CABLING AND TRANSVERSE SIMPLICITY

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ABSTRACT. We study Legendrian knots in a cabled knot type. Specifically, given a topological knot type $\mathcal{K}$, we analyze the Legendrian knots in knot types obtained from $\mathcal{K}$ by cabling, in terms of Legendrian knots in the knot type $\mathcal{K}$. As a corollary of this analysis, we show that the $(2,3)$-cable of the $(2,3)$-torus knot is *not transversely simple* and moreover classify the transverse knots in this knot type. This is the first classification of transverse knots in a non-transversely-simple knot type. We also classify Legendrian knots in this knot type and exhibit the first example of a Legendrian knot that does not destabilize, yet its Thurston-Bennequin invariant is not maximal among Legendrian representatives in its knot type.

1. INTRODUCTION

In this paper we continue the investigation of Legendrian knots in tight contact 3-manifolds using 3-dimensional contact-topological methods. In [EH1], the authors introduced a general framework for analyzing Legendrian knots in tight contact 3-manifolds. There we streamlined the proof of the classification of Legendrian unknots, originally proved by Eliashberg-Fraser in [EF], and gave a complete classification of Legendrian torus knots and figure eight knots. In [EH2], we gave the first structure theorem for Legendrian knots, namely the reduction of the analysis of connected sums of Legendrian knots to that of the prime summands. This yielded a plethora of non-Legendrian-simple knot types. (A topological knot type is *Legendrian simple* if Legendrian knots in this knot type are determined by their Thurston–Bennequin invariant and rotation number.) Moreover, we exhibited pairs of Legendrian knots in the same topological knot type with the same Thurston-Bennequin and rotation numbers, which required arbitrarily many stabilizations before they became Legendrian isotopic (see [EH2]).

The goal of the current paper is to extend the results obtained for Legendrian torus knots to Legendrian representatives of cables of knot types we already understand. On the way to this goal, we encounter the *contact width*, a new knot invariant which is related to the maximal Thurston–Bennequin invariant. It turns out that the structure theorems for cabled knots types are not as simple as one might expect, and rely on properties associated to the contact width of a knot. When these properties are not satisfied, a rather unexpected and surprising phenomenon occurs for Legendrian cables. This phenomenon allows us to show, for example, that the $(2,3)$-cable of the $(2,3)$-torus knot is not transversely simple! (A topological knot type is *transversely simple* if transverse knots in that knot type are determined by their self-linking number.) Knots which are not transversely simple were also recently found in the work of Birman and Menasco [BM]. Using braid-theoretic techniques they showed that many three-braids are not transversely simple.

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Our technique should also provide infinite families of non-transversely-simple knots (essentially certain cables of positive torus knots), but for simplicity we content ourselves with the above-mentioned example. Moreover, we give a complete classification of transverse (and Legendrian) knots for the $(2,3)$-cable of the $(2,3)$-torus knot. This is the first classification of transverse knots in a non-transversely-simple knot type.

We assume that the reader has familiarity with [EH1]. In this paper, the ambient 3-manifold is the standard tight contact (S^3, ξ_{std}), and all knots and knot types are *oriented*. Let $\mathcal{K}$ be a topological knot type and $\mathcal{L}(\mathcal{K})$ be the set of Legendrian isotopy classes of $\mathcal{K}$. For each $[L] \in \mathcal{L}(\mathcal{K})$ (we often write L to mean $[L]$), there are two so-called *classical invariants*, the *Thurston-Bennequin invariant* $tb(L)$ and the *rotation number* $r(L)$. To each $\mathcal{K}$ we may associate an oriented knot invariant

$$\overline{tb}(\mathcal{K}) = \max_{L \in \mathcal{L}(\mathcal{K})} tb(L),$$

called the *maximal Thurston-Bennequin number*.

A close cousin of $\overline{tb}(\mathcal{K})$ is another oriented knot invariant called the *contact width* $w(\mathcal{K})$ (or simply the *width*) defined as follows: First, an embedding $\phi : S^1 \times D^2 \hookrightarrow S^3$ is said to *represent* $\mathcal{K}$ if the core curve of $\phi(S^1 \times D^2)$ is isotopic to $\mathcal{K}$. (For notational convenience, we will suppress the distinction between $S^1 \times D^2$ and its image under ϕ.) Next, in order to measure the *slope* of homotopically nontrivial curves on $\partial(S^1 \times D^2)$, we make a (somewhat nonstandard) oriented identification $\partial(S^1 \times D^2) \simeq \mathbb{R}^2/\mathbb{Z}^2$, where the meridian has slope 0 and the longitude (well-defined since $\mathcal{K}$ is inside S^3) has slope ∞. We will call this coordinate system $\mathcal{C}_{\mathcal{K}}$. Finally we define

$$w(\mathcal{K}) = \sup \frac{1}{\text{slope}(\Gamma_{\partial(S^1 \times D^2)})},$$

where the supremum is taken over $S^1 \times D^2 \hookrightarrow S^3$ representing $\mathcal{K}$ with $\partial(S^1 \times D^2)$ convex.

Note that there are several notions similar to $w(\mathcal{K})$ — see [Co, Ga]. The contact width clearly satisfies the following inequality:

$$\overline{tb}(\mathcal{K}) \leq w(\mathcal{K}) \leq \overline{tb}(\mathcal{K}) + 1.$$

In general, it requires significantly more effort to determine $w(\mathcal{K})$ than it does to determine $\overline{tb}(\mathcal{K})$. Observe that $\overline{tb}(\mathcal{K}) = -1$ and $w(\mathcal{K}) = 0$ when $\mathcal{K}$ is the unknot.

1.1. **Cablings and the uniform thickness property.** Recall that a (p,q)-*cable* $\mathcal{K}_{(p,q)}$ of a topological knot type $\mathcal{K}$ is the isotopy class of a knot of slope $\frac{q}{p}$ on the boundary of a solid torus $S^1 \times D^2$ which represents $\mathcal{K}$, where the slope is measured with respect to $\mathcal{C}_{\mathcal{K}}$, defined above. In other words, a representative of $\mathcal{K}_{(p,q)}$ winds p times around the meridian of $\mathcal{K}$ and q times around the longitude of $\mathcal{K}$. A (p,q)-*torus knot* is the (p,q)-cable of the unknot.

One would like to classify Legendrian knots in a cabled knot type. This turns out to be somewhat subtle and relies on the following key notion:

Uniform thickness property (UTP). Let $\mathcal{K}$ be a topological knot type. Then $\mathcal{K}$ satisfies the *uniform thickness condition* or *is uniformly thick* if the following hold:

(1) $\overline{tb}(\mathcal{K}) = w(\mathcal{K})$.
(2) Every embedded solid torus $S^1 \times D^2 \hookrightarrow S^3$ representing $\mathcal{K}$ can be thickened to a *standard neighborhood* of a maximal tb Legendrian knot.

Here, a *standard neighborhood* $N(L)$ of a Legendrian knot L is an embedded solid torus with core curve L and convex boundary $\partial N(L)$ so that $\#\Gamma_{\partial N(L)} = 2$ and $tb(L) = \frac{1}{\text{slope}(\Gamma_{\partial N(L)})}$. Such a standard neighborhood $N(L)$ is contact isotopic to any sufficiently small tubular neighborhood N of L with ∂N convex and $\#\Gamma_{\partial N} = 2$. (See [H1].) Note that, strictly speaking, Condition 2 implies Condition 1; it is useful to keep in mind, however, that the verification of the UTP usually proceeds by outlawing solid tori representing $\mathcal{K}$ with $\frac{1}{\text{slope}(\Gamma)} > \overline{tb}(\mathcal{K})$ and then showing that solid tori with $\frac{1}{\text{slope}(\Gamma)} < \overline{tb}(\mathcal{K})$ can be thickened properly. We will often say that a solid torus N (with convex boundary) representing $\mathcal{K}$ *does not admit a thickening*, if there is no thickening $N' \supset N$ whose $\text{slope}(\Gamma_{\partial N'}) \neq \text{slope}(\Gamma_{\partial N})$.

The reason for introducing the UTP is due (in part) to:

Theorem 1.1. *Let $\mathcal{K}$ be a knot type which is Legendrian simple and satisfies the UTP. Then $\mathcal{K}_{(p,q)}$ is Legendrian simple and admits a classification in terms of the classification of $\mathcal{K}$.*

Of course this theorem is of no use if we cannot find knots satisfying the UTP. The search for such knot types has an inauspicious start as we first observe that the unknot $\mathcal{K}$ *does not* satisfy the UTP, since $\overline{tb}(\mathcal{K}) = -1$ and $w(\mathcal{K}) = 0$. In spite of this we have the following theorems:

Theorem 1.2. *Negative torus knots satisfy the UTP.*

Theorem 1.3. *If a knot type $\mathcal{K}$ satisfies the UTP, then (p, q)-cables $\mathcal{K}_{(p,q)}$ satisfies the UTP, provided $\frac{p}{q} < w(\mathcal{K})$.*

We sometimes refer to a slope $\frac{p}{q}$ as "sufficiently negative" if $\frac{p}{q} < w(\mathcal{K})$. Moreover, if $\frac{p}{q} > w(\mathcal{K})$ then we call the slope "sufficiently positive".

Theorem 1.4. *If two knot types $\mathcal{K}_1$ and $\mathcal{K}_2$ satisfy the UTP, then their connected sum $\mathcal{K}_1\#\mathcal{K}_2$ satisfies the UTP.*

In Section 3 we give a more precise description and a proof of Theorem 1.1 and in Section 4 we prove Theorems 1.2 through 1.4 (the positive results on the UTP).

1.2. **New phenomena.** While negative torus knots are well-behaved, positive torus knots are more unruly:

Theorem 1.5. *There are positive torus knots that do not satisfy the UTP.*

It is not too surprising that positive torus knots and negative torus knots have very different behavior — recall that we also had to treat the positive and negative cases separately in the proof of the classification of Legendrian torus knots in [EH1]. A slight extension of Theorem 1.5 yields the following:

Theorem 1.6. *There exist a knot type $\mathcal{K}$ and a Legendrian knot $L \in \mathcal{L}(\mathcal{K})$ which does not admit any destabilization, yet satisfies $tb(L) < \overline{tb}(\mathcal{K})$.*

Although the phenomenon that appears in Theorem 1.6 is rather common, we will specifically treat the case when $\mathcal{K}$ is a $(2, 3)$-cable of a $(2, 3)$-torus knot. The same knot type $\mathcal{K}$ is also the example in the following theorem:

Theorem 1.7. *Let $\mathcal{K}$ be the $(2,3)$-cable of the $(2,3)$-torus knot. There is a unique transverse knot in $\mathcal{T}(\mathcal{K})$ for each self-linking number n, where $n \leq 7$ is an odd integer $\neq 3$, and exactly two transverse knots in $\mathcal{T}(\mathcal{K})$ with self-linking number 3. In particular, $\mathcal{K}$ is not transversely simple.*

Here $\mathcal{T}(\mathcal{K})$ is the set of transverse isotopy classes of $\mathcal{K}$.

Previously, Birman and Menasco [BM] produced non-transversely-simple knot types by exploiting an interesting connection between transverse knots and closed braids. It should be noted that our theorem contradicts results of Menasco in [M1]. However, this discrepancy has led Menasco to find subtle and interesting properties of cabled braids (see [M2]). The earlier work of Birman-Menasco [BM] and our Theorem 1.7 both give negative answers to a long-standing question of whether the self-linking number and the topological type of a transverse knot determine the knot up to contact isotopy. The corresponding question for Legendrian knots, namely whether every topological knot type $\mathcal{K}$ is Legendrian simple, has been answered in the negative in the works of Chekanov [Ch] and Eliashberg-Givental-Hofer [EGH]. Many other non-Legendrian-simple knot types have been found since then (see for example [Ng, EH2]).

