QUANTITATIVE DARBOUX THEOREMS
IN CONTACT GEOMETRY

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ABSTRACT. This paper begins the study of relations between Riemannian geometry and contact topology on $(2n + 1)$–manifolds and continues this study on 3–manifolds. Specifically we provide a lower bound for the radius of a geodesic ball in a contact $(2n + 1)$–manifold (M, ξ) that can be embedded in the standard contact structure on $\mathbb{R}^{2n+1}$, that is on the size of a Darboux ball. The bound is established with respect to a Riemannian metric compatible with an associated contact form α for ξ. In dimension three, it further leads us to an estimate of the size for a standard neighborhood of a closed Reeb orbit. The main tools are classical comparison theorems in Riemannian geometry. In the same context, we also use holomorphic curves techniques to provide a lower bound for the radius of a PS-tight ball.

1. INTRODUCTION

Darboux's theorem in contact geometry says that any point in a $(2n + 1)$ dimensional contact manifold has a neighborhood that can be identified with an open ball in $\mathbb{R}^{2n+1}$ with its standard contact structure. In [7] it was shown that a quantitative version of Darboux's theorem can give interesting global information about a contact structure on a 3–manifold. For example one can give Riemannian geometric criteria for a contact structure to be universally tight and in addition prove a contact geometric version of the sphere theorem. Such results rely on deep theorems about contact 3–manifolds. Our understanding of contact manifold in higher dimensions is much less advanced but we will still be able to prove a quantitative version of Darboux's theorem.

Given a contact structure ξ on a $(2n + 1)$–manifold M and a Riemannian metric g we can define the *Darboux radius* of (M, ξ) at a point $p \in M$ as

$$\delta_p(\xi, g) = \sup\{r \mid \text{the geodesic ball } (B_p(r), \xi) \text{ at } p \text{ of radius } r \text{ is}$$
$$\text{contactomorphic to an open ball in } (\mathbb{R}^{2n+1}, \xi_{\text{std}})\},$$

and the *Darboux radius of M* to be

$$\delta(\xi, g) = \inf_{p \in M} \delta_p(\xi, g).$$

One would like to estimate these quantities in terms of g. In dimension 3 this was done in [7] but relied heavily on a theorem of Eliashberg [6] that says that any tight contact structure on a 3–ball is embeddable in the standard contact structure on $\mathbb{R}^3$. In particular in [7] we defined the *tightness radius at a point p*, τ_p, and the *tightness radius τ* for a contact metric 3-manifold. Eliashberg's theorem identifies τ_p and τ with δ_p and δ, respectively. As Eliashberg's theorem is unavailable to us in higher dimensions and it is even plausible that it does not hold, we will discuss direct arguments to estimate δ.

But first, a more direct generalization of the estimates proved in [7] would be to estimate the size of geodesic balls in contact manifolds that were in some sense "tight". However we need to clarify what we will mean by "tight" in higher dimensions. There is currently no tight vs. overtwisted dichotomy in higher dimensions, though one possibility for the definition of overtwisted in this context is the following: we say a contact structure ξ on M^{2n+1} is *PS-overtwisted* if it contains a bLob (see Section 3.3 for the relevant definitions). Otherwise we say ξ is *PS-tight*. It was shown

in [12, 13] that a PS-overtwisted contact manifold cannot be symplectically filled by a monotone symplectic manifold, thus sharing a property of overtwisted contact structures in dimension 3. Given a contact $(2n + 1)$–manifold (M, ξ) and a Riemannian metric g we now define the *PS–tightness radius* at $p \in M$ with respect to g to be

$$\tau_p^{\mathrm{PS}}(\xi, g) = \sup\{r \,|\, \text{the geodesic ball } B_p(r) \text{ at } p$$
$$\text{of radius } r \text{ is PS–tight}\},$$

and the *PS-tightness radius of M* to be

$$\tau^{\mathrm{PS}}(\xi, g) = \inf_{p \in M} \tau_p^{\mathrm{PS}}(\xi, g).$$

Again, in dimension 3 the above quantities all are equal

$$\tau^{\mathrm{PS}}(\xi, f) = \tau(\xi, g) = \delta(\xi, g),$$

because in that dimension *PS-tight* is equivalent to *tight*, which, for a 3–ball, in turn is equivalent to being a Darboux ball. In higher dimensions there may be no relation between $\tau^{\mathrm{PS}}(\xi, g)$, and $\delta(\xi, g)$.

1.1. **Estimates for τ^{PS}.** The first result in this article extends convexity type estimates for $\tau^{\mathrm{PS}}(M, \xi)$ from [7] to higher dimensions in the setting of compatible metrics. In higher dimensions the definition of a metric g being compatible with a contact structure ξ on M^{2n+1} is more complicated than the one considered in [7] for dimension 3. We refer to Definition 2.6 for the precise details. Here we merely note that given a contact form α for ξ used in the definition of compatibility, Proposition 2.2 gives an almost complex structure J on ξ, and the metric can be written as follows

$$g(v, w) = \frac{1}{\theta'} d\alpha(v, Jw^\xi) + \alpha(v)\alpha(w),$$

where θ' is constant and measures the instantaneous rotation speed of ξ, and w^ξ denotes the component of the vector w lying in ξ.

We obtain the following generalization of Theorem 1.3 from [7].

Theorem 1.1. *Let (M^{2n+1}, ξ) be a contact manifold and (α, g, J) be a compatible metric structure for ξ. Then,*

(1.1) $$\tau^{\mathrm{PS}}(\xi, g) \geq \mathrm{conv}(g),$$

where

$$\mathrm{conv}(g) = \sup\{r \,|\, r < \mathrm{inj}(g) \text{ and the geodesic balls of radius } r$$
$$\text{are weakly geodesically convex}\},$$

and $\mathrm{inj}(g)$ is the injectivity radius of (M, g). In particular, if $\sec(g) \leq K$, $K > 0$, then

$$\tau^{\mathrm{PS}}(\xi, g) \geq \min\{\mathrm{inj}(g), \frac{\pi}{2\sqrt{K}}\}$$

and $\tau^{\mathrm{PS}}(\xi, g) = \mathrm{inj}(g)$, if g has non-positive curvature.

As in dimension 3 the bound on τ^{PS} is especially effective in the case of non-positive curvature.

