoints can be mapped to the same place.

We define the *arc complex* of Σ, denoted $\mathcal{A}(\Sigma)$, to be the complex with vertices the isotopy classes of properly embedded essential arcs with endpoints at the chosen distinguished points on the boundary. There will be an edge between two *distinct* vertices if the interiors of the arcs are disjoint after they have been isotoped to intersect minimally. Finally k distinct vertices will bound a k-simplex if they can be isotoped to have disjoint interior.

We note that this definition is essentially equivalent to Harer's definition in [10], though he allows for marked points on the interior, and is similar to Saito and Yamamoto's definition in [16]. We did not check if our definition is equivalent to Saito and Yamamoto's, but suspect that this is the case. More specifically Saito and Yamamoto require that the arcs in their arc complex are disjoint from the chosen marked points on the boundary. Thus the orientation on the surface Σ, and the induced orientation on the boundary of Σ, sets up a one to one correspondence between their arcs and ours. Moreover they assign an edge if arcs are disjoint, so it is clear that the one complex they define is the same as the one complex we define (and when we define distance later this will imply that our notion of distance in the curve complex is the same as theirs). We note that even though Saito and Yamamoto were also studying open book decompositions we think our definition that requires arcs to have fixed endpoints is more convenient with discussing various notions such as right veering that have become important in contact geometry.

There is a natural subcomplex $\mathcal{A}_e(\Sigma)$ of $\mathcal{A}(\Sigma)$ whose vertices consist of homologically essential arcs and the higher cells are defined as above. We call this the *homologically essential arc complex*.

We define a distance function on the 0-skeleton of $\mathcal{A}(\Sigma)$

$$d\colon\ \mathcal{A}^0(\Sigma) \times \mathcal{A}^0(\Sigma) \to \mathbb{R}$$

by declaring each edge in $\mathcal{A}$ to have unit length. In particular $d([\gamma], [\gamma'])$ is k if k is the smallest integer for which there is a sequence of distinct vertices $\gamma_0, \ldots, \gamma_k$ such that $\gamma = \gamma_0$, $\gamma' = \gamma_k$ and γ_i is connected to γ_{i+1} by an edge for each $i = 0, \ldots, k-1$. It is useful to point out that while Saito and Yamamoto's definition of the arc complex is different from ours, it is clear that the vertices are in one-to-one correspondence and under this correspondence the distance function we just defined is the same as theirs. Thus the notion of translation distance below is also the same as theirs.

Similarly we define a distance function on the 0-skeleton of $\mathcal{A}_e(\Sigma)$

$$d_e\colon\ \mathcal{A}_e^0(\Sigma)\times\mathcal{A}_e^0(\Sigma)\to\mathbb{R}$$

as above.

2.3. **A simple bound on distance.** Computing distance can be very difficult, so in this subsection we indicate how to get bounds on certain distances. Given a surface Σ with more than one boundary component we say Σ' is obtained from it by *capping off a boundary component* if Σ' is obtained from Σ by gluing a disk to one of its boundary components.

Lemma 2.1. *Let Σ be a surface with more than one boundary component. If Σ' is obtained from Σ by capping off a boundary component and the distance between γ and γ' in $\mathcal{A}(\Sigma')$ is greater than 1 then the distance in $\mathcal{A}(\Sigma)$ is also greater than 1. The same statement holds in the homologically essential arc complex.*

Proof. Given arcs γ and γ' in Σ (without endpoints on the capped off boundary component) that have distance 1 or 0 as arcs in Σ we see that distance 0 implies they are isotopic and hence they would be isotopic in Σ' too (and hence distance 0 there). Distance 1 implies they are not isotopic but have disjoint interiors which clearly implies they have distance 0 or 1 in Σ'. Thus if the distance in Σ' is greater than 1, then it must also be great than 1 in Σ. □

3. Translation distance and destabilization

In this section we will define the translation distance and essential translation distance of a surface diffeomorphism and show how the later relates to destabilization of open book decompositions.

3.1. **Translation distance.** Given a diffeomorphism ϕ of a surface with boundary Σ (that fixes the boundary) we define the *translation distance* to be the minimal distance that ϕ moves a vertex in $\mathcal{A}(\Sigma)$:

$$\operatorname{dist}(\phi)=\min\{d(\alpha,\phi(\alpha)) : \alpha\in\mathcal{A}^0(\Sigma)\}.$$

This notion was originally defined in [16] using a slightly different notion for the arc complex, but as commented in Subsection 2.2 the notion of distance, and hence translation distance, is the same. We similarly define the *essential translation distance*:

$$\operatorname{dist}_e(\phi)=\min\{d_e(\alpha,\phi(\alpha)) : \alpha\in\mathcal{A}_e^0(\Sigma)\}.$$

We will see below that the essential translation distance has a closer connection to contact geometric properties and for that reason we restrict attention to it. We gave the definition of translation distance largely to tie this work with that of Saito and Yamamoto and to contrast it with essential translation distance. For now we observe in the next two lemmas one simple difference between the two different notions of translation distance, the first lemma is just Lemma 1.4 from the introduction.