The theorem which bridges the Legendrian classification and the transverse classification is the following theorem from [EH1]:

Theorem 1.8. *Transverse simplicity is equivalent to* stable simplicity, i.e., *any two $L_1, L_2 \in \mathcal{L}(\mathcal{K})$ with the same tb and r become contact isotopic after some number of positive stabilizations.*

The problem of finding a knot type which is not stably simple is much more difficult than the finding a knot type which is not Legendrian simple, especially since the Chekanov-Eliashberg contact homology invariants vanish on stabilized knots. Our technique for distinguishing stabilizations of Legendrian knots is to use the standard cut-and-paste contact topology techniques, and, in particular, the method of *state traversal*.

Theorems 1.5 through 1.6 will be proven in Section 5 while Theorem 1.7 is be proven in Section 6. More specifically, the discussion in Section 6 provides a complete classification of $(2,3)$-cables of $(2,3)$-torus knots.

Theorem 1.9. *If $\mathcal{K}'$ is the $(2,3)$-cable of the $(2,3)$-torus knot, then $\mathcal{L}(\mathcal{K}')$ is classified as in Figure 1. This entails the following:*

1. *There exist exactly two maximal Thurston-Bennequin representatives $K_\pm \in \mathcal{L}(\mathcal{K}')$. They satisfy $tb(K_\pm) = 6$, $r(K_\pm) = \pm 1$.*
2. *There exist exactly two non-destabilizable representatives $L_\pm \in \mathcal{L}(\mathcal{K}')$ which have non-maximal Thurston-Bennequin invariant. They satisfy $tb(L_\pm) = 5$ and $r(L_\pm) = \pm 2$.*
3. *Every $L \in \mathcal{L}(\mathcal{K}')$ is a stabilization of one of K_+, K_-, L_+, or L_-.*
4. *$S_+(K_-) = S_-(K_+)$, $S_-(L_-) = S_-^2(K_-)$, and $S_+(L_+) = S_+^2(K_+)$.*
5. *$S_+^k(L_-)$ is not (Legendrian) isotopic to $S_+^k S_-(K_-)$ and $S_-^k(L_+)$ is not isotopic to $S_-^k S_+(K_+)$, for all positive integers k. Also, $S_-^2(L_-)$ is not isotopic to $S_+^2(L_+)$.*

2. PRELIMINARIES

Throughout this paper, a convex surface Σ is either closed or compact with Legendrian boundary, Γ_Σ is the dividing set of Σ, and $\#\Gamma_\Sigma$ is the number of connected components of Γ_Σ.

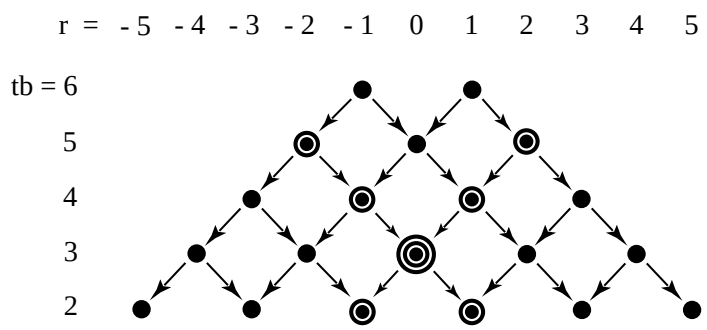

FIGURE 1. Classification of Legendrian $(2,3)$-cables of $(2,3)$-torus knots. Concentric circles indicate multiplicities, *i.e.*, the number of distinct isotopy classes with a given r and tb.

2.1. **Framings.** For convenience we relate the framing conventions that are used throughout the paper. In what follows, $X \setminus Y$ will denote the metric closure of the complement of Y in X.

Let $\mathcal{K}$ be a topological knot type and $\mathcal{K}_{(p,q)}$ be its (p,q)-cable. Let $N(\mathcal{K})$ be a solid torus which represents $\mathcal{K}$. Suppose $K_{(p,q)} \in \mathcal{K}_{(p,q)}$ sits on $\partial N(\mathcal{K})$. Take an oriented annulus A with boundary on $\partial N(\mathcal{K}_{(p,q)})$ so that $(\partial N(\mathcal{K}_{(p,q)})) \setminus A$ consists of two disjoint annuli Σ_1, Σ_2 and $A \cup \Sigma_i$, $i = 1, 2$, is isotopic to $\partial N(\mathcal{K})$. We define the following coordinate systems, *i.e.*, identifications of tori with $\mathbb{R}^2/\mathbb{Z}^2$.

(1) $\mathcal{C}_{\mathcal{K}}$, the coordinate system on $\partial N(\mathcal{K})$ where the (well-defined) longitude has slope ∞ and the meridian has slope 0.
(2) $\mathcal{C}'_{\mathcal{K}}$, the coordinate system on $\partial N(\mathcal{K}_{(p,q)})$ where the meridian has slope 0 and slope ∞ is given by $A \cap \partial N(\mathcal{K}_{(p,q)})$.

We now explain how to relate the framings $\mathcal{C}'_{\mathcal{K}}$ and $\mathcal{C}_{\mathcal{K}_{(p,q)}}$ for $\partial N(\mathcal{K}_{(p,q)})$. Suppose $K_{(p,q)} \in \mathcal{K}_{(p,q)}$ is contained in $\partial N(\mathcal{K})$. Then the Seifert surface $\Sigma(K_{(p,q)})$ is obtained by taking p parallel copies of the meridional disk of $N(\mathcal{K})$ (whose boundary we assume are p parallel closed curves on $\partial N(\mathcal{K})$ of slope 0) and q parallel copies of the Seifert surface for $\mathcal{K}$ (whose boundary we assume are q parallel closed curves on $\partial N(\mathcal{K})$ of slope ∞), and attaching a band at each intersection between the slope 0 and slope ∞ closed curves for a total of $|pq|$ bands. Therefore, the framing coming from $\mathcal{C}'_{\mathcal{K}}$ and the framing coming from $\mathcal{C}_{\mathcal{K}_{(p,q)}}$ differ by pq; more precisely, if $L_{(p,q)} \in \mathcal{L}(\mathcal{K}_{(p,q)})$ and $t(L_{(p,q)}, \mathcal{F})$ is the *twisting number* with respect to the framing $\mathcal{F}$ (or the *Thurston-Bennequin invariant* with respect to $\mathcal{F}$), then:

$$t(L_{(p,q)}, \mathcal{C}'_{\mathcal{K}}) + pq = t(L_{(p,q)}, \mathcal{C}_{\mathcal{K}_{(p,q)}}) = tb(L_{(p,q)}). \tag{1}$$

Let us also define the *maximal twisting number* of $\mathcal{K}$ with respect to $\mathcal{F}$ to be:

$$\overline{t}(\mathcal{K}, \mathcal{F}) = \max_{L \in \mathcal{L}(\mathcal{K})} t(L, \mathcal{F}).$$

2.2. **Computations of tb and r.** Suppose $L_{(p,q)} \in \mathcal{L}(\mathcal{K}_{(p,q)})$ is contained in $\partial N(\mathcal{K})$, which we assume to be convex. We compute $tb(L_{(p,q)})$ for two typical situations; the proof is an immediate consequence of Equation 1.

Lemma 2.1.

(1) *Suppose $L_{(p,q)}$ is a Legendrian divide and slope$(\Gamma_{\partial N(\mathcal{K})}) = \frac{q}{p}$. Then $tb(L_{(p,q)}) = pq$.*

(2) *Suppose $L_{(p,q)}$ is a Legendrian ruling curve and slope$(\Gamma_{\partial N(\mathcal{K})}) = \frac{q'}{p'}$. Then $tb(L_{(p,q)}) = pq - |pq' - qp'|$.*

Next we explain how to compute the rotation number $r(L_{(p,q)})$.

Lemma 2.2. *Let D be a convex meridional disk of $N(\mathcal{K})$ with Legendrian boundary on a contact-isotopic copy of the convex surface $\partial N(\mathcal{K})$, and let $\Sigma(L)$ be a convex Seifert surface with Legendrian boundary $L \in \mathcal{L}(\mathcal{K})$ which is contained in a contact-isotopic copy of $\partial N(\mathcal{K})$. (Here the isotopic copies of $\partial N(\mathcal{K})$ are copies inside an I-invariant neighborhood of $\partial N(\mathcal{K})$, obtained by applying the Flexibility Theorem to $\partial N(\mathcal{K})$.) Then*

$$r(L_{(p,q)}) = p \cdot r(\partial D) + q \cdot r(\partial \Sigma(K)).$$

Proof. Take p parallel copies $D_1, \dots, D_p$ of D and q parallel copies $\Sigma(K)_1, \dots, \Sigma(K)_q$ of $\Sigma(K)$. The key point is to use the Legendrian realization principle [H1] simultaneously on ∂D_i, $i = 1, \dots, p$, and $\partial \Sigma(K)_j$, $j = 1, \dots, q$. Provided slope$(\Gamma_{\partial N(\mathcal{K})}) \neq \infty$, the Legendrian realization principle allows us to perturb $\partial N(\mathcal{K})$ so that (i) $(\bigcup_{i=1,\dots,p} \partial D_i) \cup (\bigcup_{j=1,\dots,q} \partial \Sigma(K)_j)$ is a Legendrian graph in $\partial N(\mathcal{K})$ and (ii) each ∂D_i and $\partial \Sigma(K)_j$ intersects $\Gamma_{\partial N(\mathcal{K})}$ *efficiently, i.e.*, in a manner which minimizes the geometric intersection number. (The version of Legendrian realization described in [H1] is stated only for multicurves, but the proof for nonisolating graphs is identical.) Now, suppose $L'_{(p,q)} \in \mathcal{L}(\mathcal{K}_{(p,q)})$ and its Seifert surface $\Sigma(L'_{(p,q)})$ are constructed by resolving the intersections of $(\bigcup_{i=1,\dots,p} \partial D_i) \cup (\bigcup_{j=1,\dots,q} \partial \Sigma(K)_j)$. Recalling that the rotation number is a homological quantity (a relative half-Euler class) [H1], we readily compute that

$$r(L'_{(p,q)}) = p \cdot r(\partial D) + q \cdot r(\partial \Sigma(K)).$$

(For more details on a similar computation, see [EH1].) Finally, $L_{(p,q)}$ is obtained from $L'_{(p,q)}$ by resolving the inefficient intersections between $L'_{(p,q)}$ and $\Gamma_{\partial N(\mathcal{K})}$. Since $\partial N(\mathcal{K})$ is a torus and $\Gamma_{\partial N(\mathcal{K})}$ consists of two parallel essential curves, the inefficient intersections come in pairs, and have no net effect on the rotation number computation. This proves the lemma. □

3. From the UTP to classification

In this section we use Theorem 1.3 to give a complete classification of $\mathcal{L}(\mathcal{K}_{(p,q)})$, provided $\mathcal{L}(\mathcal{K})$ is classified, $\mathcal{K}$ satisfies the UTP, and $\mathcal{K}$ is Legendrian simple. In summary, we show:

Theorem 3.1. *If $\mathcal{K}$ is Legendrian simple and satisfies the UTP, then all its cables are Legendrian simple.*

The form of classification for Legendrian knots in the cabled knot types depends on whether or not the cabling slope $\frac{p}{q}$ is greater or less than $w(\mathcal{K})$. The precise classification for sufficiently positive slopes is given in Theorem 3.2, while the classification for sufficiently negative slopes is given in Theorem 3.6.

In particular, these results yield a complete classification of Legendrian iterated torus knots, provided each iteration is sufficiently negative (so that the UTP is preserved). We follow the strategy for classifying Legendrian knots as outlined in [EH1].

Suppose $\mathcal{K}$ satisfies the UTP and is Legendrian simple. By the UTP, every Legendrian knot $L \in \mathcal{L}(\mathcal{K})$ with $tb(L) < \overline{tb}(\mathcal{K})$ can be destabilized to one realizing $\overline{tb}(\mathcal{K})$. The Bennequin inequality [Be] gives bounds on the rotation number; hence there are only finitely many distinct $L \in \mathcal{L}(\mathcal{K})$, say $L_0, \ldots, L_n$, which have $tb(L_i) = \overline{tb}(\mathcal{K})$, $i = 0, \ldots, n$. Write $r_i = r(L_i)$, and assume $r_0 < r_1 < \cdots < r_n$. By symmetry, $r_i = -r_{n-i}$. (This is easiest to see in the front projection by rotating about the x-axis, if the contact form is $dz - ydx$.) Now, every time a Legendrian knot L is stabilized by adding a zigzag, its tb decreases by 1 and its r either increases by 1 (positive stabilization $S_+(L)$) or decreases by 1 (negative stabilization $S_-(L)$). Hence the image of $\mathcal{L}(K)$ under the map (r, tb) looks like a mountain range, where the peaks are all of the same height $\overline{tb}(\mathcal{K})$, situated at $r_0, \ldots, r_n$. The slope to the left of the peak is $+1$ and the slope to the right is -1, and the slope either continues indefinitely or hits a slope of the opposite sign descending from an adjacent peak to create a valley. See Figure 2.