Corollary 1.2. *Let (M, ξ) be a $(2n + 1)$-contact manifold and g a complete Riemannian metric compatible with ξ having non-positive sectional curvature. Then ξ pulled back to the universal cover of M is PS-tight. In particular, if (M, ξ) contains a **bLob** N, then the image of $i_* : \pi_1(N) \to \pi_1(M)$, where $i : N \to M$ is the inclusion map, is infinite.*

Remark 1.3. Sasakian (or, more generally, K-contact) manifolds are contact metric manifolds satisfying some extra conditions, see [3]. One can naturally wonder whether Theorem 1.1 is relevant to their study. But those manifolds do not contain any bLob since one can combine the main results of [14] and [16] to prove that they are symplectically fillable. The same conclusion actually holds for the wider class of integrable CR-contact metric manifolds in dimension at least 5, see Definition 2.7. As explained in [5, Theorem 5.60] this can be proved using deep results in complex analysis due to Lempert, Hironaka and Rossi. However, Theorem 1.4 below still gives non-trivial information in this context. □

1.2. **Direct geometric methods for estimating the Darboux radius.** We now discuss a method for estimating $\delta(M^{2n+1}, \xi)$ in higher dimensions. This strategy is more geometric and direct that the one used in [7] to bound the Darboux radius in dimension 3, as it does not use holomorphic curves or classification results. We want a control on the Darboux radius using control on curvature and the rotation speed θ'. Note that, if $\mathcal{K}$ is a bound on sectional curvature then both $1/\theta'$ and $1/\sqrt{\mathcal{K}}$ behave like lengths under homothety (constant rescaling of the metric). This explains the appearance of such terms in the following estimate. It is also expected that the estimate deteriorates when the rotation speed increases or when one widen the sectional curvature interval.

Theorem 1.4. *Let (M^{2n+1}, ξ) be a $(2n + 1)$–dimensional contact manifold and (α, g, J) be a complete compatible metric structure for ξ with rotation speed θ'. If the sectional curvature of g is contained in the interval $[-\mathcal{K}, \mathcal{K}]$ for some positive $\mathcal{K}$ then*

$$\delta(\xi, g) \geq \min\left(\frac{\text{inj}(g)}{2}, \frac{c_n}{\max(\sqrt{\mathcal{K}}, \theta')}\right)$$

where $c_n := 1/(192(1 + 2n(n-1))\sqrt{n})$ depends only on dimension.

Theorem 4.1 is a better, but more technical, bound on $\delta(\xi, g)$ and is proven in Section 4 after the explicit quantities used in the better estimate are explained, but the main point is the existence of an *explicit* bound rather than its precise expression.

Remark 1.5. Since every contact structure has a compatible metric structure, this theorem always produces an explicit bound on the Darboux radius in terms of Riemannian curvature information.

In dimension 3 we can improve the bound on $\delta(\xi, g)$ coming from these geometric methods.

Theorem 1.6. *Let (M, ξ) be a contact 3–manifold and g a complete Riemannian metric of bounded curvature that is compatible with ξ and has rotation speed θ'. Then,*

(1.2) $$\delta(\xi, g) = \tau(M, \xi) \geq \min\left\{\frac{1}{2}\text{inj}(g), \frac{\pi}{2\sqrt{K}}, \frac{2}{\sqrt{2A + B^2} + B}\right\},$$

where

$$A = \max_{p \in M} \frac{4}{3}|\sec_p(g)|, \quad B = \frac{\theta'}{2} + \sqrt{\frac{\theta'^2}{4} - \frac{1}{2}\min_{p \in M}(\text{Ric}_p(R_\alpha))}$$

and Ric *is the Ricci tensor.*

Theorems 1.4 and 1.6 provide bounds on $\delta(\xi, g)$ which are weaker than the one given in Theorem 1.1 in dimension 3. But the ideas underlying its proof extends to higher dimensions and can be used when both the convexity of the boundary assumption of Theorem 1.1 and the absence of closed Reeb orbits fail, thus ruining the strategy used to prove Theorem 1.1. As an example of such situation we show the following estimate on the size of a standard neighborhood of a closed Reeb orbit in a contact 3–manifold.

Theorem 1.7. *Let (M, ξ) be a contact 3–manifold and g a complete Riemannian metric that is compatible with ξ. Let γ be a closed Reeb orbit and $\mathsf{T}(r)$ an embedded geodesic tube of radius r about γ. If r is below the bound of Equation* (1.2) *then the contact structure restricted to $\mathsf{T}(r)$ is tight and, moreover, can be embedded in $(S^1 \times D^2, \xi_{std} = \ker(d\phi + r^2 \, d\theta))$.*

1.3. **Outline.** In Section 2 we define the notions of compatibility between metrics and contact structures in an arbitrary dimension. We also recall some facts about convexity in Riemannian geometry in Section 3.1. The paper then splits into two logically independent threads which both depend on Section 2. The first one is covered in Section 3 which compares Riemannian and almost complex convexity and proves Theorem 1.1. The second one starts in Section 4 which states the precise version of Theorem 1.4 and proves it modulo a number of propositions which are proved in Subsections 4.1 to 4.4. Section 5 explains how geometrical methods of Section 4 can be strengthened using topological methods which are specific to dimension 3 and proves Theorems 1.6 and 1.7.

2. METRICS COMPATIBLE WITH CONTACT STRUCTURES

Throughout this section and the rest of the paper we will interchangeably use the notation $g(u, v)$ and $\langle u, v \rangle$ to denote the Riemannian metric evaluated on the vectors u and v. Let V be a vector space, ω a symplectic pairing on V, and g is an inner product on V. There are isomorphisms $\phi_\omega : V \to V^*$ and $\phi_g : V \to V^*$ given by contracting a vector v into ω and, respectively, g. We say that ω and g are *weakly compatible* if there is some non-zero constant c such that the linear map $A = \phi_g^{-1} \circ \phi_\omega : V \to V$ satisfies

$$A^2(v) = -c^2 v$$

for all $v \in V$. Note that $\frac{1}{c} A$ clearly defines an almost complex structure on V. (It is more standard in the literature to deal to g and ω being compatible, in that case $c = 1$). We notice that if V is 2-dimensional then g and ω are automatically compatible.

We also recall that a symplectic pairing ω on V and an almost complex structure J on V are called *compatible* if for all u and v in V we have $\omega(v, Jv) > 0$ for $v \neq 0$ and $\omega(Ju, Jv) = \omega(u, v)$.

Definition 2.1. We say a contact structure ξ and a Riemannian metric g on a $(2n+1)$–manifold M are *weakly compatible* if there is a contact form α for ξ such that its Reeb vector field R_α is orthogonal to ξ and $g|_\xi$ and $(d\alpha)|_\xi$ are, pointwise, weakly compatible.

We notice that if g and α are weekly compatible, then we have an endomorphism $A : \xi \to \xi$ that is defined pointwise using $g|_\xi$ and $(d\alpha)|_\xi$. By definition we have $A^2 = -c^2 \, id_\xi$, where c is a positive function on M. Generalizing a result from dimension 3 in [7] we have the following result.