Lemma 3.1. *Let (Σ,ϕ) be an open book decomposition. Then $\operatorname{dist}_e(\phi)=0$ implies that $M_{(\Sigma,\phi)}$ has an $S^2\times S^1$ summand.*

Proof. Suppose that $\operatorname{dist}_e(\phi)=0$. In this case there is an essential arc γ in Σ such that $d_e(\gamma,\phi(\gamma))=0$. That is ϕ can be assumed to fix γ. Since γ is essential there is an embedded closed curve γ' in Σ that intersects γ exactly once and the intersection is transverse. Notice that $\gamma\times[0,1]$ in $M_{(\Sigma,\phi)}$ is a 2–sphere S (see Subsection 2.1 to recall the definition of $M_{(\Sigma,\phi)}$). We can think of γ' as sitting on one of the pages of the open book and this gives a simple closed curve in $M_{(\Sigma,\phi)}$ that intersects S exactly once. Let N be a neighborhood of $S\cup\gamma'$ in $M_{(\Sigma,\phi)}$ and N' be the closure of its complement. One can glue a closed 3–ball to N and N' to obtain closed 3–manifolds M and M'. It is now easy to see that $M_{(\Sigma,\phi)}=M\#M'$ and that $M=S^2\times S^1$. □

Lemma 3.2. *Let (Σ,ϕ) be an open book decomposition. Then $\operatorname{dist}(\phi)=0$ implies that either*

(1) *$M_{(\Sigma,\phi)}$ has an $S^2\times S^1$ summand,*
(2) *$M_{(\Sigma,\phi)}$ can be written as a non-trivial connected sum of two manifolds, or*

(3) *Σ can be written as the boundary sum of two surface $\Sigma_1 \natural \Sigma_2$, ϕ can be isotoped to preserve Σ_1 and Σ_2 and (Σ_i, ϕ_i) is an open book for S^3 for $i = 1$ or 2, where $\phi_i = \phi|_{\Sigma_i}$.*

The proof of this lemma is very similar to the proof of the previous lemma and is left to the reader. So we see from these two lemmas that the essential translation distance can tell us about the manifold $M_{(\Sigma,\phi)}$ whereas the translation distance can only tell us about the open book decomposition.

3.2. **Destabilizing open book decompositions.** The connection between the essential translation distance of diffeomorphism and destabilization of an open book decomposition is given in the following theorem.

Theorem 3.3. *Let (Σ, ϕ) be an open book decomposition that is right veering. If the 3–manifold $M_{(\Sigma,\phi)}$ associated to the open book does not have an $S^2 \times S^1$ summand, then (Σ, ϕ) (positively) destabilizes if and only if $\mathrm{dist}_e(\phi) = 1$.*

We note that the obvious analogous theorem for left veering monodromies and negative destabilization also holds. The proof is left to the reader. Before proving this theorem we note that Theorem 1.5 from the introduction is an obvious corollary.

Proof. We first assume that (Σ, ϕ) destabilizes. Thus there is an open book (Σ', ϕ') of which (Σ, ϕ) is a stabilization. More specifically Σ is Σ' with a 1–handle attached and $\phi = \phi' \circ \tau_\alpha$ for some α that intersects the co-core of the attached 1–handle once. Let γ denote this co-core.

Now in the definition of the arc complex choose the marked points on the boundary to contain $\partial\gamma$ if $\partial\gamma$ is contained in two separate boundary components and to be one of the endpoints of γ otherwise. In the latter case isotope one of the endpoints of γ so that it is also at a marked point. One may easily check that $d_e(\gamma, \phi(\gamma)) = 1$. Thus $\mathrm{dist}_e(\phi) \leq 1$. If $\mathrm{dist}_e(\phi) = 0$ then $M_{(\Sigma,\phi)}$ would have an $S^2 \times S^1$ summand by Lemma 3.1 and this is ruled out by the hypothesis of the theorem. Thus $\mathrm{dist}_e(\phi) = 1$.

Now suppose that $\mathrm{dist}_e(\phi) = 1$. Then there is an arc γ such that $d_e(\gamma, \phi(\gamma)) = 1$. That is γ and $\phi(\gamma)$ have disjoint interiors. If γ has both its endpoints at the same point then we can clearly isotope γ (moving one of its endpoints) so that γ is properly embedded and has distinct endpoints (recall our *important convention* above) and γ and $\phi(\gamma)$ are still disjoint. Now let α be the simple closed curve in Σ formed from $\gamma \cup \phi(\gamma)$ by connecting their endpoints and isotoping so that the curve is in the interior of Σ. Notice that due to the right veeringness of ϕ it is clear that we can choose α so that it intersects both γ and $\phi(\gamma)$ each exactly once. One may easily verify that $\tau_\alpha^{-1} \circ \phi$ fixes γ. Thus we may form a surface Σ' by cutting along γ and $\tau_\alpha^{-1} \circ \phi$ will induce a diffeomorphism ϕ' of Σ'. It is also clear that (Σ, ϕ) is a stabilization of (Σ', ϕ'). $\square$

Remark 3.4. We note that the proof actually shows that, under the hypothesis of the theorem, an arc is the co-core of a 1–handled used to stabilize an open book if and only if the essential distance between it and its image under ϕ is 1. The co-core of a stabilizing 1–handle will be called a *stabilizing arc*.

4. Non-destabilizable planar open book decompositions of S^3

In this section we will prove Theorem 1.1. To that end let Σ_n be a compact planar surface with $(n+1)$-boundary components, $n \geq 4$, and $\phi_n \colon \Sigma_n \to \Sigma_n$ be a diffeomorphism obtained as the composition of right handed Dehn twists along the curves shown in Figure 1. Since all other curves, except one, are disjoint, the conjugacy class of ϕ_n is independent of the order of the product. Let c_i denote the boundary components of Σ_n as indicated in Figure 1.