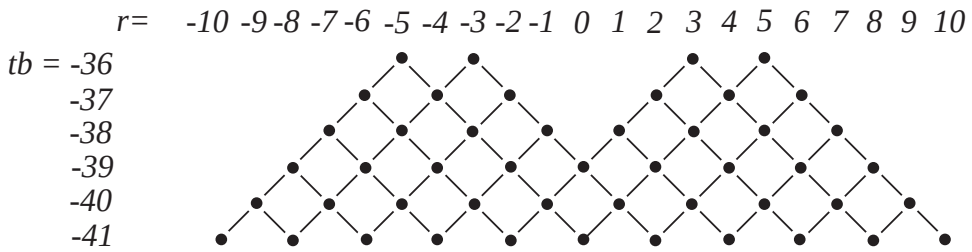

FIGURE 2. The (r, tb)-mountain range for the $(-9, 4)$-torus knot.

The following notation will be useful in the next few results. Given two slopes $s = \frac{r}{t}$ and $s' = \frac{r'}{t'}$ on a torus T with r, t relatively prime and r', t' relatively prime, we denote:

$$s \bullet s' = rt' - tr'.$$

This quantity is the minimal number of intersections between two curves of slope s and s' on T.

Theorem 3.2. *Suppose $\mathcal{K}$ is Legendrian simple and satisfies the UTP. If p, q are relatively prime integers with $\frac{p}{q} > w(\mathcal{K})$, then $\mathcal{K}_{(p,q)}$ is also Legendrian simple. Moreover,*

$$\overline{tb}(\mathcal{K}_{(p,q)}) = pq - \left| w(\mathcal{K}) \bullet \frac{p}{q} \right|,$$

and the set of rotation numbers realized by $\{L \in \mathcal{L}(\mathcal{K}_{(p,q)}) | tb(L) = \overline{tb}(\mathcal{K}_{(p,q)})\}$ is

$$\{q \cdot r(L) | L \in \mathcal{L}(\mathcal{K}), tb(L) = w(\mathcal{K})\}.$$

This theorem is established through the following three lemmas.

Lemma 3.3. *Under the hypotheses of Theorem 3.2, $\overline{tb}(\mathcal{K}_{(p,q)}) = pq - |w(\mathcal{K}) \bullet \frac{p}{q}|$ and any Legendrian knot $L \in \mathcal{L}(\mathcal{K}_{(p,q)})$ with $tb(L) < \overline{tb}(\mathcal{K}_{(p,q)})$ destabilizes.*

Proof. We first claim that $t(L, \mathcal{C}'_K) < 0$ for any $L \in \mathcal{L}(\mathcal{K}_{(p,q)})$. If not, there exists a Legendrian knot $L' \in \mathcal{L}(\mathcal{K}_{(p,q)})$ with $t(L', \mathcal{C}'_K) = 0$. Let S be a solid torus representing $\mathcal{K}$ such that $L' \subset \partial S$ (as a Legendrian divide) and the boundary torus ∂S is convex. Then slope$(\Gamma_{\partial S}) = \frac{q}{p}$ when measured with respect to $\mathcal{C}_K$. However, since $\frac{p}{q} > w(\mathcal{K})$, this contradicts the UTP.

Since $t(L,\mathcal{C}'_{\mathcal{K}}) < 0$, there exists an S so that $L \subset \partial S$ and ∂S is convex. Let s be the slope of $\Gamma_{\partial S}$. Then we have the following inequality:

$$\left|\frac{1}{s} \bullet \frac{p}{q}\right| \geq \left|w(\mathcal{K}) \bullet \frac{p}{q}\right|,$$

with equality if and only if $\frac{1}{s} = w(\mathcal{K})$. To see this, use an oriented diffeomorphism of the torus ∂S that sends slope 0 to 0 and slope $\frac{1}{w(\mathcal{K})} = \frac{1}{\overline{tb}(\mathcal{K})}$ to ∞ (this forces $-\infty \leq s < 0$ and $\frac{q}{p} > 0$), and compute determinants. (Alternatively, this follows from observing that there is an edge from 0 to $\frac{1}{w(\mathcal{K})}$ in the Farey tessellation, and $\frac{1}{s} \in (-\infty, w(\mathcal{K})]$, whereas $\frac{p}{q} \in (w(\mathcal{K}), \infty)$.) Thus $t(L,\mathcal{C}'_{\mathcal{K}}) \leq -|w(\mathcal{K}) \bullet \frac{p}{q}|$ for all $L \in \mathcal{L}(\mathcal{K}_{(p,q)})$. But now, if S is a solid torus representing $\mathcal{K}$ of maximal thickness, then a Legendrian ruling curve on ∂S easily realizes the equality. Converting from $\mathcal{C}'_{\mathcal{K}}$ to $\mathcal{C}_{\mathcal{K}}$, we obtain $\overline{tb}(\mathcal{K}_{(p,q)}) = pq - |w(\mathcal{K}) \bullet \frac{p}{q}|$.

Now consider a Legendrian knot $L \in \mathcal{L}(\mathcal{K}_{(p,q)})$ with $tb(L) < \overline{tb}(\mathcal{K}_{(p,q)})$. Placing L on a convex surface ∂S, if the intersection between L and $\Gamma_{\partial S}$ is not efficient (*i.e.*, does not realize the geometric intersection number), then there exists a bypass which allows us to destabilize L. Otherwise L is a Legendrian ruling curve on ∂S with $\frac{1}{s} \neq w(\mathcal{K})$. Now, since $\mathcal{K}$ satisfies the UTP, there is a solid torus S' with $S \subset S'$, where $\partial S'$ is convex and $\text{slope}(\Gamma_{\partial S'}) = \frac{1}{w(\mathcal{K})}$. By comparing with a Legendrian ruling curve of slope $\frac{q}{p}$, *i.e.*, taking a convex annulus $A = L \times [0,1]$ in $\partial S \times [0,1] = S' \setminus S$ and using the Imbalance Principle, we may easily find a bypass for L. Therefore, if $t(L,\mathcal{C}'_{\mathcal{K}}) < -|w(\mathcal{K}) \bullet \frac{p}{q}|$, then we may destabilize L. □

Lemma 3.4. *Under the hypotheses of Theorem 3.2, Legendrian knots with maximal tb in $\mathcal{L}(\mathcal{K}_{(p,q)})$ are determined by their rotation number. Moreover, the rotation numbers associated to maximal tb Legendrian knots in $\mathcal{L}(\mathcal{K}_{(p,q)})$ are*

$$\{q \cdot r(L) | L \in \mathcal{L}(\mathcal{K}), tb(L) = w(\mathcal{K})\}$$

Proof. Given a Legendrian knot $L \in \mathcal{L}(\mathcal{K}_{(p,q)})$ with maximal tb, there exists a solid torus S with convex boundary, where $\text{slope}(\Gamma_{\partial S}) = \frac{1}{w(\mathcal{K})}$ and L is a Legendrian ruling curve on ∂S. The torus S is a standard neighborhood of a Legendrian knot K in $\mathcal{L}(\mathcal{K})$. From Lemma 2.2 one sees that

$$r(L) = q \cdot r(K).$$

Thus the rotation number of L determines the rotation number of K.

If L and L' are two Legendrian knots in $\mathcal{L}(\mathcal{K}_{(p,q)})$ with maximal tb, then we have the associated solid tori S and S' and Legendrian knots K and K' as above. If L and L' have the same rotation numbers then so do K and K'. Since $\mathcal{K}$ is Legendrian simple, K and K' are Legendrian isotopic. Thus we may assume that K and K' are the same Legendrian knot and that S and S' are two standard neighborhoods of $K = K'$. Inside $S \cap S'$ we can find another standard neighborhood S'' of $K = K'$ with convex boundary having dividing slope $\frac{1}{w(\mathcal{K})}$ and ruling slope $\frac{q}{p}$. The sets $S \setminus S''$ and $S' \setminus S''$ are both diffeomorphic to $T^2 \times [0,1]$ and have $[0,1]$-invariant contact structures. Thus we can assume that L and L' are both ruling curves on $\partial S''$. One may now use the other ruling curves on $\partial S''$ to Legendrian isotop L to L'. □

Lemma 3.5. *Under the hypotheses of Theorem 3.2, Legendrian knots in $\mathcal{L}(\mathcal{K}_{(p,q)})$ are determined by their Thurston-Bennequin invariant and rotation number.*

Proof. Here one simply needs to see that there is a unique Legendrian knot in the valleys of the (r, tb)-mountain range; that is, if L and L' are maximal tb Legendrian knots in $\mathcal{L}(\mathcal{K}_{(p,q)})$ and $r(L) = r(L') + 2qn$ (note the difference in their rotation numbers must be even and a multiple of q) then $S_+^{qn}(L') = S_-^{qn}(L)$. To this end, let K and K' be the Legendrian knots in $\mathcal{L}(\mathcal{K})$ associated to L and L' as in the proof of the previous lemma. The knots K and K' have maximal tb and $r(K) = r(K') + 2n$. Since $\mathcal{K}$ is Legendrian simple we know $S_+^n(K') = S_-^n(K)$. Using the fact that $S_-^q(L)$ sits on a standard neighborhood of $S_-(K)$ (and the corresponding fact for K' and L') it easily follows that $S_+^{qn}(L') = S_-^{qn}(L)$. □

We now focus our attention on sufficiently negative cablings of a knot type $\mathcal{K}$.

Theorem 3.6. *Suppose $\mathcal{K}$ is Legendrian simple and satisfies the UTP. If p, q are relatively prime integers with $q > 0$ and $\frac{p}{q} < w(\mathcal{K})$, then $\mathcal{K}_{(p,q)}$ is also Legendrian simple. Moreover $\overline{tb}(\mathcal{K}_{(p,q)}) = pq$ and the set of rotation numbers realized by $\{L \in \mathcal{L}(\mathcal{K}_{(p,q)}) | tb(L) = \overline{tb}(\mathcal{K}_{(p,q)})\}$ is*

$$\{\pm(p + q(n + r(L))) \mid L \in \mathcal{L}(\mathcal{K}), tb(L) = -n\},$$

where n is the integer that satisfies

$$-n - 1 < \frac{p}{q} < -n.$$

We begin with two lemmas.