Proposition 2.2. *Let ξ be a contact structure on a (2n+1)–manifold M and g a weakly compatible metric. Then fixing the contact form α implicated in the definition of weakly compatible there is a unique almost complex structure $J : \xi \to \xi$ on ξ that is compatible with $d\alpha|_\xi$ and satisfies*

$$g(u, Jv) = -g(Ju, v) \quad and \quad g(u, v) = g(Ju, Jv),$$

for all $u, v \in \xi$. Moreover $\|v\| = \|Jv\|$ and v and Jv are orthogonal.

The almost complex structure J can be extended to a linear map on TM as follows

$$\text{(2.1)} \qquad \phi : TM \to TM : v \to Jv^{\xi},$$

where

$$\text{(2.2)} \qquad v^{\xi} = v - g(v, n)n$$

is the component of v lying in ξ (here $n = R_{\alpha}/\|R_{\alpha}\|$ is the unit normal vector to ξ). The metric g may be expressed

$$\text{(2.3)} \qquad g(u, v) = \frac{\rho}{\theta'} d\alpha(u, \phi(v)) + \rho^2 \alpha(u)\alpha(v),$$

where $\rho = \|R_{\alpha}\|$ and $\theta' = \rho c$. Here c is the function coming from $A^2 = -c^2 \, id_{\xi}$.

Remark 2.3. We expressed Equation 2.3 using θ' instead of c because θ' is a quantity intrinsically associated to a contact structure ξ and a metric g. Specifically, it is the *instantaneous rotation of ξ with respect to g* defined in [7]. We do not give the definition here as we do not use it in this paper, but merely mention it as a means of justifying the notation used in the equaiton.

Remark 2.4. It is useful to notice that ϕ^2 can be expressed as

$$\text{(2.4)} \qquad \phi^2(v) = -v + \alpha(v)R_{\alpha},$$

for all vectors $v \in TM$.

Proof. Using the notation above we have the endomorphism $A : \xi \to \xi$ with $A^2 = -c^2 \, id_{\xi}$. Set $J = \frac{1}{c}A$, by the definition of A we see

$$g(Au, v) = \phi_g(A(u))(v) = \phi_{d\alpha}(u)(v) = d\alpha(u, v)$$
$$= -d\alpha(v, u) = -g(Av, u) = -g(u, Av).$$

This clearly implies that J is also skew-symmetric with respect to g. Skew-symmetry of an almost complex structure and a metric is equivalent to the metric being invariant under the complex structure so we also have $g(Ju, Jv) = g(u, v)$. From the above we also see that $d\alpha(u, Jv) = c\, g(u, v)$ for $u, v \in \xi$, from which the compatibility of J and $d\alpha|_{\xi}$ easily follows. The orthogonality of v and Jv and the fact that those two vectors have the same norm is an immediate consequence of these formulas.

Given any vector $v \in TM$ we write $v = v^{\xi} + v^n$ where v^{ξ} is the component of v in ξ and v^n is the component of v normal to ξ. Noting that $\rho\alpha = \iota_n g$, we compute

$$g(u, v) = g(u^{\xi}, v^{\xi}) + g(u^n, v^n) = \frac{1}{c}d\alpha(u^{\xi}, Jv^{\xi}) + g(n, u)g(n, v)$$
$$= \frac{1}{c}d\alpha(u, \phi(v)) + \rho^2\alpha(u)\alpha(v),$$

where the last equality follows by noting that $d\alpha(u, v) = d\alpha(u^{\xi}, v)$. $\qquad \square$

It is difficult to say much about weakly compatible metrics in higher dimensions, but if we assume the length of R_{α} is constant then we can understand something about the covariant derivatives of vectors with respect to R_{α} and observe that the flow of the Reeb vector field traces out geodesics.

Proposition 2.5. *Let g be a metric weakly compatible with the contact structure ξ on the $(2n+1)$–manifold M. Let R_α be the Reeb vector field associated to the form α implicated in the definition of g being weakly compatible with ξ. If $\rho = \|R_\alpha\|$ is constant, then*

$$(2.5) \qquad\qquad\qquad\qquad \nabla_{R_\alpha} R_\alpha = 0.$$

In particular, flow lines of R_α are geodesics. Moreover, if v is a vector field tangent to ξ then $\nabla_{R_\alpha} v$ is also tangent to ξ.

Proof. We first notice that α and R_α are are related by

$$(2.6) \qquad\qquad\qquad\qquad \rho^2 \alpha = \iota_{R_\alpha} g,$$

since they agree on ξ and on R_α. For any vector field v the definition of the Reeb vector field implies $d\alpha(R_\alpha, v) = 0$. On the other hand, using the assumption that ρ is (a non-zero) constant, we compute

$$
\begin{aligned}
d\alpha(R_\alpha, v) &= R_\alpha \cdot \alpha(v) - v \cdot \alpha(R_\alpha) - \alpha([R_\alpha, v]) \\
&= R_\alpha \cdot \alpha(v) - \alpha(\nabla_{R_\alpha} v - \nabla_v R_\alpha) \\
&= \rho^{-2}(R_\alpha \cdot \langle R_\alpha, v \rangle - \langle R_\alpha, \nabla_{R_\alpha} v - \nabla_v R_\alpha \rangle) \\
&= \rho^{-2}(\langle \nabla_{R_\alpha} R_\alpha, v \rangle + \langle R_\alpha, \nabla_{R_\alpha} v \rangle - \langle R_\alpha, \nabla_{R_\alpha} v \rangle + \langle R_\alpha, \nabla_v R_\alpha \rangle) \\
&= \rho^{-2}(\langle \nabla_{R_\alpha} R_\alpha, v \rangle),
\end{aligned}
$$

where the last equality follows since $2\langle R_\alpha, \nabla_v R_\alpha \rangle = v \cdot \langle R_\alpha, R_\alpha \rangle = 0$. So $\langle \nabla_{R_\alpha} R_\alpha, v \rangle = 0$ for every v and thus $\nabla_{R_\alpha} R_\alpha = 0$.

If v is a vector field tangent to ξ then

$$\langle R_\alpha, \nabla_{R_\alpha} v \rangle = \langle \nabla_{R_\alpha} R_\alpha, v \rangle + \langle R_\alpha, \nabla_{R_\alpha} v \rangle = R_\alpha \cdot \langle R_\alpha, v \rangle = 0,$$

so $\nabla_{R_\alpha} v$ is tangent to ξ. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We will not say more about weak compatibility in higher dimensions and restrict our attention to the stronger notion of compatibility.