Lemma 4.1. *The open book (Σ_n, ϕ_n) supports the standard tight contact structure on S^3.*

Proof. Since ϕ_n is a composition of right handed Dehn twists it is well known that the supported contact structure is tight, [7]. Since S^3 has a unique tight contact structure, [4], we are left to see that (Σ_n, ϕ_n) is an open book for S^3. One may readily verify that Figure 2 is a Kirby diagram for $M_{(\Sigma_n,\phi_n)}$. After sliding the left most 0–framed unknot over right most 0–framed unknot (first

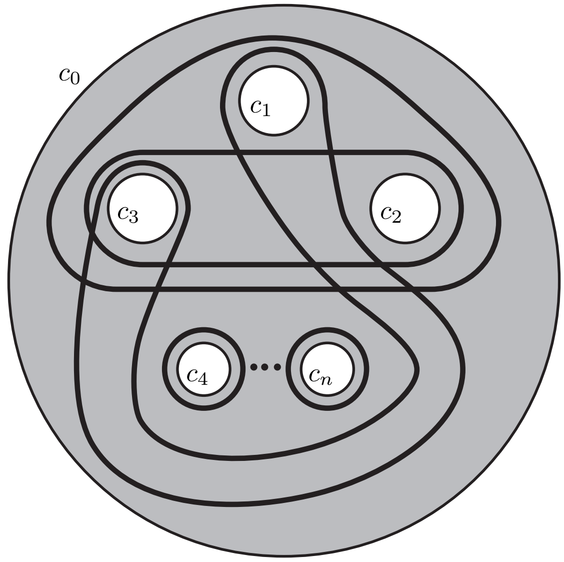

FIGURE 1. The surface Σ_n with boundary components $c_0, \ldots, c_n$ labeled.

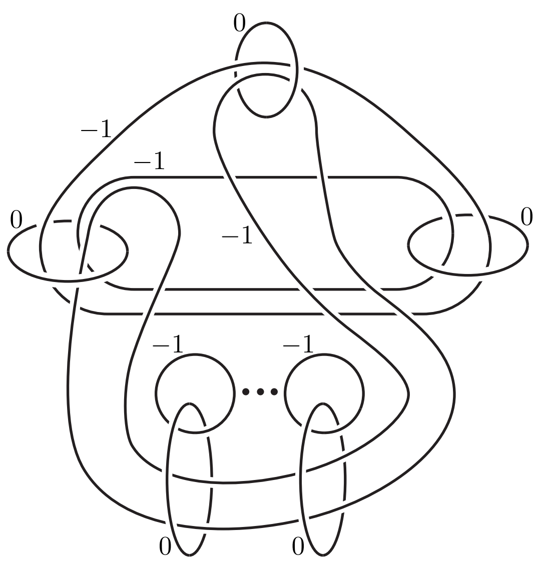

FIGURE 2. Surgery diagram for the manifold described by the open book decomposition (Σ_n, ϕ_n).

isotope the right unknot to be concentric with the left unknot) one sees that all but one non-zero framed curve has a zero framed meridian and so may be cancelled from the picture. Once this is done we are left with a Hopf link whose components are framed 0 and -1, thus giving S^3. □

To see that (Σ_n, ϕ_n) does not destabilize we compute the essential translation distance.

Theorem 4.2. *With the notation above*

$$\mathrm{dist}_e(\phi_n) = 2.$$

Before proving this we establish Theorem 1.1.

Proof of Theorem 1.1. The proof immediately follows from Theorem 4.2 and Theorem 1.5. □

Proof of Theorem 4.2. First note that if γ is a horizontal arc connecting boundary c_0 to c_2 then

$$d_e(\gamma, \phi_n(\gamma)) = 2$$

since they intersect but it is easy to find an arc disjoint from both. Thus $\mathrm{dist}_e(\phi_n) \leq 2$. We are left to see it is not 1 or 0. Of course it cannot be zero by Lemma 3.1. We will see that it is not 1 in Lemmas 4.3, 4.5 and 4.6 below. More specifically the lemmas show that for any arc γ we have $d_e(\gamma, \phi_n(\gamma)) > 1$. Notice that we only need to check arcs connecting different boundary components since on a planar surface a stabilizing arc cannot have both endpoints on the same boundary component since it would separate the surface. □

Lemma 4.3. *Let γ be any arc connecting c_0 to c_2 in Σ_n, then $d_e(\gamma, \phi_n(\gamma)) > 1$. Similarly any arc γ connecting c_i to c_j with $4 \leq i \neq j \leq n$ will have $d_e(\gamma, \phi_n(\gamma)) > 1$.*

Proof. Let (Σ', ϕ') be the open book obtained from (Σ_n, ϕ_n) by capping off all boundary components except c_0 and c_2. Notice that Σ' is an annulus and ϕ' is the square of the Dehn twist about the core curve in Σ'. Thus it is clear that $\mathrm{dist}_e(\phi') > 1$ and hence by Lemma 2.1 we know $d_e(\gamma, \phi_n(\gamma)) > 1$. The other cases are follow similarly. □

Remark 4.4. One can bound the distance most arcs in Σ_n are moved by ϕ_n by capping off all but two boundary components. This is much simpler than the arguments made in Lemmas 4.5 and 4.6, but this argument will not work for certain arcs thus one must resort to the more complicated arguments given below which subsume some of these easier cases too.