Lemma 3.7. *Under the hypotheses of Theorem 3.6, every $L_{(p,q)} \in \mathcal{L}(\mathcal{K}_{(p,q)})$ with $tb(L_{(p,q)}) < \overline{tb}(\mathcal{K}_{(p,q)})$ can be destabilized and*

$$\overline{tb}(\mathcal{K}_{(p,q)}) = pq.$$

Proof. By Theorem 1.3, $\mathcal{K}_{(p,q)}$ also satisfies the UTP. Therefore every $L_{(p,q)} \in \mathcal{L}(\mathcal{K}_{(p,q)})$ with $tb(L_{(p,q)}) < \overline{tb}(\mathcal{K}_{(p,q)})$ can be destabilized to a Legendrian knot realizing $\overline{tb}(\mathcal{K}_{(p,q)})$. Moreover, since $\frac{p}{q}$ is sufficiently negative, there exist $L_{(p,q)} \in \mathcal{L}(\mathcal{K}_{(p,q)})$ with $tb(L_{(p,q)}) = pq$, which appear as Legendrian divides on a convex torus $\partial N(\mathcal{K})$. By Lemma 2.1 we have $\overline{tb}(\mathcal{K}_{(p,q)}) \geq pq$. Equality (the hard part) follows from Claim 4.2 below. □

Lemma 3.8. *Under the hypotheses of Theorem 3.2, Legendrian knots with maximal tb in $\mathcal{L}(\mathcal{K}_{(p,q)})$ are determined by their rotation number. Moreover, the set of rotation numbers attained by $\{L_{(p,q)} \in \mathcal{L}(\mathcal{K}_{(p,q)}) \mid tb(L_{(p,q)}) = pq\}$ is*

$$\{\pm(p + q(n + r(L))) \mid L \in \mathcal{L}(\mathcal{K}), tb(L) = -n\}.$$

Another way of stating the range of rotation numbers (and seeing where they come from) in Lemma 3.8 is as follows: To each $L \in \mathcal{L}(\mathcal{K})$, there correspond two elements $L^{\pm} \in \mathcal{L}(\mathcal{K}_{(p,q)})$ with $tb(L^{\pm}) = pq$ and $r(L^{\pm}) = q \cdot r(L) \pm s$, where s is the remainder $s = -p - qn > 0$. $L^{\pm}$ is obtained by removing a standard neighborhood of $N(S_{\pm}(L))$ from $N(L)$, and considering a Legendrian divide on a torus with slope$(\Gamma) = \frac{q}{p}$ inside $T^2 \times [1, 2] = N(L) \setminus N(S_{\pm}(L))$.

Proof. The proof that Legendrian knots with maximal tb in $\mathcal{L}(\mathcal{K}_{(p,q)})$ are determined by their rotation numbers is similar to the proof of Lemma 3.4 (also see [EH1]).

The range of rotation numbers follows from Lemma 2.2 as well as some considerations of tight contact structures on thickened tori. First let $T_{1.5} = \partial N(\mathcal{K})$ which contains $L_{(p,q)}$ with $tb(L_{(p,q)}) =$

pq. We will use the coordinate system $\mathcal{C}_{\mathcal{K}}$. Then there exists a thickened torus $T^2 \times [1,2]$ with convex boundary, where $T^2 \times [1,1.5] \subset N(\mathcal{K})$, slope$(\Gamma_{T_1}) = -\frac{1}{n+1}$, slope$(\Gamma_{T_{1.5}}) = \frac{q}{p}$, and slope$(\Gamma_{T_2}) = -\frac{1}{n}$. Here we write $T_i = T^2 \times \{i\}$. Observe that $T^2 \times [1,2]$ is a *basic slice* in the sense of [H1], since the shortest integral vectors $(-n,1)$ and $(-(n+1),1)$ form an integral basis for $\mathbb{Z}^2$. This means that the tight contact structure must be one of two possibilities, distinguished by the *relative half-Euler class* $e(\xi)$. (It is called the "relative Euler class" in [H1], but "relative half-Euler class" is more appropriate.) Their Poincaré duals are given by $PD(e(\xi)) = \pm((-n,1) - (-(n+1),1)) = \pm(1,0)$. Now, by the universal tightness of $T^2 \times [1,2]$, it follows from the classification of [Gi2, H1] that:

(1) either $PD(e(\xi), T^2 \times [1,1.5]) = (p,q) - (-n-1,1)$ and $PD(e(\xi), T^2 \times [1.5,2]) = (-n,1) - (p,q)$,
(2) or $PD(e(\xi), T^2 \times [1,1.5]) = -(p,q) + (-n-1,1)$ and $PD(e(\xi), T^2 \times [1.5,2]) = -(-n,1) + (p,q)$.

In view of Lemma 2.2, we want to compute (i) $r(\partial D)$, where D is a convex meridional disk for $N(\mathcal{K})$ with Legendrian boundary on $T_{1.5} = \partial N(\mathcal{K})$, and (ii) $r(\partial\Sigma)$, where Σ is a convex Seifert surface for a Legendrian ruling curve ∞ on $T_{1.5}$. Write $D = D' \cup A$, where D' is a meridional disk with *efficient* Legendrian boundary for $N(\mathcal{K}) \setminus (T^2 \times [1,1.5])$, and $A \subset T^2 \times [1,1.5]$. (An *efficient* closed curve on a convex surface intersects the dividing set Γ minimally.) Also write $\Sigma = \Sigma' \cup B$, where $B \subset T^2 \times [1.5,2]$ and $\Sigma' \subset S^3 \setminus (T^2 \times [1,2])$ has efficient Legendrian boundary L on T_2.

By additivity,

$$r(\partial\Sigma) = r(L) + \chi(B_+) - \chi(B_-) = r(L) + \langle e(\xi), B\rangle.$$

Here S_+ (resp. S_-) denotes the positive (resp. negative) region of a convex surface S, divided by Γ_S. Similarly,

$$r(\partial D) = r(\partial D') + \langle e(\xi), A\rangle = \langle e(\xi), A\rangle.$$

Therefore, either $r(\partial\Sigma) = r(L) + p + n$ and $r(\partial D) = -q + 1$, or $r(\partial\Sigma) = r(L) - p - n$ and $r(\partial D) = q - 1$. In the former case,

$$r(L_{(p,q)}) = p(-q+1) + q(r(L) + p + n) = p + q(r(L) + n).$$

In the latter case, we have $r(L_{(p,q)}) = -p + q(r(L) - n)$ and we use the fact that $\{r(L) \mid L \in \mathcal{L}(\mathcal{K}), tb(L) = -n\}$ is invariant under the map $r \mapsto -r$. □

Proof of Theorem 3.6. By Lemma 3.7, every $L'_{(p,q)} \in \mathcal{L}(\mathcal{K}_{(p,q)})$ can be written as $S_+^{k_1} S_-^{k_2}(L_{(p,q)})$ for some $L_{(p,q)}$ with maximal tb. To complete the classification, we need to show that every $L'_{(p,q)}$ which is a "valley" of the image of (r, tb) (i.e., $L'_{(p,q)}$ for which $(r(L'_{(p,q)}) \pm 1, tb(L'_{(p,q)}) + 1)$ is in the image of (r, tb) but $(r(L'_{(p,q)}), tb(L'_{(p,q)}) + 2)$ is not) destabilizes to two maximal tb representatives $L^+_{(p,q)}$ and $L^-_{(p,q)}$ (the "peaks"). Observe that there are two types of valleys: type (i) has a depth of $s = -p - qn$ and type (ii) has a depth of $kq - s$, $k \in \mathbb{Z}^+$.

We start with valleys of type (i). Such valleys occur when $r(L^-) = q \cdot r(L) - s$, $r(L^+) = q \cdot r(L) + s$, and $tb(L^-) = tb(L^+) = pq$. It is clear that the valley between L^- and L^+ corresponds to a Legendrian ruling curve of slope $\frac{q}{p}$ on the boundary of the standard neighborhood $N(L)$ of L with $tb(L) = -n$. By stabilizing L in two ways, we see that any element $L'_{(p,q)}$ with $r(L'_{(p,q)}) = q \cdot r(L)$ and $tb(L'_{(p,q)}) = pq - s$ satisfies $L'_{(p,q)} = S^s_-(L^+) = S^s_+(L^-)$.

Next we explain the valleys of type (ii) which have depth $kq - s$, $k \in \mathbb{Z}^+$. The peaks L^- and L^+ correspond to "adjacent" $L, L' \in \mathcal{L}(\mathcal{K})$ which have $tb(L) = tb(L') = -n$ and $r(L) < r(L')$, and such that there is no Legendrian $L'' \in \mathcal{L}(\mathcal{K})$ with $tb(L'') = -n$ and $r(L) < r(L'') < r(L')$. Hence $r(L^-) = q \cdot r(L) + s$ and $r(L^+) = q \cdot r(L') - s$. The k in the expression $kq - s$ above satisfies $r(L') - r(L) = 2k$. The valley $L'_{(p,q)}$ with $tb(L'_{(p,q)}) = pq - (kq - s)$ and $r(L'_{(p,q)}) = q \cdot r(L) + kq = q \cdot r(L') - kq$ occurs as a Legendrian ruling curve of slope $\frac{q}{p}$ on the standard tubular neighborhood of $S^k_+(L) = S^k_-(L')$. Therefore, $L'_{(p,q)} = S^{kq-s}_+(L^-) = S^{kq-s}_-(L^+)$. This proves the Legendrian simplicity of $\mathcal{L}(\mathcal{K}_{(p,q)})$. □

4. Verification of uniform thickness

In this section we prove that many knot types satisfy the UTP. Let us begin with negative torus knots.

Theorem 1.2. *Negative torus knots satisfy the UTP.*

Proof. Let $\mathcal{K}$ be the unknot and $\mathcal{K}_{(p,q)}$ be its (p, q)-cable, *i.e.*, the (p, q)-torus knot, with $pq < 0$. It was shown in [EH1] that $\overline{tb}(\mathcal{K}_{(p,q)}) = pq$. Unless indicated otherwise, we measure the slopes of tori isotopic to $\partial N(\mathcal{K}_{(p,q)})$ with respect to $\mathcal{C}'_{\mathcal{K}}$. Then $\overline{tb}(\mathcal{K}_{(p,q)}) = pq$ is equivalent to $\overline{t}(\mathcal{K}_{(p,q)}) = \overline{t}(\mathcal{K}_{(p,q)}, \mathcal{C}'_{\mathcal{K}}) = 0$. In other words, the standard neighborhood of $L \in \mathcal{L}(\mathcal{K}_{(p,q)})$ satisfying $tb(L) = pq$ has boundary slope ∞ with respect to $\mathcal{C}'_{\mathcal{K}}$.

We will first verify Condition 1 of the UTP, arguing by contradiction. (In fact, the argument that follows can be used to prove that $\overline{t}(\mathcal{K}_{(p,q)}) = 0$.) Suppose there exists a solid torus $N = N(\mathcal{K}_{(p,q)})$ which has convex boundary with $s = \text{slope}(\Gamma_{\partial N}) > 0$ and $\#\Gamma_{\partial N} = 2$. After shrinking N if necessary, we may assume that s is a large positive integer. Next, using the Giroux Flexibility Theorem, ∂N can be isotoped into *standard form*, with Legendrian rulings of slope ∞. Now let A be a convex annulus with Legendrian boundary on ∂N and $A \times [-\varepsilon, \varepsilon]$ its invariant neighborhood. Here A is chosen so that $R = N \cup (A \times [-\varepsilon, \varepsilon])$ is a thickened torus whose boundary $\partial R = T_1 \cup T_2$ is parallel to $\partial N(\mathcal{K})$. Here, the relative positions of T_1 and T_2 are that if $T_2 = \partial N(\mathcal{K})$, then $T_1 \subset N(\mathcal{K})$.

Let us now analyze the possible dividing sets for A. First, ∂-parallel dividing curves are easily eliminated. Indeed, if there is a ∂-parallel arc, then we may attach the corresponding bypass onto ∂N and increase s to ∞,after isotopy. This would imply excessive twisting inside N, and the contact structure would be overtwisted. Hence we may assume that A is in standard form, with two parallel nonseparating arcs. Now choose a suitable identification $\partial N(\mathcal{K}) \simeq \mathbb{R}^2/\mathbb{Z}^2$ so that the ruling curves of A have slope ∞, $\text{slope}(\Gamma_{T_1}) = -s$ and $\text{slope}(\Gamma_{T_2}) = 1$. (This is possible since a holonomy computation shows that Γ_{T_1} is obtained from Γ_{T_2} by performing $s + 1$ right-handed Dehn twists.)