Definition 2.6. We say a contact structure ξ and a Riemannian metric g on a $(2n + 1)$–manifold M are *compatible* if there is a contact form α for ξ such that its Reeb vector field R_α orthogonal to ξ, has unit length $\|R_\alpha\| = 1$, and $g|_\xi$ and $(d\alpha)|_\xi$ are, pointwise, weakly compatible so that the function θ', defined in Proposition 2.2, is constant.

Equivalently, we can say g and ξ are *compatible* if there is a contact form α for ξ and an almost complex structure J on ξ that is compatible with $d\alpha$ on ξ, satisfies

$$(2.7) \qquad\qquad\qquad\qquad g(u, v) = \frac{1}{\theta'} d\alpha(u, Jv),$$

for all $u, v \in \xi$, where θ' is some positive constant and for which the Reeb vector field R_α is the unit normal to ξ.

Recall [3, p. 63] that the *Nijenhuis torsion* $[T, T]$ of a $(1, 1)$–tensor field T is a skew–symmetric tensor field of type $(1, 2)$ defined as

$$(2.8) \qquad\qquad [T, T](X, Y) = T^2[X, Y] + [TX, TY] - T[TX, Y] - T[X, TY]$$

Definition 2.7. Let ξ be a contact structure on a $(2n+1)$–manifold M. An almost complex structure J on ξ is called a *CR–structure* (or more specifically a *strictly pseudo-convex integrable CR–structure*) if and only if the Nijenhuis torsion $[J, J]$ of J on ξ vanishes. This explicitly says that for all sections v and w of ξ

$$(2.9) \qquad [J, J](v, w) = -[v, w] + [Jv, Jw] - J([Jv, w] + [v, Jw]) = 0,$$

Notice that $[J, J]$ is well defined as a $(1, 2)$–tensor field on ξ because

$$\alpha([Jv, w] + [v, Jw]) = -d\alpha(Jv, w) - d\alpha(v, Jw) = -d\alpha(Jv, w) + d\alpha(Jv, w) = 0.$$

thus

$$(2.10) \qquad\qquad [Jv, w] + [v, Jw] \text{ is tangent to } \xi.$$

Similarly one can see that $[Jv, Jw] - [v, w]$ is tangent to ξ.

After [3], we call (M, α, g, J) a *contact metric structure* whenever α and J define the compatible metric g as above, equivalently we say that (α, g, J) is a compatible metric structure to (M, ξ). Additionally, if J and ξ define a CR-structure we say that (M, α, g, J) is a *CR–contact metric structure*. The integrability condition in Equation (2.9) can be expressed in terms of the torsion of $(1, 1)$–tensor field ϕ given in Equation (2.1). Indeed, for any $v, w \in \xi$, using $J = \phi|_\xi$ and Equation (2.10), we have

$$
(2.11)\qquad
\begin{aligned}
[\phi, \phi](v, w) &= \phi^2([v, w]) + [\phi(v), \phi(w)] - \phi([\phi(v), w]) - \phi([v, \phi(w)]) \\
&= -[v, w] + \alpha([v, w])R_\alpha + [Jv, Jw] - J([Jv, w] + [v, Jw]) \\
&= [J, J](v, w) - d\alpha(v, w)R_\alpha = [J, J](v, w) + \langle v, Jw \rangle R_\alpha.
\end{aligned}
$$

Therefore, Equation (2.9) is equivalent to

$$(2.12) \qquad\qquad ([\phi, \phi](v, w))^\xi = 0,$$

for any $v, w \in \xi$ and the projection $(\,\cdot\,)^\xi$ onto ξ is defined in Equation (2.2).

Note that the above integrability condition is automatic for any almost complex structure on a plane field in dimension 3. Indeed, we may choose a basis $\{v, \phi(v)\}$ of ξ and observe

$$(2.13) \qquad\qquad [J, J](v, v) = 0, \qquad [J, J](v, Jv)) = 0,$$

from which Equation (2.9) follows.

We are now ready to state relations between the various operators and their derivatives. Most these formulas are well known in the literature, *cf.* [3], though extra terms occur due to the generality we are considering here. We also note that there are some sign discrepancies with [3], coming from the fact that we are always using positive contact structures. Due to this, and for the convenience of the reader, we provide proofs of these formulas here.

Recall that *second fundamental form II* of ξ is the quadratic form on ξ defined as follows [15]: for vectors u and v in $\xi_p = T_pM \cap \xi$,

$$(2.14) \qquad\qquad II(u, v) = \frac{1}{2} \langle \nabla_u v + \nabla_v u, n \rangle,$$

where n is the oriented unit normal to ξ. We also define the endomorphism

$$(2.15) \qquad\qquad h = \frac{1}{2}\mathcal{L}_{R_\alpha}\phi : TM \to TM,$$

which is essentially a repackaging of II. (Here $\mathcal{L}$ denotes the Lie derivative.)

Lemma 2.8. *Let (α, g, J) be a compatible metric structure on (M, ξ). Then $h(TM) \subset \xi$.*

Proof. First compute

$$(2.16) \qquad h(v) = \frac{1}{2}(\mathcal{L}_{R_\alpha}\phi)(v) = \frac{1}{2}(\mathcal{L}_{R_\alpha}(\phi(v)) - \phi(\mathcal{L}_{R_\alpha}v)) = \frac{1}{2}([R_\alpha, \phi(v)] - \phi([R_\alpha, v])).$$

Since for any section u of ξ we have $\alpha([R_\alpha, u]) = -d\alpha(R_\alpha, u) = 0$, we see that $[R_\alpha, u]$ is in ξ for any vector field u in ξ. The result follows since the image of ϕ is contained in ξ. $\qquad \square$

The basic relations between metric and contact geometric quantities are derived in the following proposition.

Proposition 2.9. *Let (α, g, J) be a compatible metric structure on the contact manifold (M, ξ). Let R_α be the Reeb vector field of α, h the endomorphism defined above and II the second fundamental form of ξ. Then the following equations hold*

$$(2.17) \qquad \langle h(u), v \rangle = \langle u, h(v) \rangle,$$

$$(2.18) \qquad \nabla_v R_\alpha = \phi\left((\theta'/2)\,v - h(v)\right),$$

$$(2.19) \qquad (\phi h + h\phi)(v) = 0,$$

$$(2.20) \qquad II(v, v) + II(Jv, Jv) = 0,$$

and

$$(2.21) \qquad \nabla_{R_\alpha}(Jv) = J(\nabla_{R_\alpha}v),$$

for any $v, u \in \xi$.

We will also need the following relations between curvature and contact geometric properties.