Lemma 4.5. *Let (Σ', ϕ') be obtained from (Σ_n, ϕ_n) by capping off c_2. Then $\mathrm{dist}_e(\phi') > 1$. In particular, any arc γ in Σ_n with endpoints on any boundary component except c_2 must satisfy $d_e(\gamma, \phi_n(\gamma)) > 1$.*

Proof. We will show that the open book obtained from (Σ', ϕ') by capping off all but one of the boundary components $c_4, \ldots, c_n$ satisfies $\mathrm{dist}_e(\phi'') > 1$. Combining this with Lemma 2.1 and 4.3 we will clearly have that $\mathrm{dist}_e(\phi') > 1$.

The open book decomposition (Σ'', ϕ'') is shown on the lefthand side of Figure 3. A Kirby diagram for the corresponding manifold is shown on the righthand side of Figure 3 and an easy

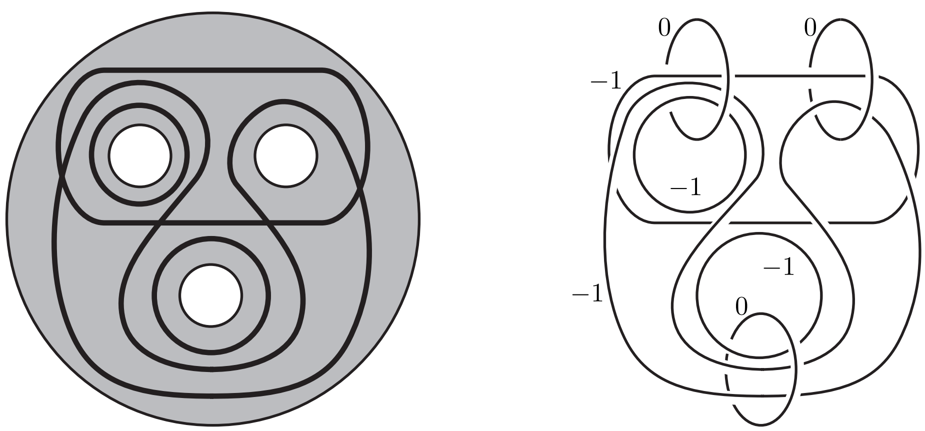

FIGURE 3. The open book decomposition (Σ'', ϕ'') on the lefthand side. On the righthand side is a Kirby picture for the manifold supported by (Σ'', ϕ'').

exercise in Kirby calculus yields that this manifold is the one obtained from (-2)–surgery on the right handed trefoil in S^3. Since all the Dehn twist are positive we also see that the contact structure is Stein fillable and hence tight [5, 7]. We claim that the open book is not a stabilization. If it were then it would be a stabilization of an open book decomposition (P, ψ), where P is a pair of pants and ψ is a product of some right handed Dehn twists, since if one ever used a left handed Dehn twist the monodromy would be left veering and the contact structure would be overtwisted [12]. Label the boundary components of P and suppose $\psi = \tau_0^p \tau_1^q \tau_2^r$, where τ_i is the right handed

Dehn twist along a simple closed curve parallel to the ith boundary component, and p, q, r are all nonnegative integers. Then the open book decomposition (P, ψ) supports the 3-manifold obtained by 0–surgery on the unknot followed by p, q and r surgeries on three meridians to the unknot. The first homology of this manifold has size $pq + qr + rp$. On the other hand, the first homology of (-2)–surgery on the trefoil is $\mathbb{Z}_2$. So $pq + qr + rp = 2$, in particular $\{p, q, r\} = \{0, 1, 2\}$. Hence (P, ψ) supports the lens space $L(2, 1)$. It is well known, see [15], that the only surgeries on the right handed trefoil knot yielding a Lens space are the 5 and 7 surgeries (in addition one can easily check that the fundamental group the manifold described by (Σ'', ϕ'') is not $\mathbb{Z}_2$). Thus (Σ'', ϕ'') does not destabilize. □

Lemma 4.6. *Let (Σ', ϕ') be obtained from (Σ_n, ϕ_n) by capping off c_0. Then $\mathrm{dist}_e(\phi') > 1$. In particular, any arc γ in Σ_n with endpoints on any boundary component except c_0 must satisfy $d_e(\gamma, \phi_n(\gamma)) > 1$.*

Proof. Arguing as in the proof of Lemma 4.5 it suffices to consider the open book (Σ'', ϕ'') obtained from (Σ', ϕ') by capping off all but one of the boundary components $c_4, \ldots, c_n$ satisfies $\mathrm{dist}_e(\phi'') > 1$. That open book decomposition is shown on the lefthand side of Figure 4 and supports a Stein