We briefly explain the classification of tight contact structures on R with the boundary condition $\text{slope}(\Gamma_{T_1}) = -s$, $\text{slope}(\Gamma_{T_2}) = 1$, $\#\Gamma_{T_1} = \#\Gamma_{T_2} = 2$. For more details, see [H1]. Corresponding to the slopes $-s$, 1, are the shortest integer vectors $(-1, s)$ and $(1, 1)$. Any tight contact structure on R can naturally be layered into *basic slices* $(T^2 \times [1, 1.5]) \cup (T^2 \times [1.5, 2])$, where $\text{slope}(\Gamma_{T_{1.5}}) = \infty$ (corresponding to the shortest integer vector $(0, 1)$) and $\#\Gamma_{T_{1.5}} = 2$. There are two possibilities for each basic slice — the Poincaré duals of the relative half-Euler classes are given by $\pm$ the difference of the shortest integer vectors corresponding to the dividing sets on the boundary. For

$T^2 \times [1, 1.5]$, the possible $PD(e(\xi))$ are $\pm$ of $(0,1) - (-1, s) = (1, 1-s)$; for $T^2 \times [1.5, 2]$, the possibilities are $\pm$ of $(1,1) - (0,1) = (1,0)$. Since $s >> 1$, the 4 possible tight contact structures on R are given by $\pm(1,0) \pm (1, 1-s)$. Of the 4 possibilities, 2 of them are universally tight and 2 of them are virtually overtwisted. The contact structure ξ is universally tight when there is *no mixing of sign, i.e.*, $PD(e(\xi)) = +(1,0) + (1,1-s)$ or $-(1,0) - (1,1-s)$; when there is mixing of sign $+(1,0) - (1,1-s)$ or $-(1,0) + (1,1-s)$, the contact structure is virtually overtwisted.

To determine the half-Euler class, consider $\Sigma = \gamma \times [-\varepsilon, \varepsilon] \subset A \times [-\varepsilon, \varepsilon]$, where γ is a Legendrian ruling curve of slope ∞. Since Σ is $[-\varepsilon, \varepsilon]$-invariant, $\langle e(\xi), \Sigma \rangle = \chi(\Sigma_+) - \chi(\Sigma_-) = 0$, where χ is the Euler characteristic and Σ_+ (resp. Σ_-) is the positive (resp. negative) part of $\Sigma \setminus \Gamma_\Sigma$. Therefore, $PD(e(\xi))$ must be $\pm(0, s-1)$, implying a *mixture of sign*.

Let us now recast the slopes of Γ_{T_i} in terms of coordinates $\mathcal{C}_\mathcal{K}$, where $\mathcal{K}$ is the unknot. With respect to $\mathcal{C}_\mathcal{K}$, slope$(\Gamma_{T_{1.5}}) = \frac{q}{p}$, where $\frac{q}{p}$ is neither a negative integer nor the reciprocal of one. One of the consequences of the classification of tight contact structures on solid tori in [Gi2, H1] is the following: if S is a convex torus in the standard tight contact (S^3, ξ_{std}) which bounds solid tori on both sides, then the only slopes for Γ_S at which there can be a sign change are negative integers or reciprocals of negative integers. Therefore, we have a contradiction, proving Condition 1.

Next we prove Condition 2, keeping the same notation as in the proof of Condition 1. Suppose that $N = N(\mathcal{K}_{(p,q)})$ now has boundary slope s, where $-\infty < s < 0$ and slopes are measured with respect to $\mathcal{C}'_\mathcal{K}$. If Γ_A has a ∂-parallel arc, then s approaches $-\infty$ (in terms of the Farey tessellation) when we attach a corresponding bypass onto N. Therefore, as usual, we may take A to be in standard form and Γ_A to consist of parallel nonseparating dividing arcs. Now observe that $\frac{q}{p}$ cannot lie between slope(Γ_{T_1}) and slope(Γ_{T_2}), where the slopes are measured with respect to $\mathcal{C}_\mathcal{K}$. This implies that there are no convex tori in R which are isotopic to T_i and have slope $\frac{q}{p}$. In the complement $S^3 \setminus R$, there is a convex torus isotopic to T_i with slope $\frac{q}{p}$. Using this, we readily find a thickening of N to have slope ∞, measured with respect to $\mathcal{C}'_\mathcal{K}$.

Once we thicken N to have boundary slope ∞, there is one last thing to ensure, namely that $\#\Gamma_{\partial N} = 2$; in other words, we want N to be the standard neighborhood of a Legendrian curve with twisting number 0 with respect to $\mathcal{C}'_\mathcal{K}$.

Claim 4.1. *Any solid torus N with convex boundary, slope$(\Gamma_{\partial N}) = \infty$, and $\#\Gamma_{\partial N} = 2n$, $n > 1$, extends to a solid torus $\overline{N}$ with convex boundary, slope ∞, and $\#\Gamma_{\partial \overline{N}} = 2$.*

Proof. There exists a thickened torus R with $\partial R = T_2 - T_1$, where $N \subset R$, the T_i, $i = 1, 2$, bound solid tori on both sides, and slope$(\Gamma_{T_i}) = \frac{q}{p}$ with respect to $\mathcal{C}_\mathcal{K}$. By shrinking N somewhat, we may take $R \setminus N$ to be a pair-of-pants Σ_0 times S^1. Since there is twisting on both sides of the exterior of R, we may also arrange that $\#\Gamma_{T_i} = 2$. Moreover, as $\Gamma_{\partial(R \setminus N)}$ is parallel to the S^1-fibers, the tight contact structure on $R \setminus N$ is necessarily *vertical*, *i.e.*, isotopic to an S^1-invariant contact structure, after appropriately modifying the boundary to be Legendrian-ruled. (See [H2] for a proof.)

The data for this tight contact structure are encoded in Γ_{Σ_0}. (Here we are assuming without loss of generality that Σ_0 is convex with Legendrian boundary.) Let $\partial \Sigma_0 = \gamma \sqcup \gamma_1 \sqcup \gamma_2$, where $\gamma_i = \Sigma_0 \cap T_i$ and $\gamma = \partial N \cap \Sigma_0$. There are $2n$ endpoints of Γ_{Σ_0} on γ, and 2 for each of γ_i. If there is an arc between γ_1 and γ_2, then an imbalance occurs and there is necessarily a ∂-parallel arc along γ. This would allow a thickening of N to one whose boundary has fewer dividing curves.

The situation from which we have no immediate escape is when all the arcs from γ_i go to γ, and the extra endpoints along γ connect up without creating ∂-parallel arcs. We need to look externally (*i.e.*, outside of R) to obtain the desired bypass. The key features we take advantage of are:

(1) There is twisting on both sides of the exterior of R.
(2) There is no mixing of sign about R.

One of the (nontrivial) bypasses found along T_1 and T_2 therefore can be extended into R to give a bypass to reduce $\#\Gamma_{\partial N}$. □

This completes the proof of Theorem 1.2. □

Recall a fraction $\frac{p}{q}$ is *sufficiently negative* if

$$\frac{p}{q} < w(\mathcal{K}).$$

(Observe that $\frac{p}{q}$ is the reciprocal of the slope of a curve ∂N corresponding to (p, q).)

Theorem 1.3. If a knot type $\mathcal{K}$ satisfies the UTP, then (p, q)-cables $\mathcal{K}_{(p,q)}$ satisfies the UTP, provided $\frac{p}{q}$ is sufficiently negative.

Let $\mathcal{K}$ be a knot type that satisfies the UTP. We write $N = N(\mathcal{K})$ and $N_{(p,q)} = N(\mathcal{K}_{(p,q)})$. The coordinates for ∂N and $\partial N_{(p,q)}$ will be $\mathcal{C}_{\mathcal{K}}$ and $\mathcal{C}'_{\mathcal{K}}$, respectively. The proof of Theorem 1.3 is virtually identical to that of Theorem 1.2.

Proof. We prove that the contact width $w(\mathcal{K}_{(p,q)}, \mathcal{C}'_{\mathcal{K}})$, measured with respect to $\mathcal{C}'_{\mathcal{K}}$, and $\overline{t}(\mathcal{K}_{(p,q)}, \mathcal{C}'_{\mathcal{K}})$ both equal 0, and that any $N_{(p,q)}$ with convex boundary can be thickened to a standard neighborhood of a Legendrian knot with $t(L_{(p,q)}, \mathcal{C}'_{\mathcal{K}}) = 0$.

It is easy to see that $t(L_{(p,q)}, \mathcal{C}'_{\mathcal{K}}) = 0$ can be attained: Since $\frac{p}{q}$ is sufficiently negative, inside any N (with convex boundary) of maximal thickness there exists a Legendrian representative $L_{(p,q)} \in \mathcal{L}(\mathcal{K}_{(p,q)})$ of twisting number $t(L_{(p,q)}) = 0$, which appears as a Legendrian divide on a convex torus parallel to ∂N.

Suppose $N_{(p,q)}$ has convex boundary and slope$(\Gamma_{\partial N_{(p,q)}}) = s$. As before, arrange the characteristic foliation on $\partial N_{(p,q)}$ to be in standard form with Legendrian rulings of slope ∞, and consider the convex annulus A with Legendrian boundary on $\partial N_{(p,q)}$, where the thickening R of $N_{(p,q)} \cup A$ is a thickened torus whose boundary $\partial R = T_1 \cup T_2$ is isotopic to ∂N. We assume that Γ_A consists of parallel nonseparating arcs, since otherwise we can further thicken $N_{(p,q)}$ by attaching the bypass corresponding to a ∂-parallel arc.

Now let N be a maximally thickened solid torus which contains R, where the thickness is measured in terms of the *contact width*.

Claim 4.2. $w(\mathcal{K}_{(p,q)}, \mathcal{C}'_{\mathcal{K}}) = \overline{t}(\mathcal{K}_{(p,q)}, \mathcal{C}'_{\mathcal{K}}) = 0.$

Proof. If $s > 0$, then by shrinking the solid torus $N_{(p,q)}$, we may take s to be a large positive integer and $\#\Gamma_{\partial N_{(p,q)}} = 2$. Then, as in the proof of Theorem 1.2, (i) inside R there exists a convex torus parallel to T_i with slope $\frac{q}{p}$ (with respect to $\mathcal{C}_{\mathcal{K}}$), (ii) the tight contact structure on R must have mixing of sign, and (iii) this mixing of sign cannot happen inside the maximally thickened torus N. This contradicts slope$(\Gamma_{\partial N_{(p,q)}}) = s > 0$. □

Claim 4.3. *Every $N_{(p,q)}$ can be thickened to a standard neighborhood of a Legendrian knot $L_{(p,q)}$ with $t(L_{(p,q)}) = 0$.*

Proof. If $-\infty < s < 0$, then there cannot be any convex tori in R isotopic to T_i and with slope ∞. Hence there is a convex torus parallel to T_i with slope ∞ and $\#\Gamma = 2$ *outside of* R. By an application of the Imbalance Principle, we can thicken N to have slope ∞. The proof of the reduction to $\#\Gamma_{\partial N} = 2$ is identical to the proof of Claim 4.1 — the key point is that there is twisting on both sides of $N \setminus R$. □

This completes the proof of Theorem 1.3. □

We now demonstrate that the UTP is well-behaved under connected sums.

Theorem 1.4. If two knot types $\mathcal{K}_1$ and $\mathcal{K}_2$ satisfy the UTP, then their connected sum $\mathcal{K}_1\#\mathcal{K}_2$ satisfies the UTP.

Proof. The following is the key claim:

Claim 4.4. *Every solid torus N with convex boundary which represents $\mathcal{K}_1\#\mathcal{K}_2$ can be thickened to a standard neighborhood N' of a Legendrian curve in $\mathcal{L}(\mathcal{K}_1\#\mathcal{K}_2)$.*

Proof. Applying the Giroux Flexibility Theorem, ∂N can be put in *standard form*, with meridional Legendrian rulings. Let S be the separating sphere for $\mathcal{K}_1\#\mathcal{K}_2$ — we arrange S so it (1) is convex, (2) intersects N along two disks, and (3) intersects ∂N in a union of Legendrian rulings. Moreover, on the annular portion of $S \setminus (\mathcal{K}_1\#\mathcal{K}_2)$, we may assume that (4) there are no ∂-parallel arcs, since otherwise N can be thickened further by attaching the corresponding bypasses. Now, cutting S^3 along S and gluing in copies of the standard contact 3-ball B^3 with convex boundary, we obtain solid tori N_i, $i = 1, 2$, (with convex boundary) which represent $\mathcal{K}_i$.