Proposition 2.10. *Let (α, g, J) be a compatible metric structure on (M, ξ), then*

$$(2.22) \qquad \|h\|^2 \leq n\left(\frac{\theta'}{2}\right)^2 - \frac{1}{2}\mathrm{Ric}(R_\alpha),$$

where h is defined above, $\mathrm{Ric}(R_\alpha)$ is the Ricci curvature in the direction of R_α, and θ' a constant defined in Equation (2.7). In dimension 3, we obtain equality in Formula (2.22). In addition, for a vector u, one has

$$(2.23) \qquad \|\nabla_u R_\alpha\| \leq B\|u\|$$

and

$$(2.24) \qquad \|\nabla_u \phi\| \leq 2B\|u\|,$$

where

$$B = \frac{\theta'}{2} + \sqrt{n\frac{\theta'^2}{4} - \frac{1}{2}\min_M\left(\mathrm{Ric}(R_\alpha)\right)}.$$

A key to proving the above results is the computation of the covariant derivative of ϕ.

Lemma 2.11. *Let (α, g, J) be a compatible metric structure on (M, ξ). Then for any vectors u, v and w the following equation holds.*

$$(2.25) \qquad \langle (\nabla_u \phi)(v), w \rangle = \frac{1}{2}\left(\langle [\phi, \phi](v, w), \phi(u) \rangle - \theta'\alpha(w)\left\langle u^\xi, v^\xi \right\rangle + \theta'\alpha(v)\left\langle u^\xi, w^\xi \right\rangle\right).$$

Proof of Proposition 2.9. Throughout this proof we will repeatedly use the formula for ϕ^2 given in Equation (2.4) without further notice.

We specialize Equation (2.25) to $u = R_\alpha$ and $v, w \in \xi$, we get $\langle (\nabla_{R_\alpha} \phi)(v), w \rangle = 0$. Thus using Equation (2.16) we see that

$$\langle 2\, h(v), w \rangle = \langle [R_\alpha, \phi(v)] - \phi([R_\alpha, v]), w \rangle = \langle (\nabla_{R_\alpha} \phi)(v) - \nabla_{\phi(v)} R_\alpha + \phi(\nabla_v R_\alpha), w \rangle$$
$$= \langle \phi(\nabla_v R_\alpha) - \nabla_{\phi(v)} R_\alpha, w \rangle = \langle R_\alpha, \nabla_{\phi(v)} w + \nabla_v \phi(w) \rangle = \alpha(\nabla_{\phi(v)} w + \nabla_v \phi(w)).$$

Note that since $d\alpha(\phi(v), w) = -d\alpha(v, \phi(w))$, we have $\alpha([\phi(v), w]) + \alpha([v, \phi(w)]) = 0$ and therefore

$$\langle 2\, h(v), w \rangle = \alpha(\nabla_w \phi(v) + \nabla_{\phi(w)} v) = \langle v, 2\, h(w) \rangle.$$

This proves Equation (2.17).

For Equation (2.18), observe $[\phi, \phi](R_\alpha, w) = \phi^2([R_\alpha, w]) - \phi([R_\alpha, \phi(w)])$ thus specializing Equation (2.25) to $v = R_\alpha$ above we obtain

$$\langle (\nabla_u \phi)(R_\alpha), w \rangle = \frac{1}{2} \langle \phi^2([R_\alpha, w]) - \phi([R_\alpha, \phi(w)], \phi(u) \rangle + \theta'/2 \langle u^\xi, w^\xi \rangle$$
$$= \langle h(w), \phi^2(u) \rangle + \theta'/2 \langle u^\xi, w^\xi \rangle$$
$$= -\langle h(w), u \rangle + \alpha(u) \langle h(w), R_\alpha \rangle + \theta'/2 \langle u^\xi, w^\xi \rangle$$
$$= -\langle h(u), w \rangle + \theta'/2 \langle u^\xi, w^\xi \rangle$$

where the we used Equation (2.16) in the second identity and $h(TM) \subset \xi$ together with the symmetry Equation (2.17) of h in the last one. Since $\phi(R_\alpha) = 0$, the left hand-side of the above is $\langle \nabla_u(\phi(R_\alpha)) - \phi(\nabla_u R_\alpha), w \rangle = -\langle \phi(\nabla_u R_\alpha), w \rangle$. Thus $\phi(\nabla_u R_\alpha) = h(u) - (\theta'/2) u^\xi$. Applying ϕ, noting that $\nabla_u R_\alpha$ is tangent to ξ and recalling that $\phi = J$ on ξ, we establish Equation (2.18).

We now prove that h anti-commutes with ϕ, that is we prove Equation (2.19)

$$-\theta' \langle \phi(v), u \rangle + \langle u, h(\phi(v)) + \phi(h(v)) \rangle = \left\langle \frac{\theta'}{2} \phi(u) - \phi(h(u)), v \right\rangle - \left\langle \frac{\theta'}{2} \phi(v) - \phi(h(v)), u \right\rangle$$
$$= \langle \nabla_u R_\alpha, v \rangle - \langle \nabla_v R_\alpha, u \rangle = -\langle R_\alpha, [u, v] \rangle = d\alpha(u, v) = -\theta' \langle u, \phi(v) \rangle.$$

The second equality follows from Equation (2.18) and the last equality follows from the fact that for all u and v in TM we have

(2.26)
$$g(u, \phi(v)) = -\frac{1}{\theta'} d\alpha(u, v),$$

which in turn follows from Equation (2.3) and the fact that $\phi(v) \in \xi$. Continuing we see $\langle u, h(\phi(v)) + \phi(h(v)) \rangle = 0$ for all $u \in \xi$, thus establishing Equation (2.19).

To prove Equation (2.20) (which is the only one which appears to be new), we compute for any $v \in \xi$ (i.e. vector field extension of v)

$$II(v, v) + II(Jv, Jv) = \langle \nabla_v v, R_\alpha \rangle + \langle \nabla_{Jv} Jv, R_\alpha \rangle = -\langle v, \nabla_v R_\alpha \rangle - \langle Jv, \nabla_{Jv} R_\alpha \rangle$$
$$= \langle v, -\theta'/2 Jv - h(Jv) \rangle + \langle Jv, -\theta'/2 J^2 v - h(J^2 v) \rangle$$
$$= -\langle v, h(Jv) \rangle + \langle Jv, h(v) \rangle = -\langle v, h(Jv) \rangle + \langle v, h(Jv) \rangle = 0,$$

where the third equality follows from Equation (2.18) and Equation (2.19) has been used repeatedly.