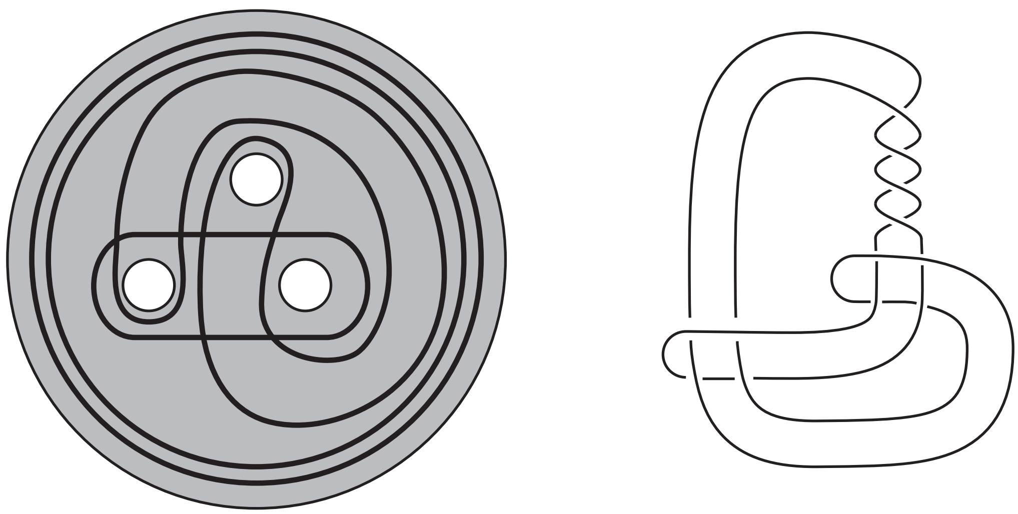

FIGURE 4. The open book decomposition (Σ'', ϕ'') on the lefthand side. On the righthand side is the knot K.

fillable contact structure on the 3–manifold. One can produce a Kirby picture for this manifold as we have done above and then an exercise in Kirby calculus, see Figure 5, shows this manifold is obtained from 2–surgery on the knot K shown on the righthand side of Figure 4.

The fundamental group of this manifold can be presented as

$$\langle a, b, c | acb^{-1}a^{-1}c^{-1}acbc^{-1}, aab^{-1}cbabc^{-1}, aba^{-1}b^{-1}cc \rangle.$$

(This may be worked out from a presentation of the fundamental group of the knot complement or using SnapPy.) Notice that adding the relations $a = 1$ and $b^3 = 1$ to the presentation gives a presentation for the dihedral group D_3 and so the group is not $\mathbb{Z}_2$. Hence by the same argument as in the proof of Lemma 4.5, (Σ'', ϕ'') is not a stabilization. □

5. DESTABILIZABLE OPEN BOOK DECOMPOSITIONS OF S^3

This section is devoted to showing that all planar open books with 4 or fewer boundary components that support the standard tight contact structure on S^3 destabilize.

Proof of Theorem 1.2. We begin by making some general observations about planar open book decompositions for the tight contact structure on S^3. Suppose (Σ, ϕ) is such a planar open book

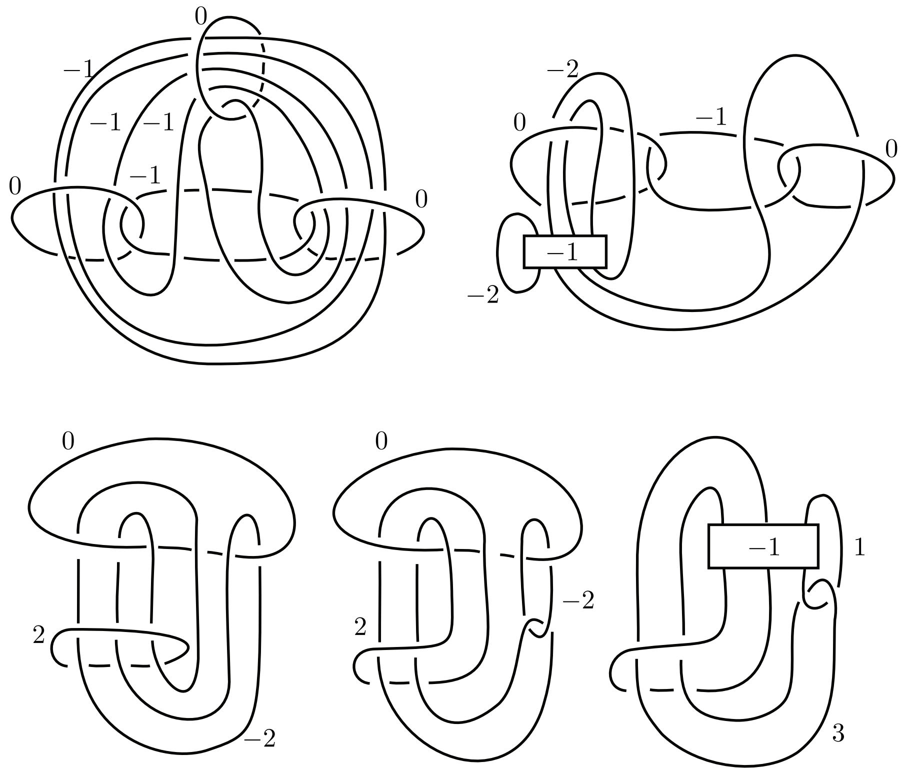

FIGURE 5. The Kirby Calculus to get from a surgery presentation of the manifold supported by (Σ'', ϕ''), shown in the upper left, to 2 surgery on the knot K. To go from the upper left figure to the upper right figure, slide the outer most (-1)–framed 2–handle over the parallel one, also slide the non-oval (-1)–framed 2–handle over the this 2–handle three times (twice over arcs contained in the leftmost 0–framed 2–handle and once over an arc in the uppermost 0–framed 2–handle), and finally cancel two 2–handles. To go from the upper right figure to the lower left, slide the non-oval 2–handle over the (-1)–framed 2–handle and cancel two 2–handles. To get to the middle diagram on the bottom row, just isotope. The final diagram on the lower right is then obtained by blowing up a $(+1)$–framed unknot to unlink the 0 and (-2)–framed handles, blowing down the resulting (-1)–framed handle and one of the $(+1)$–framed handles. Finally blowing down the $(+1)$–framed unknot in the lower right diagram yields 2–surgery on the knot K.

decomposition. Let $\partial\Sigma = c_0 \cup c_1 \cup \ldots \cup c_n$, $n \geq 0$. We will think of Σ as a disk with n disjoint sub-disks removed and think of c_0 as the boundary of the original disk.