Since $\mathcal{K}_i$ satisfies the UTP, there exists a thickening of N_i to N_i', where N_i' is the standard neighborhood of a Legendrian knot $L_i \in \mathcal{L}(\mathcal{K}_i)$. Also arranging $\partial N_i'$ so that it admits meridional Legendrian rulings, we take an annulus from a Legendrian ruling γ_i' on $\partial N_i'$ to a Legendrian ruling γ_i on $\partial N_i \cap \partial N$. If $tb(\gamma_i) < -1$, then the Imbalance Principle, together with the fact that $tb(\gamma_i') = -1$, yields enough bypasses which can be attached onto ∂N_i to thicken N_i into the standard neighborhood of a Legendrian knot.

However, upon closer inspection, it is evident that the bypasses produced can be attached onto N inside the original S^3. This produces a thickening of N to N', which has boundary slope $\frac{1}{m}$ (*i.e.*, is the standard neighborhood of a Legendrian knot in $\mathcal{L}(\mathcal{K})$) measured with respect to $\mathcal{C}_{\mathcal{K}_1\#\mathcal{K}_2}$. □

Condition 1 of the UTP follows immediately from the claim. To prove Condition 2, we need to show that a standard neighborhood N' of a Legendrian knot in $\mathcal{L}(\mathcal{K}_1\#\mathcal{K}_2)$ can be thickened to N'' which is the standard neighborhood of a maximal tb representative of $\mathcal{L}(\mathcal{K}_1\#\mathcal{K}_2)$. This is equivalent to showing any Legendrian knot L' in $\mathcal{L}(\mathcal{K}_1\#\mathcal{K}_2)$ can be destabilized to a maximal tb representative. Given $L' \in \mathcal{L}(\mathcal{K}_1\#\mathcal{K}_2)$, then L' can be written as $L_1'\#L_2'$, with $L_i' \in \mathcal{L}(\mathcal{K}_i)$, $i = 1, 2$. Each L_i' can be destabilized to a maximal tb representative L_i'' by the UTP for each $\mathcal{K}_i$. Since

$$\overline{tb}(\mathcal{K}_1\#\mathcal{K}_2) = \overline{tb}(\mathcal{K}_1) + \overline{tb}(\mathcal{K}_2) + 1,$$

by [EH2], we simply take $L'' = L_1''\#L_2''$. This proves Theorem 1.4. □

5. NON-UNIFORMLY-THICK KNOTS AND NON-DESTABILIZABILITY

We prove the following more precise version of Theorem 1.5.

Theorem 1.5. *The $(2,3)$-torus knot does not satisfy the UTP.*

Although our considerations will work for any (p,q)-torus knot with $q > p > 0$, we assume for simplicity that $\mathcal{K}$ is a $(2,3)$-torus knot, in order to keep the arguments simpler in a few places.

Proof. The goal is to exhibit solid tori N representing $\mathcal{K}$, which cannot be thickened to the maximal thickness. The overall strategy is not much different from the strategy used in [EH3] and [EH4] to classify and analyze tight contact structures on Seifert fibered spaces over S^2 with three singular fibers. The plan is as follows: we work backwards by starting with an arbitrary solid torus N which represents $\mathcal{K}$ and attempting to thicken it. This gives us a list N_k of potential non-thickenable candidates, as well as tight contact structures on their complements $S^3 \setminus N_k$ (Lemma 5.1). In Lemma 5.2 we prove that the decomposition into N_k and $S^3 \setminus N_k$ actually exists inside the standard tight (S^3, ξ_{std}) and in Lemma 5.3 we prove the N_k indeed resist thickening.

Let T be an oriented standardly embedded torus in S^3 which bounds solid tori V_1 and V_2 on opposite sides and which contains a $(2,3)$-torus knot $\mathcal{K}$. Suppose $T = \partial V_1$ and $T = -\partial V_2$. Also let F_i, $i = 1, 2$, be the core curve for V_i. In [EH1] it was shown that $\overline{tb}(\mathcal{K}) = pq - p - q = 1$. Measured with respect to the coordinate system $\mathcal{C}'_{F_i}$ for either i, $\overline{t}(\mathcal{K}, \mathcal{C}'_{F_i}) = -p - q = -5$, which corresponds to a slope of $-\frac{1}{5}$.

Lemma 5.1. *Suppose the solid torus N representing $\mathcal{K}$ resists thickening. Then slope$(\Gamma_{\partial N}) = -\frac{k+1}{6k+5}$, where k is a nonpositive integer and the slope is measured with respect to $\mathcal{C}'_{F_i}$.*

Proof. Let L_i, $i = 1, 2$, be a Legendrian representative of F_i with Thurston-Bennequin invariant $-m_i$, where $m_i > 0$. If $N(L_i)$ is the standard neighborhood of L_i, then slope$(\Gamma_{\partial N(L_i)}) = -\frac{1}{m_i}$ with respect to the coordinate system $\mathcal{C}_{F_i}$. We recast these slopes with respect to a new coordinate system $\mathcal{C}$ which identifies $T \xrightarrow{\sim} \mathbb{R}^2/\mathbb{Z}^2$, where $\mathcal{K}$ (viewed as sitting on T) corresponds to $(0,1)$.

First we change coordinates from $\mathcal{C}_{F_1}$ to $\mathcal{C}$. Consider the oriented basis $((2,3),(1,2))$ with respect to $\mathcal{C}_{F_1}$; we map it to $((0,1),(-1,0))$ with respect to $\mathcal{C}$. This corresponds to the map $A_1 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$. (Here we are viewing the vectors as column vectors and multiplying by A_1 on the left.) Then A_1 maps $(-m_1, 1) \mapsto (-3m_1 - 2, -2m_1 - 1)$. Since we are only interested in slopes, let us write it instead as $(3m_1 + 2, 2m_1 + 1)$.

Similarly, we change from $\mathcal{C}_{F_2}$ to $\mathcal{C}$. The only thing we need to know here is that $(-m_2, 1)$ with respect to $\mathcal{C}_{F_2}$ maps to $(2m_2 + 3, m_2 + 2)$ with respect to $\mathcal{C}$.

Given a solid torus N which resists thickening, let L_i, $i = 1, 2$, be a Legendrian representative of F_i which maximizes $tb(L_i)$ in the complement of N (subject to the condition that $L_1 \sqcup L_2$ is isotopic to $F_1 \sqcup F_2$ in the complement of N). View $S^3 \setminus (N(L_1) \cup N(L_2) \cup N)$ as a Seifert fibered space over the thrice-punctured sphere, where the annuli which connect among $N(L_1)$, $N(L_2)$, and N admit fibrations by the Seifert fibers. Now suppose $3m_1 + 2 \neq 2m_2 + 3$. Then we apply the Imbalance Principle to a convex annulus A' between $N(L_1)$ and $N(L_2)$ to find a bypass along $N(L_i)$. This bypass in turn gives rise to a thickening of $N(L_i)$, allowing the increase of $tb(L_i)$ by one. Eventually we arrive at $3m_1 + 2 = 2m_2 + 3$ and a convex annulus A' which has no ∂-parallel

arcs (hence we may assume A' is in *standard form*). Moreover, the denominator of slope$(\Gamma_{\partial N})$ must also equal $3m_1+2 = 2m_2+3$, since otherwise N admits a thickening. Since $m_i > 0$, the smallest solution to $3m_1+2 = 2m_2+3$ is $m_1 = 1$, $m_2 = 1$. All the other positive integer solutions are therefore obtained by taking $m_1 = 2k+1$, $m_2 = 3k+1$, with k a nonnegative integer.

We now compute the slope of the dividing curves on $\partial(N(L_1)\cup N(L_2)\cup N(A'))$, measured with respect to $\mathcal{C}'_{F_1} = \mathcal{C}'_{F_2}$, after edge-rounding. Here $N(A')$ stands for the I-invariant neighborhood of the convex annulus A'. We have:

$$-\frac{2m_1+1}{3m_1+2} + \frac{m_2+2}{2m_2+3} - \frac{1}{6k+5} = -\frac{4k+3}{6k+5} + \frac{3k+3}{6k+5} - \frac{1}{6k+5} = -\frac{k+1}{6k+5}.$$

For small k we get $-\frac{1}{5} < -\frac{2}{11} < -\frac{3}{17} < -\frac{4}{23} < \cdots < -\frac{1}{6}$. □

Let N_k be a tight solid torus representing $\mathcal{K}$ so that the boundary slope is $-\frac{k+1}{6k+5}$ with respect to $\mathcal{C}'_{F_i}$ and $\#\Gamma_{N_k} = 2$. (There are exactly two tight contact structures on N_k which satisfy the given boundary conditions, and they are both universally tight.) Let $M_k = S^3 \setminus N_k$. From the above discussion, if N_k is to resist thickening, then we know that M_k must be contactomorphic to the manifold obtained from $N(L_1)\cup N(L_2)$ by adding a standard neighborhood of a convex annulus A'. M_k is a Seifert fibered space and has a degree 6 cover $\widetilde{M_k}$ diffeomorphic to S^1 times a punctured torus (*cf.* [EH1]). One may easily check that the pullback of the tight contact structure to $\widetilde{M_k}$ admits an isotopy where the S^1 fibers become Legendrian and have twisting number $-(6k+5)$ with respect to the product framing.

Lemma 5.2. *The standard tight contact structure on S^3 splits into a (universally) tight contact structure on N_k with boundary slope $-\frac{k+1}{6k+5}$ and the tight contact structure on M_k described above.*

Proof. Let N_k be a (universally) tight solid torus described above and let A be a convex annulus in standard form from N_k to itself, such that the complement of $R = N_k \cup N(A)$ in S^3 consist of standard neighborhoods $N(L_i)$, $i = 1,2$. Here $N(A)$ is the I-invariant neighborhood of A. (Observe that R is also contact isotopic to $N_k \cup N(A')$.)

For either choice of contact structure on N_k, the contact structure on R can be isotoped to be transverse to the fibers of R (where the fibers are parallel to $\mathcal{K}$), while preserving the dividing set on ∂R. Such a *horizontal* contact structure is universally tight. (For more details of this standard argument, see for example [H2].)

Once we know that the contact structure on R is tight, we just need to apply the classification of tight contact structures on solid tori and thickened tori. In fact, any tight contact structure on $R = T^2 \times [1,2]$ with boundary conditions $\#\Gamma_{T_1} = \#\Gamma_{T_2} = 2$ and slope$(\Gamma_{T_1}) = -\frac{1}{m_1}$, slope$(\Gamma_{T_2}) = -m_2$ (here m_i are positive integers) glues together with $N(L_1)$ and $N(L_2)$ to give the tight contact structure on S^3. □

Lemma 5.3. *The tight solid torus N_k does not admit a thickening to a solid torus $N_{k'}$ whose boundary slope is $-\frac{k'+1}{6k'+5}$, where $k' < k$. More generally, N_k does not admit any nontrivial thickenings,* i.e., *no thickenings with a boundary slope different from $-\frac{k+1}{6k+5}$.*

Proof. If N_k can be thickened to $N_{k'}$, then there exists a Legendrian curve isotopic to the regular fiber of the Seifert fibered space $M_k = S^3 \setminus N_k$ with twisting number $> -(6k+5)$, measured with respect to the Seifert fibration. (Take a ruling curve on $\partial N_{k'} \subset M_k$.) Pulling back to the sixfold

cover $\widetilde{M_k}$, we have a Legendrian knot which is topologially isotopic to a fiber but has twisting number $> -(6k+5)$. However, we claim that the maximal twisting number for a fiber in $\widetilde{M_k}$ is $-(6k+5)$. One way to see this is to add a solid torus to $\widetilde{M_k}$ to obtain T^3 and extend the contact structure so that all the S^1 fibers in T^3 are Legendrian with twisting $-(6k+5)$. We can now apply the classification of tight contact structures on T^3 due to Giroux and Kanda (see [K]) to conclude that the maximal twisting number for a fiber is $-(6k+5)$.