Equation (2.21) follows from Equation (2.25) and Lemma 2.8. Indeed, for $u \in \xi$ and any vector v we see that

$$\langle \nabla_{R_\alpha}(Ju), v \rangle = \langle (\nabla_{R_\alpha}\phi)(u), v \rangle + \langle \phi(\nabla_{R_\alpha}u), v \rangle$$
$$= \langle \phi(\nabla_{R_\alpha}u), v \rangle = \langle J(\nabla_{R_\alpha}u), v \rangle,$$

where the last equality follows since $\nabla_{R_\alpha}u$ is tangent to ξ by Proposition 2.5. $\qquad \square$

Proof of Lemma 2.11. To establish Equation (2.25) we notice that both sides of the equation are tensors in u, v and w, so it suffices to establish the result when u, v and w are chosen to be elements of a basis for TM. We choose vectors $v_1, \dots, v_n$ in ξ such that $v_1, Jv_1, \dots, v_n, Jv_n$ is an oriented orthonormal basis for ξ. In the computation below we assume that u, v and w are chosen from the set of vectors $\{v_1, Jv_1, \dots, v_n, Jv_n, R_\alpha\}$. Notice that this implies that the lengths of $u, v, w, \phi(u), \phi(v)$ and $\phi(w)$ are constant as are their inner products with each other.

Equation (2.26) says that $\theta' g(u, \phi(v)) = -d\alpha(u, v)$ for any vectors u and v. Using this and the fact that $d\alpha$ is closed we have

$$(2.27) \qquad \langle [u, v], \phi(w) \rangle + \langle [w, u], \phi(v) \rangle + \langle [v, w], \phi(u) \rangle = 0.$$

Recall, the Koszul formula states that for any vector fields u, v and w

$$2 \langle \nabla_u v, w \rangle = u \cdot \langle v, w \rangle + v \cdot \langle u, w \rangle - w \cdot \langle u, v \rangle + \langle [u, v], w \rangle + \langle [w, u], v \rangle + \langle [w, v], u \rangle.$$

Using this, Equation (2.4) to compute ϕ^2 and the previous equation we can begin our computation of $\nabla \phi$ as follows

$$2 \langle (\nabla_u \phi)(v), w \rangle = 2 \langle \nabla_u(\phi(v)) - \phi(\nabla_u v), w \rangle = 2 \langle \nabla_u(\phi(v)), w \rangle + 2 \langle \nabla_u v - \alpha(\nabla_u v)R_\alpha, \phi(w) \rangle$$
$$= \langle [u, \phi(v)], w \rangle + \langle [w, u], \phi(v) \rangle + \langle [w, \phi(v)], u \rangle$$
$$+ \langle [u, v], \phi(w) \rangle + \langle [\phi(w), u], v \rangle + \langle [\phi(w), v], u \rangle$$
$$= \langle [u, \phi(v)], w \rangle + \langle [w, \phi(v)], u \rangle$$
$$+ \langle [\phi(w), u], v \rangle + \langle [\phi(w), v], u \rangle - \langle [v, w], \phi(u) \rangle.$$

Substituting $\phi(v)$ for v and $\phi(w)$ for w in Equation (2.27) and using Equation (2.4) to compute ϕ^2 we learn

$$- \langle [u, \phi(v)], w \rangle + \alpha(w) \langle [u, \phi(v)], R_\alpha \rangle - \langle [\phi(w), u], v \rangle$$
$$+ \alpha(v) \langle [\phi(w), u], R_\alpha \rangle + \langle [\phi(v), \phi(w)], \phi(u) \rangle = 0.$$

This may be used to eliminate the first and third term in the preceding equation which becomes

$$(2.28) \qquad 2 \langle (\nabla_u \phi)(v), w \rangle = \langle [w, \phi(v)], u \rangle + \langle [\phi(w), v], u \rangle - \langle [v, w], \phi(u) \rangle + \alpha(w) \langle [u, \phi(v)], R_\alpha \rangle$$
$$+ \alpha(v) \langle [\phi(w), u], R_\alpha \rangle + \langle [\phi(v), \phi(w)], \phi(u) \rangle.$$

To obtain (2.25), substitute $u = -\phi^2(u) + \alpha(u)R_\alpha$ in the first two terms and use $\langle \cdot, \phi^2(\cdot) \rangle = \langle \phi(\cdot), \phi(\cdot) \rangle$ together with $\langle [v, w], \phi(u) \rangle = \langle \phi^2([v, w]), \phi(u) \rangle$ to obtain (after cancelation of terms involving R_α)

$$2 \langle (\nabla_u \phi)(v), w \rangle = - \langle \phi([\phi(v), w]), \phi(u) \rangle - \langle \phi([v, \phi(w)]), \phi(u) \rangle - \langle \phi^2([v, w]), \phi(u) \rangle$$
$$+ \alpha(w) \langle [u, \phi(v)], R_\alpha \rangle + \alpha(v) \langle [\phi(w), u], R_\alpha \rangle + \langle [\phi(v), \phi(w)], \phi(u) \rangle.$$

Note that

$$(2.29) \qquad \langle R_\alpha, [u, \phi(v)] \rangle = \alpha([u, \phi(v)]) = -d\alpha(u, \phi(v))$$
$$= -d\alpha(u^\xi, Jv^\xi) = -\theta' \langle u^\xi, v^\xi \rangle.$$

Thus collecting terms $\langle \cdot, \phi(u) \rangle$ in the previous identity and applying the definition of $[\phi, \phi]$, yields Equation (2.25). □

Proof of Proposition 2.10. To establish Estimate (2.22) we compute $\mathrm{Ric}(R_\alpha)$. Because h restricted to ξ is symmetric with respect to g we can find a local unit eigenvector v, $h(v) = \lambda v$, for some function λ. Since h anti-commutes with J we see $h(Jv) = -\lambda Jv$. Let $\{v_1, Jv_1, \dots, v_n, Jv_n\}$ be the orthonormal basis of eigenvectors of h (and thus a frame for ξ.) Let $v = v_i$, and $\lambda = \lambda_i$ for some $1 \leq i \leq n$. From Equation (2.18) we have

$$\nabla_v R_\alpha = (\theta'/2 - \lambda)Jv,$$
$$\nabla_{Jv} R_\alpha = -(\theta'/2 + \lambda)v.$$