Since (S^3, ξ_{std}) is Stein fillable we can use Corollary 2 in [20] to see that the monodromy ϕ has a positive factorization $\tau_{\gamma_1} \circ \tau_{\gamma_2} \circ \ldots \circ \tau_{\gamma_m}$, where γ_i, $i = 1, 2, \ldots, m$, is an essential simple closed curve in Σ. From this positive factorization, we can construct a Lefschetz fibration of a Stein filling of (S^3, ξ_{std}) whose Euler characteristic is $2 - (n+1) + m$, see [7]. According to [3], D^4 is the unique Stein filling of (S^3, ξ_{std}). Since $\chi(D^4) = 1$ we have $2 - (n+1) + m = 1$, and hence, $m = n$.

If γ_i, $1 \leq i \leq n$, encloses the boundary components $c_{i_1}, \ldots, c_{i_k}$, where $i_1, \ldots, i_k \in \{1, 2, \ldots, n\}$, then denoting the homology class of a curve c by $[c]$ we see that

$$[\gamma_i] = [c_{i_1}] + \ldots + [c_{i_k}]$$

in $H_1(\Sigma)$. Let M be the $n \times n$ matrix whose i, j entry is 1 if $[c_j]$ is part of the homology expansion of γ_i and 0 otherwise. Thus we see that $[\gamma_i] = \sum_j m_{i,j}[c_j]$. One may easily check that a presentation for the first homology of the 4–manifold described as a Lefschetz fibration by (Σ, ϕ), and the factorization of ϕ, is

$$\langle [c_1], \ldots, [c_n] | [\gamma_1], \ldots, [\gamma_n] \rangle.$$

Thus the homology will be trivial if and only if $\det M = \pm 1$.

We now return to the proof of the theorem. If Σ has 2 or 3 boundary components then it is clear that it must destabilize. So we are left to consider the case when Σ has 4 boundary components, that is when $n = 3$. By the above argument, there are 3 essential simple closed curves $\gamma_1, \gamma_2, \gamma_3$ such that $\phi = \tau_{\gamma_1} \circ \tau_{\gamma_2} \circ \tau_{\gamma_3}$. Moreover there is a 3×3 matrix M with entries 0's and 1's representing the homological relations between the γ_i and c_j. In addition $\det M = \pm 1$.

Notice that since Σ has only 4 boundary components, each γ_i is either parallel to one of the boundary components or bounds a region containing exactly two boundary components.

Case 1: each γ_i is boundary parallel. We get a destabilizing arc by taking an arc from the boundary component not parallel to a γ_i to one of the other boundary components.

Case 2: there are exactly two γ_i which are boundary parallel. Without loss of generality we suppose γ_1 and γ_2 are parallel to c_1 and c_2, respectively. We notice that γ_3 cannot enclose c_1 and c_2, since if it does $\det M \neq \pm 1$. So γ_3 enclose c_3 and one of c_1 and c_2, say c_1. We can now find a stabilizing arc connecting c_1 to c_3.

Case 3: each γ_i is not boundary parallel. Then each γ_i encloses 2 boundary components. Either two γ_i enclose the same two boundary components, or $\gamma_1, \gamma_2, \gamma_3$ enclose 3 different pairs of boundary components. In either cases, $\det M \neq \pm 1$.

Case 4: there is exactly one γ_i which is boundary parallel. Without loss of generality we suppose that γ_1 is parallel to the boundary component c_1. In order for $\det M$ to equal ± 1 we cannot have γ_2 and γ_3 enclosing the same boundary components. Up to diffeomorphism (and relabeling the boundary components), we can assume that γ_2 is a convex curve enclosing c_1 and c_2, see Figure 6. We will assume that γ_3 encloses c_2 and c_3 (the only other possibility is that it encloses c_1 and c_3, but

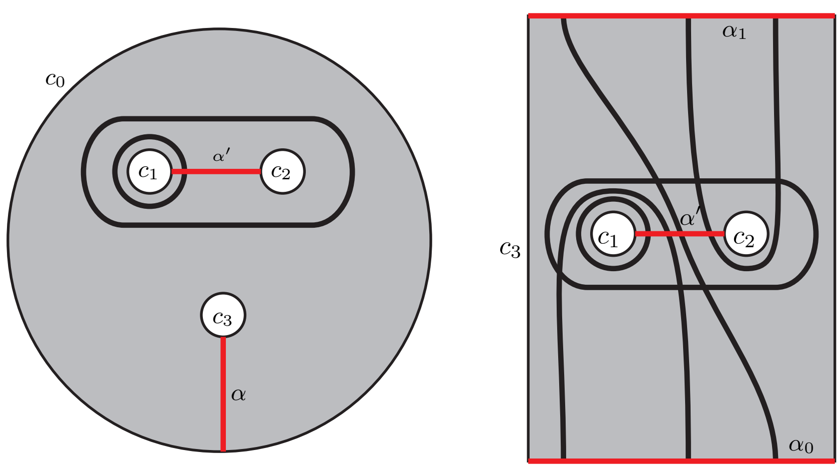

FIGURE 6. The surface Σ with boundary components labeled and the curves $\alpha, \alpha', \gamma_1$ and γ_2 on the lefthand side. On the righthand side is the surface cut open with a potential γ_3 curve.

in this case either γ_3 intersects α one time and we can destabilize the open book or it does not and an argument exactly like the one below will show that the resulting manifold is not S^3).