Next, suppose N_k admits a nontrivial thickening N' (not necessarily of type $N_{k'}$). Then we use the argument in Lemma 5.1 to find Legendrian curves $L_i \subset S^3 \setminus N'$ which maximize the twisting number amongst Legendrian curves isotopic to F_i in $S^3 \setminus N'$, and a convex annulus from $N(L_1)$ to $N(L_2)$, so that $\partial(N(L_1) \cup N(L_2) \cup N(A'))$ has some slope $-\frac{k'+1}{6k'+5}$, $k' < k$. This puts us in the case treated in the previous paragraph. □

This completes the proof of Theorem 1.5. □

As a corollary of the above investigation we have:

Theorem 1.6. *Let $\mathcal{K}'$ be the $(2,3)$-cable of the $(2,3)$-torus knot $\mathcal{K}$. Then there exists a Legendrian knot $L \in \mathcal{L}(\mathcal{K}')$ which does not admit any destabilization, yet satisfies $tb(L) < \overline{tb}(\mathcal{K}')$.*

Proof. Let N_k be a solid torus which resists thickening; say $k = 1$. Then the boundary slope of N_1 is $-\frac{2}{11}$, measured with respect to $\mathcal{C}'_{F_i}$. We choose a slope $-\frac{a}{b} < -\frac{2}{11}$ whose corresponding simple closed curve, denoted $(-b, a)$, has fewer intersections with the simple closed curve $(-11, 2)$ than with any other simple closed curve whose corresponding slope $-\frac{c}{d}$ satisfies $-\frac{2}{11} < -\frac{c}{d} < 0$. To verify that $-\frac{a}{b} = -\frac{3}{16}$ works, consider the standard Farey tessellation of the hyperbolic unit disk. Since there mutually are edges among $-\frac{1}{5} < -\frac{3}{16} < -\frac{2}{11}$, $-\frac{a}{b} = -\frac{3}{16}$ is shielded from any $-\frac{c}{d} > -\frac{2}{11}$ by the edge from $-\frac{1}{5}$ to $-\frac{2}{11}$. Therefore, to get from $-\frac{3}{16}$ to $-\frac{c}{d}$ we need at least two steps, implying that $(-16, 3)$ and $(-11, 2)$ have fewer intersections than $(-16, 3)$ and $(-d, c)$. Now, by changing coordinates from $\mathcal{C}'_{F_i}$ to $\mathcal{C}_{\mathcal{K}}$, we see that the slope $-\frac{a}{b} = -\frac{3}{16}$ corresponds to the $(2,3)$-cable of the $(2,3)$-torus knot.

First observe that there is a Legendrian knot $L' \in \mathcal{L}(\mathcal{K}')$ which sits inside the solid torus N_0 with slope$(\Gamma_{\partial N_0}) = -\frac{1}{5}$ (with respect to $\mathcal{C}'_{F_i}$), as a Legendrian divide on a convex torus which is isotopic to (but not contact isotopic to) ∂N_0 and which has slope $-\frac{3}{16}$. By the classification of tight contact structures on solid tori, such a convex torus exists because $-\frac{3}{16} > -\frac{1}{5}$. This proves that $\overline{t}(\mathcal{K}', \mathcal{C}'_{\mathcal{K}}) \geq 0$.

Next we exhibit $L \in \mathcal{L}(\mathcal{K}')$ which cannot be destabilized to twisting number 0 with respect to $\mathcal{C}'_{F_i}$. Let L be a Legendrian ruling curve on ∂N_1, where the ruling is of slope $-\frac{3}{16}$. By construction, the twisting number $t(L, \mathcal{C}'_{\mathcal{K}}) = -1$, computed by intersecting $(-11, 2)$ and $(-16, 3)$.

Lemma 5.4. *L cannot be destabilized.*

Proof. The proof is an application of the state transition technique [H3]. Suppose that L admits a destabilization. Then there exists a convex torus Σ isotopic to ∂N_1 which contains L as well as a bypass to L. More conveniently, instead of isotoping both L and the torus, we fix L and isotop the torus from ∂N_1 to Σ. Then the annulus $B_0 = (\partial N_1) \setminus L$ is isotoped to $B = \Sigma \setminus L$ relative to the boundary. Observe that Γ_{B_0} consists of two parallel nonseparating arcs. To get to B, we perform isotopy discretization, *i.e.*, a sequence of bypass moves (which may well be trivial bypass

attachments). There can be no nontrivial bypasses attached onto B_0 from the exterior of N_1, since N_1 has maximal thickness.

We claim there are no nontrivial bypasses from the interior as well. First of all, since there are no Legendrian knots isotopic to L with twisting number zero inside N_1, no ∂-parallel dividing curves (on B) can be created by attaching bypasses from the interior. On the other hand, the slope (or holonomy) of the two separating arcs on B_0 cannot be changed, since the only slope $-\frac{c}{d}$ with $-\frac{c}{d} \geq -\frac{2}{11}$ with an edge (in the Farey tessellation) to $-\frac{3}{16}$ is $-\frac{2}{11}$. This proves that all the state transitions for B_0 are trivial state transitions. We are unable to reach B. □

This completes the proof of Theorem 1.6. □

6. NON-TRANSVERSE-SIMPLICITY

Theorem 1.7. *Let $\mathcal{K}'$ be the $(2,3)$-cable of the $(2,3)$-torus knot $\mathcal{K}$. Then $\mathcal{K}'$ is not transversely simple.*

We first gather some preliminary lemmas.

Lemma 6.1. $\overline{tb}(\mathcal{K}') = w(\mathcal{K}') = 6$.

The proof of this lemma is identical to that of Theorem 1.2.

Lemma 6.2. *There are precisely two maximal Thurston-Bennequin representatives in $\mathcal{L}(\mathcal{K}')$, which we call $K_\pm$ and which have $tb(K_\pm) = 6$, $r(K_\pm) = \pm 1$.*

Proof. Any $K \in \mathcal{L}(\mathcal{K}')$ with $tb(K) = 6$ can be realized as a Legendrian divide on the boundary of a solid torus N representing $\mathcal{K}$. By Lemma 5.1, N can be thickened to a solid torus N' with slope$(\Gamma_{\partial N'}) = -\frac{1}{5}$, measured with respect to $\mathcal{C}'_{F_i}$. This means that there are two possible tight contact structures on N, both universally tight, and the extension to N' is determined by the tight contact structure on N. Once N' is determined, the tight contact structure on $S^3 \setminus N'$ is unique up to isotopy, since N' is the standard tubular neighborhood of the unique maximal tb representative of $\mathcal{K}$. This proves that there are at most two maximal tb representatives of $\mathcal{L}(\mathcal{K})$.

We now show that there are indeed two representatives by computing their rotation numbers to be $r(K) = \pm 1$ (and hence showing they are distinct). To use Lemma 2.2, we need to know the rotation number of a ruling curve λ isotopic to $\mathcal{K}$ on ∂N and the rotation number of a meridional ruling curve μ on ∂N. A ruling curve isotopic to $\mathcal{K}$ on $\partial N'$ has rotation number 0 (by the Bennequin inequality). The region R between ∂N and $\partial N'$ (in $\mathcal{C}_\mathcal{K}$ coordinates) has relative half-Euler class

$$PD(e(\xi), R) = \pm((1,1) - (2,3)) = \pm(-1,-2).$$

So $r(\lambda) = \mp 1$. One similarly sees that $r(\mu) = \pm 2$. Thus

$$r(K) = 2(\pm 2) + 3(\mp 1) = \pm 1.$$

□

Lemma 6.3. *The only non-destabilizable representatives of $\mathcal{L}(\mathcal{K}')$ besides those which attain $\overline{tb}(\mathcal{K}')$ are $L_\pm$ which have $tb(L_\pm) = 5$ and $r(L_\pm) = \pm 2$. They are realized as Legendrian ruling curves on a convex torus isotopic to T with dividing curves of slope $-\frac{2}{11}$ (with respect to $\mathcal{C}'_{F_i}$), and which does not admit a thickening.*

Proof. Let K be a non-destabilizable representative of $\mathcal{L}(\mathcal{K}')$. Since $\overline{tb}(\mathcal{K}') = 6$, we can always place K on the (convex) boundary $\Sigma = \partial N$ of a solid torus N representing $\mathcal{K}$. If K is a Legendrian divide on Σ, then we are in the case of Lemma 6.2. If K is not a Legendrian divide, then K must intersect Γ_Σ efficiently, and we may assume that K is a Legendrian ruling curve on Σ. Slopes of Σ will usually be measured with respect to $\mathcal{C}'_{F_i}$.

We now show that if $s = \text{slope}(\Gamma_\Sigma) \neq -\frac{2}{11}$, then K can be destabilized (contradicting our assumption). Note that s must be in $[-\frac{1}{5}, 0)$ and $s = -\frac{3}{16}$ corresponds to the situation in Lemma 6.2. In the following cases, we find a convex torus Σ' isotopic to and disjoint from Σ so that a simple closed curve of slope $-\frac{3}{16}$ has smaller geometric intersection with $\Gamma_{\Sigma'}$ than with Γ_Σ. The destabilization is then a consequence of the Imbalance Principle. If $s \in [-\frac{1}{5}, -\frac{3}{16})$, then there is $\Sigma' \subset N$ with $\text{slope}(\Gamma_{\Sigma'}) = -\frac{3}{16}$. If $s \in (-\frac{3}{16}, -\frac{2}{11})$, then there exists Σ' of slope $-\frac{3}{16}$ outside N (since N can be thickened to maximal width by Lemma 5.1). Similarly, if $s \in (-\frac{1}{6}, 0)$, then there exists a Σ' with $\text{slope}(\Gamma_{\Sigma'}) = -\frac{1}{6}$, by using Lemma 5.1. Next, if $s \in (-\frac{2}{11}, -\frac{1}{6})$, there exists a Σ' of slope $-\frac{1}{6}$ inside N (it is not difficult to see that this Σ' works by referring to the Farey tessellation). Therefore we are left with $s = -\frac{1}{6}$. But then we use the classification of $\mathcal{L}(\mathcal{K})$ to deduce that N can be thickened to N' with boundary slope $-\frac{1}{5}$, corresponding to a representative of $\mathcal{L}(\mathcal{K})$ of maximal Thurston-Bennequin invariant. We can now compare Σ with Σ' of slope $-\frac{3}{16}$ to destabilize. This proves that the only two places where we get stuck and cannot destabilize are $-\frac{2}{11}$ and $-\frac{3}{16}$.

Now let $L \in \mathcal{L}(\mathcal{K}')$ be non-destabilizable representatives with $tb(L) = 5$. Then they are Legendrian ruling curves on the boundary of a solid torus N_1, where $\text{slope}(\Gamma_{\partial N_1}) = -\frac{2}{11}$ with respect to $\mathcal{C}'_{F_i}$. There are two possible tight contact structures on N_1, and they are both universally tight. Since the tight contact structures on their complements $S^2 \setminus N_1$ are always contact isotopic, there are at most two non-destabilizable, non-maximal representatives. Using Lemma 2.2, we obtain:

$$r(L) = 2(\pm 1) + 3(0) = \pm 2.$$

(Since μ intersects $\Gamma_{\partial N_1}$ in four points, $r(\mu) = \pm 1$. It is also not hard to compute $r(\lambda) = 0$ by using the fact that there are no ∂-parallel arcs on the Seifert surface for λ.) Therefore L_+ and L_- are distinguished by the contact structures on the solid torus N_1. □

Lemma 6.4. *$S_-(L_-) = S_-^2(K_-)$ and $S_+(L_+) = S_+^2(K_+)$.*

Proof. Since L_- is a Legendrian ruling curve on N_1 with $\text{slope}(\Gamma_{\partial N_1}) = -\frac{2}{11}$, $S_-(L_-)$ is a Legendrian ruling curve on $\partial N_1' \subset N_1$, where N_1' is a solid torus representing $\mathcal{K}$ and $\text{slope}(\Gamma_{\partial N_1'}) = -\frac{1}{6}$. Similarly, since K_- is a Legendrian divide on N with $\text{slope}(\Gamma_{\partial N}) = -\frac{3}{16}$, $S_-^2(K_-)$ is a Legendrian ruling curve on $\partial N' \subset N$, where N' is a solid torus representing $\mathcal{K}$ and $\text{slope}(\Gamma_{\partial N'}) = -\frac{1}{6}$. Now N' and N_1' are neighborhoods of Legendrian knots in $\mathcal{L}(\mathcal{K})$ with $tb = 0$. If the associated rotation numbers are the same, then they are contact isotopic (by the Legendrian simplicity of the $(2,3)$-torus knot). One may easily check that the rotation numbers are indeed the same. Therefore, there is an ambient contact isotopy taking N' to N_1', and it simply remains to Legendrian isotop $S_-(L_-)$ to $S_-^2(K_-)$ through ruling curves. □

We are now ready to proceed with the proof of Theorem 1.7.