We now compute sectional curvature $K(v, R_\alpha)$ by repeatedly using that h is symmetric and anti-commutes with J and J is antisymmetric:

$$
\begin{aligned}
K(v, R_\alpha) &= \langle \nabla_{[R_\alpha, v]} R_\alpha, v \rangle - \langle \nabla_{R_\alpha} \nabla_v R_\alpha, v \rangle \\
&= \langle \theta'/2 J[R_\alpha, v] - Jh([R_\alpha, v]), v \rangle - \langle \nabla_{R_\alpha} (\theta'/2 - \lambda)Jv, v \rangle \\
&= \theta'/2 \langle J[R_\alpha, v], v \rangle + \lambda \langle J[R_\alpha, v], v \rangle - \langle \nabla_{R_\alpha} (\theta'/2 - \lambda)Jv, v \rangle \\
&= (\theta'/2 + \lambda) \langle J[R_\alpha, v], v \rangle - (\theta'/2 - \lambda) \langle \nabla_{Jv} R_\alpha + [R_\alpha, Jv], v \rangle + \langle (\nabla_{R_\alpha} \lambda) Jv, v \rangle \\
&= (\theta'/2 + \lambda) \langle J[R_\alpha, v], v \rangle - (\theta'/2 - \lambda) \langle [R_\alpha, Jv], v \rangle - (\theta'/2 - \lambda) \langle -(\theta'/2 + \lambda)v, v \rangle \\
&= (\theta'/2 + \lambda) \langle J[R_\alpha, v], v \rangle - (\theta'/2 - \lambda) \langle [R_\alpha, Jv], v \rangle + \theta'^2/4 - \lambda^2.
\end{aligned}
$$

Similarly, we see that

$$K(Jv, R_\alpha) = -(\theta'/2 + \lambda) \langle J[R_\alpha, v], v \rangle + (\theta'/2 - \lambda) \langle [R_\alpha, Jv], v \rangle + \theta'^2/4 - \lambda^2.$$

Therefore

$$\mathrm{Ric}(R_\alpha) = \sum_{i=1}^{n} \left(K(v_i, R_\alpha) + K(Jv_i, R_\alpha) \right) = \sum_{i=1}^{n} 2 \left(\frac{\theta'^2}{4} - \lambda_i^2 \right).$$

Noting that $\|h\|^2 = \lambda_{\max}^2$, where $\lambda_{\max}^2 = \max_{1 \leq i \leq n} \lambda_i^2$, we establish Inequality (2.22).

To prove Estimate (2.24) we first observe that Equation (2.25) immediately implies

$$\|\nabla \phi\| \leq 2 \Big(\frac{\theta'}{2} + \|h\| \Big) \|\alpha\|.$$

Estimate (2.24) then follows from the estimate $\|h\| \leq B - \frac{\theta'}{2}$ and $\|\alpha\| = 1$. Analogously, Estimate (2.23) is a direct consequence of Equation (2.18). □

3. A TIGHTNESS RADIUS ESTIMATE

This section is devoted to the proof of Theorem 1.1. It uses classical holomorphic curves arguments and a key comparison of Riemannien and almost complex convexity in symplectizations of contact metric manifolds. This comparison is explained in Subsection 3.2 after we recall a few results about Riemannian convexity in Subsection 3.1. We then recall the definition of *PS*-overtwisted manifolds and their relevant properties in Subsection 3.3 before proving the theorem in Subsection 3.4.

3.1. **Convexity in Riemannian geometry.** Let S be a hypersurface in an Riemannian manifold (M^n, g) that bounds a region U. We say that U is *geodesically convex at p in S* if any (local) geodesic in a direction $v \in T_p S$ intersects U only at p. The region U is *geodesically convex* if it is geodesically convex at every point $p \in S$.

Lemma 3.1. *Let $f : M^n \to \mathbb{R}$ be a smooth function on a Riemannian manifold (M^n, g) and let U be a sublevel set of f at a regular value. Then U is geodesically convex at $p \in S = \partial U$ if and only if the Hessian of f is positive definite:*

$$\nabla^2 f(v, v) > 0,$$

for all non-zero $v \in T_p S$.

We recall the convexity of the distance function, see *e.g.* [4, Section IX.6]. To any real number k, one associates the reference function

(3.1)
$$\mathrm{ct}_k(r) = \begin{cases} \sqrt{k} \cot(\sqrt{k} r), & \text{if } k > 0 \\ \frac{1}{r}, & \text{if } k = 0 \\ \sqrt{-k} \coth(\sqrt{-k} r), & \text{if } k < 0. \end{cases}$$

Proposition 3.2. *Let (M^n, g) be a Riemannian manifold with $\sec(g) \leq K$ for some real number K. Let r be any radius below the injectivity radius $\mathrm{inj}(g)$ and p any point in M. If K is non-negative, then the Hessian of the distance function*

$$\mathsf{r}_p : M^n \to \mathbb{R} : q \mapsto d(p, q)$$

is positive definite on the ball of radius r about p, $B_p(r)$. If K is positive then the same holds provided r is less than $\frac{\pi}{2\sqrt{K}}$.

More generally, the Hessian of r satisfies

$$\nabla^2 \mathsf{r}_p \geq \mathrm{ct}_K(r) g$$

and $\mathrm{ct}_K(r)$ is positive whenever K is non-positive or K is positive and $r < \frac{\pi}{2\sqrt{K}}$. $\qquad\square$

3.2. **Pseudo-convexity in symplectizations.** We consider the setup analogous to the one in [7] but in dimension ≥ 3. The symplectization of a contact manifold (M, ξ) equipped with a distinguished contact form α is the product $W := \mathbb{R}_+ \times M$, equipped with the symplectic form $\omega = d(t\alpha)$, where t is the coordinate on $\mathbb{R}_+$. Let J be a complex structure on ξ that is compatible with $(d\alpha)|_\xi$ and extended to TW by setting $J\partial_t = R_\alpha$.

Let U be a regular sublevel set of some function $f : M \to \mathbb{R}$ and $S = \partial U$. We can think of f as a function on W (by composing with the projection $W \to M$) and thus we get the regular sublevel set $\Omega = \mathbb{R}_+ \times U$ with boundary $\Sigma = \mathbb{R}_+ \times S$. The complex tangencies to Σ,

$$\mathcal{C}_\Sigma = T\Sigma \cap J(T\Sigma),$$

can be described as the kernel of the 1–form $df \circ J$. The form

$$L(u, v) = -d(df \circ J)(u, Jv)$$

is called the *Levi form* of Σ. Recall that Σ is said to be *pseudoconvex*, or *strictly pseudoconvex* if $L(v, v) \geq 0$, respectively $L(v, v) > 0$, for all $v \in \mathcal{C}_\Sigma$.

The following result which relates Riemannian and symplectic convexity should be compared with the analogous result in Kähler geometry [10, Lemma page 646] and in dimension 3, [7].