Let α be an arc in Σ which connects c_0 and c_3 and is disjoint from γ_2, see Figure 6. If γ_3 intersects α once then it is a stabilizing arc. We are left to consider the case when γ_3 intersects α more than once (we note that since γ_3 is homologous to $[c_1] + [c_3]$ the algebraic intersection with α is 1). We claim in this case that (Σ, ϕ) is not an open book for S^3. To see this notice that if ϕ' is ϕ without the Dehn twist about γ_3 then (Σ, ϕ') is an open book decomposition for $S^2 \times S^1$ and γ_3, sitting on one of the pages of the open book, is a knot in $S^2 \times S^1$, see Figure 7. Moreover, (-1)–surgery on γ_3 (here we mean that we are doing surgery with framing 1 less than the page framing) gives the manifold described by the open book (Σ, ϕ). Recall that Gabai [6] proved that the Property R conjecture is true, that is the only knot in $S^2 \times S^1$ that can be surgered to yield S^3 is the "trivial knot" $\{p\} \times S^1$. In particular if K is a knot in $S^2 \times S^1$ which can be surgered to give S^3 then its complement has fundamental group $\mathbb{Z}$. We will complete the proof of our claim by showing that the complement of γ_3 thought of as a knot in $S^2 \times S^1$ as above does not have fundamental group $\mathbb{Z}$.

Consider the arc α' that connects c_1 to c_2. Isotope γ_3 on Σ to intersect α and α' minimally. To understand how γ_3 sits in $S^2 \times S^1$ we proceed as follows. Let Σ' be Σ cut open along α. In the boundary of Σ' there are two copies of α which we call α_0 and α_1. Notice that γ_3 will be cut into a collection of arcs in Σ'. Each such arc is of one of three kinds. Its endpoints are either both on α_0, both on α_1 or and one end point on each. We call these Type I, II, and III arcs, respectively. Notice there must be an odd number of Type III arcs due to γ_3's homological intersection with α. Also notice that each Type I and II arc must intersect α' exactly once. (This is clear since if one were disjoint then that arc would "shield" c_1 and c_2 from all arcs of a different type and that will contradict the minimality of the intersection of γ_3 with α. Moreover if an arc intersected more than once then it will contradict the minimality of the intersection of γ_3 with α'.) We also notice that a Type III arc must also intersect α' exactly once since if it were disjoint then some of the Type I or II arcs would not intersect α' contradicting the above observation and if it intersected more than once we would again violate the minimality of the intersection of γ_3 and α'.

Let S_α be the non-separating 2-sphere in $S^2 \times S^1$ defined by α (see the proof of Lemma 3.1 for the construction of S_α). Let X denote the complement of γ_3 in $S^2 \times S^1$ and $F_\alpha = S_\alpha \cap X$. We claim that F_α is an incompressible surface in X. Since F_α is planar and has at least 3 boundary components the fundamental group of F_α is a free group on at least 2 generators. The incompressibility implies that $\pi_1(X)$ contains this free group and hence is not $\mathbb{Z}$. Thus our claim is complete once we see that F_α is incompressible.

Cutting $S^2 \times S^1$ open along S_α yields $S^2 \times [0, 1]$ and is depicted on the righthand side of Figure 7 along with some representative Type I, II and III arcs. While the arcs do not have to look exactly as in the figure the salient feature is that, due to the discussion above, they will all go through the leftmost box in the figure indicating the (-1)–twisting.

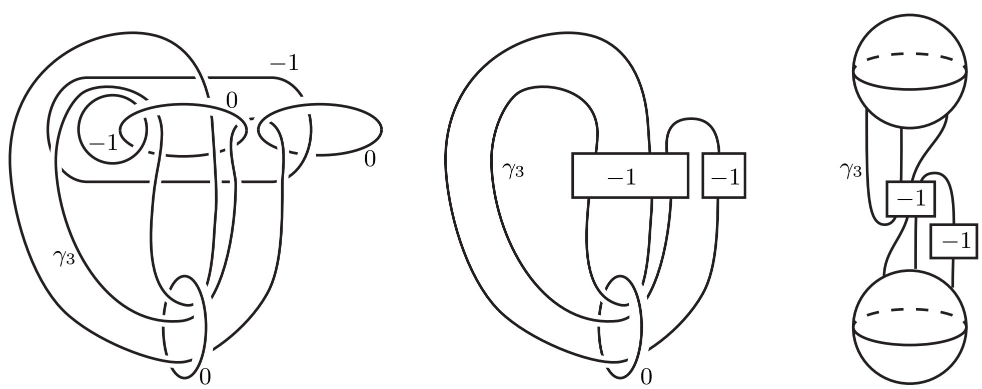

FIGURE 7. The 3–manifold $S^2 \times S^1$ described by the open book (Σ, ϕ') along with a sample γ_3 on the lefthand side. The same manifold in the center and on the right hand side this manifold is cut open along S_α. Each box indicates that a complete lefthanded twist has been added to the strands going through the box.