Proof of Theorem 1.7. In view of Theorem 1.8, it suffices to show that $S_+^k(L_-)$ is never equal to $S_+^k S_-(K_-)$ for all positive integers k (and likewise $S_-^k(L_+)$ is never equal to $S_-^k S_+(K_+)$).

Throughout this proof we use coordinates $\mathcal{C}'_{\mathcal{K}}$, unless otherwise stated. As above, let N_1 be a solid torus which represents $\mathcal{K}$, does not admit a thickening, and has boundary $\Sigma_0 = \partial N_1$, where $\#\Gamma_{\Sigma_0} = 2$ and slope$(\Gamma_{\Sigma_0}) = -\frac{2}{11}$. Assuming we have already chosen the correct N_1 (there were two choices), place the knot $L = S^k_+(L_-)$ on Σ_0 as follows: if $A_0 = \Sigma_0 \setminus L$, then there are k negative ∂-parallel arcs on the left-hand edge L_l of A_0 and k positive ∂-parallel arcs on the right-hand edge L_r of A_0. Here A_0 is oriented so that $\partial A_0 = L_r - L_l$, where L_r and L_l are oriented copies of L. (The sign of a ∂-parallel arc is the sign of the region it cuts off.) See Figure 3 for a possible Γ_{A_0}. When we draw annuli, we will usually present rectangles, with the understanding that the top and the bottom are identified.

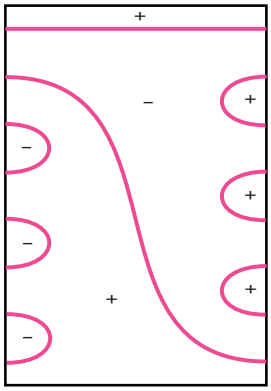

FIGURE 3. The "initial configuration" Γ_{A_0}. The left-hand boundary is L_l and the right-hand boundary is L_r. They glue to give Σ_0.

The key claim is the following:

Claim 6.5. *Every convex torus which contains L and is isotopic to Σ_0 has slope $-\frac{2}{11}$.*

This would immediately show that $S^k_+(L_-)$ is never equal to $S^k_+S_-(K_-)$. To prove this fact, we use the state traversal technique. If Σ also contains L and is isotopic to Σ_0 (not necessarily relative to L), then we can use the standard properties of incompressible surfaces in Seifert fibered spaces to conclude that Σ must be isotopic to Σ_0 *relative to* L. Therefore, it suffices to show that the slope of the dividing set does not change under any isotopy of Σ_0 relative to L. Although we would like to say that the isotopy leaves the dividing set of Σ_0 invariant, this is not quite true. It is not difficult to see (see Figure 4) that the number of dividing curves can increase, although the slope should always remain the same according to Claim 6.5. Starting with $\Sigma = \Sigma_0$, we inductively assume the following:

Inductive Hypothesis

(1) Σ is a convex torus which contains L and satisfies $2 \leq \#\Gamma_\Sigma \leq 2k+2$ and slope$(\Gamma_\Sigma) = -\frac{2}{11}$.
(2) Σ is "sandwiched" in a $[0,1]$-invariant $T^2 \times [0,1]$ with slope$(\Gamma_{T_0}) =$ slope$(\Gamma_{T_1}) = -\frac{2}{11}$ and $\#\Gamma_{T_0} = \#\Gamma_{T_1} = 2$. (More precisely, $\Sigma \subset T^2 \times (0,1)$ and is parallel to $T^2 \times \{i\}$.)
(3) There is a contact diffeomorphism $\phi : S^3 \xrightarrow{\sim} S^3$ which takes $T^2 \times [0,1]$ to a standard I-invariant neighborhood of Σ_0 and matches up their complements.

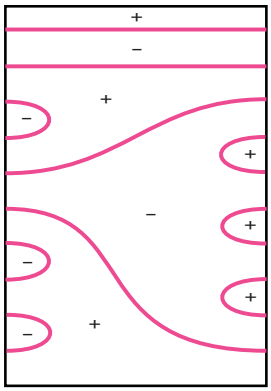

FIGURE 4. A potential Σ in the inductive step.

Suppose we isotop Σ relative to L into another convex torus Σ'. Then the standard state traversal machinery [H3] implies that we may assume that the isotopy is performed in discrete steps, where each step is given by the attachment of a bypass. Σ bounds a solid torus N on one side, and we say that the bypass is attached "from the inside" or "from the back" if the bypass is in the interior of N and the bypass is attached "from the outside" or "from the front" if the bypass is in the exterior of N. (Also for convenience assume that T_0 is inside N and T_1 is outside N.) We prove the inductive hypothesis still holds after all existing bypass attachments.

Lemma 6.6. *The Legendrian knot L cannot sit on a convex torus Σ in N_1 that is isotopic to ∂N_1 and satisfies $\#\Gamma_\Sigma = 2$ and slope$(\Gamma_\Sigma) = -\frac{1}{6}$.*

Proof. The convex torus Σ bounds the standard neighborhood of a Legendrian knot in $\mathcal{L}(\mathcal{K})$ with $tb = 0$ and $r = -1$ (*i.e.*, S_- of the maximal tb representative of $\mathcal{L}(\mathcal{K})$). Computing as in Lemma 6.3, we find that a Legendrian ruling curve of slope $-\frac{3}{16}$ on Σ must be $S_-(L_-)$. Therefore, if $L \subset \Sigma$, then L must be a stabilization of $S_-(L_-)$. However, this contradicts the fact that $L = S_+^k(L_-)$ by a simple (r, tb)-count. □

Lemma 6.7. *Given a torus Σ satisfying the inductive hypothesis, any bypass attached to $A = \Sigma \setminus L$ will not change the slope of the dividing set.*

Proof. If the bypass is attached from the outside, then the slope cannot change or this would give a thickening of our non-thickenable solid torus. If the bypass is attached from the inside, then let Σ' be the torus obtained after the bypass is attached. By examining the Farey tessellation, we see that $s = \text{slope}(\Gamma_{\Sigma'})$ must lie in $[-\frac{2}{11}, -\frac{1}{6}]$. Since Lemma 6.6 disallows $s = -\frac{1}{6}$, suppose that $s \in (-\frac{2}{11}, -\frac{1}{6})$. Let Σ'' be a convex torus of slope $-\frac{1}{6}$ and $\#\Gamma = 2$ in the interior of the solid torus bounded by Σ'. Take a Legendrian curve L' on Σ' which is parallel to and disjoint from L, and intersects $\Gamma_{\Sigma'}$ minimally. Similarly, consider L'' on Σ''. Using the Farey tessellation, it is clear that $|\Gamma_{\Sigma'} \cap L'| > |\Gamma_{\Sigma''} \cap L''|$. Thus the Imbalance Principle gives bypasses for Σ' that are disjoint from L. After successive attachments of such bypasses, we eventually obtain Σ''' of slope $-\frac{1}{6}$ containing L, contradicting Lemma 6.6. □

Therefore we see that Condition (1) is preserved.

Lemma 6.8. *Given a torus Σ satisfying the inductive hypothesis, any bypass attached to A will preserve Conditions (2) and (3).*

Proof. Suppose Σ', is obtained from Σ by a single bypass move. We already know that slope$(\Gamma_{\Sigma'}) =$ slope(Γ_Σ), and, assuming the bypass move was not trivial, $\#\Gamma$ is either increased or decreased by 2. Suppose first that $\Sigma' \subset N$, where N is the solid torus bounded by Σ. For convenience, suppose $\Sigma = T_{0.5}$ inside $T^2 \times [0,1]$ satisfying Conditions (2) and (3) of the inductive hypothesis. Then we form the new $T^2 \times [0.5,1]$ by taking the old $T^2 \times [0.5,1]$ and adjoining the thickened torus between Σ and Σ'. Now, Σ' bounds a solid torus N', and, by the classification of tight contact structures on solid tori, we can factor a nonrotative outer layer which is the new $T^2 \times [0,0.5]$.

On the other hand, suppose $\Sigma' \subset (S^3 \setminus N)$. We prove that there exists a nonrotative outer layer $T^2 \times [0.5,1]$ for $S^3 \setminus N'$, where $\#\Gamma_{T_1} = 2$. This follows from repeating the procedure in the proof of Theorem 1.5, where Legendrian representatives of F_1 and F_2 were thickened and then connected by a vertical annulus — this time the same procedure is carried out with the provision that the representatives of F_1 and F_2 lie in $S^3 \setminus N'$. Once the maximal thickness for representatives of F_1 and F_2 is obtained, after rounding we get a convex torus in $S^3 \setminus N'$ parallel to Σ' but with $\#\Gamma = 2$. Therefore we obtain a nonrotative outer layer $T^2 \times [0.5,1]$. □

This completes the proof of Theorem 1.7. □

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REFERENCES

[Be] D. Bennequin, *Entrelacements et équations de Pfaff*, Third Schnepfenried geometry conference, Vol. 1 (Schnepfenried, 1982), 87–161, Astérisque **107–108**, Soc. Math. France, Paris, 1983.

[BM] J. Birman and W. Menasco, *Stabilization in the braid groups*, preprint 2002.

[Ch] Y. Chekanov, *Differential algebra of Legendrian links*, Invent. Math. **150** (2002), 441–483.

[Co] V. Colin, Ph. D. Thesis.

[EGH] Y. Eliashberg, A. Givental and H. Hofer, *Introduction to symplectic field theory*, GAFA 2000 (Tel Aviv, 1999), Geom. Funct. Anal. 2000, Special Volume, Part II, 560–673.

[El] Y. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier (Grenoble) **42** (1992), 165–192.

[EF] Y. Eliashberg and M. Fraser, *Classification of topologically trivial Legendrian knots*, in *Geometry, topology, and dynamics* (Montreal, PQ, 1995), 17–51, CRM Proc. Lecture Notes **15**, Amer. Math. Soc., Providence, RI, 1998.

[EH1] J. Etnyre and K. Honda, *Knots and contact geometry I: torus knots and the figure eight knot*, J. Symplectic Geom. **1** (2001), 63–120.

[EH2] J. Etnyre and K. Honda, *On connected sums and Legendrian knots*, to appear in Adv. Math.

[EH3] J. Etnyre and K. Honda, *On the nonexistence of tight contact structures*, Ann. of Math. (2) **153** (2001), 749–766.
[EH4] J. Etnyre and K. Honda, *Tight contact structures with no symplectic fillings*, Invent. Math. **148** (2002), 609–626.
[Ga] D. Gay, *Symplectic 2-handles and transverse links*, Trans. Amer. Math. Soc. **354** (2002), 1027–1047.
[Gi1] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helv. **66** (1991), 637–677.
[Gi2] E. Giroux, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*, Invent. Math. **141** (2000), 615–689.
[H1] K. Honda, *On the classification of tight contact structures I*, Geom. Topol. **4** (2000), 309–368.
[H2] K. Honda, *On the classification of tight contact structures II*, J. Differential Geom. **55** (2000), 83–143.
[H3] K. Honda, *Gluing tight contact structures*, Duke Math. J. **115** (2002), 435–478.
[H4] K. Honda, *Factoring nonrotative $T^2 \times I$ layers. Erratum to "On the classification of tight contact structures I"*, Geom. Topol. **5** (2001), 925–938.
[K] Y. Kanda, *The classification of tight contact structures on the 3-torus*, Comm. Anal. Geom. **5** (1997), 413–438.
[M1] W. Menasco, *On iterated torus knots and transversal knots*, Geom. Topol. **5** (2001), 651–682.
[M2] W. Menasco, *Erratum to: On iterated torus knots and transversal knots*, in preparation.
[Ng] L. Ng, *Computable Legendrian invariants*, Topology **42** (2002), 55–82.

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