Now let D be a compressing disk for F_α in X. We can cut X along F_α to get a manifold $Y \subset S^2 \times [0, 1]$ with two copies of F_α in its boundary, we denote them by F_0 and F_1. Notice that D will be a disk in Y with boundary on either F_0 or F_1, without loss of generality assume it is on F_0. There is a disk D' in S_α such that $D \cup D'$ bounds a ball in $S^2 \times S^1$ (and in $S^2 \times S^1$ cut open along S_α). Notice that if a Type I arc has one endpoint in D', respectively $S_\alpha \setminus D'$, then the other endpoint has to be in D', respectively $S_\alpha \setminus D'$, too since the arc must be disjoint from D. Similarly all the endpoints of Type III arcs must be in $S_\alpha \setminus D'$ since they too must be disjoint from D. Finally we notice that some of the endpoints of Type I arcs must be in D' or else D is not a compressing disk for F_α. But this is a contradiction. Indeed glue two 3–balls to $S^2 \times S^1 \setminus S_\alpha$ to get S^3. We can close each Type I and II arc in these added 3–balls to get unknots and connect the endpoints of each Type III arc by a curve disjoint from all the arcs in the righthand diagram of Figure 7. We can also push D' slightly into one of the 3–balls so that it is disjoint from the added arcs. Now $D \cup D'$ is an embedded sphere in S^3 that bounds a ball containing some of the unknots associated to the Type I arcs and does not contain any of the unknots associated to the Type III arcs. Thus these respective unknots must be unlinked, but it is clear from the picture that they have linking ± 1. □

REFERENCES

[1] Sebastian Baader and Masaharu Ishikawa, *Legendrian graphs and quasipositive diagrams*, Ann. Fac. Sci. Toulouse Math. (6) **18** (2009), no. 2, 285–305. MR 2562830 (2011c:57005)

[2] Kenneth L. Baker, John B. Etnyre, and Jeremy Van Horn-Morris, *Cabling, contact structures and mapping class monoids*, J. Differential Geom. **90** (2012), no. 1, 1–80. MR 2891477

[3] Yakov Eliashberg, *Filling by holomorphic discs and its applications*, Geometry of low-dimensional manifolds, 2 (Durham, 1989), London Math. Soc. Lecture Note Ser., vol. 151, Cambridge Univ. Press, Cambridge, 1990, pp. 45–67. MR MR1171908 (93g:53060)

[4] ———, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 1-2, 165–192. MR MR1162559 (93k:57029)

[5] John B. Etnyre, *Lectures on open book decompositions and contact structures*, Floer homology, gauge theory, and low-dimensional topology, Clay Math. Proc., vol. 5, Amer. Math. Soc., Providence, RI, 2006, pp. 103–141. MR MR2249250

[6] David Gabai, *Foliations and the topology of 3-manifolds. III*, J. Differential Geom. **26** (1987), no. 3, 479–536. MR 910018 (89a:57014b)

[7] Emmanuel Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 405–414. MR MR1957051 (2004c:53144)

[8] Emmanuel Giroux and Noah Goodman, *On the stable equivalence of open books in three-manifolds*, Geom. Topol. **10** (2006), 97–114 (electronic). MR MR2207791 (2006k:57046)

[9] John Harer, *How to construct all fibered knots and links*, Topology **21** (1982), no. 3, 263–280. MR 649758 (83e:57007)

[10] John L. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. **84** (1986), no. 1, 157–176. MR 830043 (87c:32030)

[11] Matthew Hedden, *Notions of positivity and the Ozsváth-Szabó concordance invariant*, J. Knot Theory Ramifications **19** (2010), no. 5, 617–629. MR 2646650 (2011j:57020)

[12] Ko Honda, William H. Kazez, and Gordana Matić, *Right-veering diffeomorphisms of compact surfaces with boundary*, Invent. Math. **169** (2007), no. 2, 427–449. MR MR2318562 (2008e:57028)

[13] J. Johnson, *Heegaard splittings and open books*, ArXiv e-prints (2011). arXiv: 1110.2142

[14] P. M. Melvin and H. R. Morton, *Fibred knots of genus 2 formed by plumbing Hopf bands*, J. London Math. Soc. (2) **34** (1986), no. 1, 159–168. MR 859157 (87m:57006)

[15] Louise Moser, *Elementary surgery along a torus knot*, Pacific J. Math. **38** (1971), 737–745. MR 0383406 (52 #4287)

[16] Toshio Saito and Ryosuke Yamamoto, *Complexity of open book decompositions via arc complex*, J. Knot Theory Ramifications **19** (2010), no. 1, 55–69. MR 2640991 (2011e:57030)

[17] John R. Stallings, *Constructions of fibred knots and links*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978, pp. 55–60. MR MR520522 (80e:57004)

[18] W. P. Thurston and H. E. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. **52** (1975), 345–347. MR MR0375366 (51 #11561)

[19] Andy Wand, *Factorizations of diffeomorphisms of compact surfaces with boundary*, (2009).arXiv: 0910.5691v3

[20] Chris Wendl, *Strongly fillable contact manifolds and J-holomorphic foliations*, Duke Math. J. **151** (2010), no. 3, 337–384. MR 2605865 (2011e:53147)

SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY
E-mail address: etnyre@math.gatech.edu
URL: http://www.math.gatech.edu/~etnyre

DEPARTMENT OF MATHEMATICS, SHANGHAI JIAO TONG UNIVERSITY
E-mail address: liyoulin@sjtu.edu.cn