Proposition 3.3. *Let (α, g, J) be a compatible metric structure on the contact manifold (M^{2n+1}, ξ). Then, using the notation above, for any $v \in \mathcal{C}_\Sigma$ we have*

$$L(v, v) = \nabla^2 f(v, v) + \nabla^2 f(Jv, Jv).$$

The rest of this subsection is devoted to the proof of the preceding proposition and can be safely skipped on first reading. For convenience of notation we denote R_α by n in the computations. We also extend the metric g on M to W by $g + dt \otimes dt$. We note that the extended J and g are still compatible. We define two bundle maps $A : \xi \to TW$ and $B : \xi \to TW$ by

$$A(v) = J[Jv, v] - \nabla_v v - \nabla_{Jv} Jv,$$

and

$$B(v) = J[v, n] + \nabla_{Jv} n + \nabla_n Jv.$$

One can easily check that A and B are tensorial, meaning that their value at a point depends only on the vector at the point, not on the local extension to local vector fields. We begin with the following observation (which originally appeared for metric contact 3-manifolds in [7]).

Lemma 3.4. *Under the hypotheses of Proposition 3.3, we have*

$$-L(v, v) + \nabla^2 f(v, v) + \nabla^2 f(Jv, Jv) = df\left(A(v^\xi) + aB(Jv^\xi) - bB(v^\xi) - (a^2 + b^2)\nabla_n n\right),$$

where the vector $v \in \mathcal{C}_\Sigma$ is written as $v = v^\xi + a\,n + b\,\partial_t$, with $v^\xi \in \xi$, $\mathbb{R}$-invariant and a and b are constants.

Proof. We first compute

$$\nabla^2 f(v, v) + \nabla^2 f(Jv, Jv) = v \cdot (df(v)) - (\nabla_v v) \cdot f + (Jv) \cdot (df(Jv)) - (\nabla_{Jv} Jv) \cdot f$$
$$= v \cdot (df(v)) + (Jv) \cdot (df(Jv)) - df(\nabla_v v + \nabla_{Jv} Jv).$$

And, using the formula $d\alpha(u, w) = u \cdot \alpha(w) - w \cdot \alpha(u) - \alpha([u, w])$ we have

$$d(df \circ J)(v, Jv) = -v \cdot (df(v)) - (Jv) \cdot (df(Jv)) + df(J[Jv, v]).$$

Adding the two preceding equations, we obtain

$$-L(v, v) + \nabla^2 f(v, v) + \nabla^2 f(Jv, Jv) = df(J[Jv, v] - \nabla_v v - \nabla_{Jv} Jv).$$

Decomposing v as $v^\xi + a\,n + b\partial_t$ as in the statement of the lemma and using $\nabla_{\partial_t} v = 0$ we compute

$$J[Jv, v] = J[Jv^\xi, v^\xi] + aJ[Jv^\xi, n] + bJ[n, v^\xi],$$
$$\nabla_v v = \nabla_{v^\xi} v^\xi + a(\nabla_{v^\xi} n + \nabla_n v^\xi) + a^2 \nabla_n n,$$
$$\nabla_{Jv} Jv = \nabla_{Jv^\xi} Jv^\xi + b(\nabla_{Jv^\xi} n + \nabla_n Jv^\xi) + b^2 \nabla_n n.$$

Thus we see that $-L(v, v) + \nabla^2 f(v, v) + \nabla^2 f(Jv, Jv)$ equals

$$df\left(A(v^\xi) + aB(Jv^\xi) - bB(v^\xi) - (a^2 + b^2)\nabla_n n\right),$$

giving the announced formula. $\qquad\square$

Now we establish Proposition 3.3. We begin by computing an expression for the operators A and B from Lemma 3.4.

Lemma 3.5. *Under the hypotheses of Proposition 3.3, for any vector v in ξ we have*

$$A(v) = -\theta' \|v\|^2 \, \partial_t \quad \text{and} \quad B(v) = -\theta' \, v.$$

Proof. We can use Equations (2.18), (2.19), and (2.21) to compute $B(v)$

$$B(v) = J[v, n] + \nabla_{Jv} n + \nabla_n Jv = J\nabla_v n - J\nabla_n v + \nabla_{Jv} n + \nabla_n Jv$$
$$= J\nabla_v n + \nabla_{Jv} n = (-\theta'/2\, v + h(v)) + J(\theta'/2\, Jv - h(Jv)) = -\theta'\, v.$$

We now compute $A(v)$ by projecting it to n, ∂_t and ξ. Starting with the projection to n we use Equation (2.20) to conclude that

$$\langle A(v), n \rangle = \langle J[Jv, v], n \rangle - \langle \nabla_v v, n \rangle - \langle \nabla_{Jv} Jv, n \rangle$$
$$= \langle [Jv, v], \partial_t \rangle - II(v, v) - II(Jv, Jv) = 0.$$

Continuing with the projection to ∂_t we have

$$\langle A(v), \partial_t \rangle = \langle J[Jv, v], \partial_t \rangle = -\langle [Jv, v], n \rangle$$
$$= -\langle \nabla_{Jv} v, n \rangle + \langle \nabla_v Jv, n \rangle = \langle v, \nabla_{Jv} n + J\nabla_v n \rangle.$$

From the last line of our computations of $B(v)$ above we conclude that $\langle A(v), \partial_t \rangle = -\theta' \|v\|^2$. Next, note that thanks to Equations (2.11) and (2.13) for any v tangent to ξ we have

(3.2) $$[\phi, \phi](v, v) = 0, \qquad [\phi, \phi](v, \phi(v)) = -\|v\|^2 n.$$

Finally, we can compute

$$\langle A(v), v \rangle = \langle J[Jv, v] - \nabla_v v - \nabla_{Jv} Jv, v \rangle = \langle \phi([\phi(v), v]) - \nabla_v v - \nabla_{\phi(v)} \phi(v), v \rangle$$
$$= \langle \phi(\nabla_{\phi(v)} v) - \phi(\nabla_v \phi(v)) - \nabla_v v - \nabla_{\phi(v)} \phi(v), v \rangle$$
$$= -\langle (\nabla_{\phi(v)} \phi)(v), v \rangle + \langle (\nabla_v \phi)(\phi(v)), v \rangle = 0,$$

where for the last equality we used Equations (2.25) and (3.2). Similarly $\langle A(v), Jv \rangle = 0$. $\qquad \square$

Proof of Proposition 3.3. Combining Equation (2.5) and Lemmas 3.4 and 3.5 we see that

$$-L(v, v) + \nabla^2 f(v, v) + \nabla^2 f(Jv, Jv) = df\Big(-\theta'(\|v^\xi\|^2 \partial_t + aJv_\xi - bv_\xi) \Big)$$
$$= -\theta' df(\|v\|^2 \partial_t + aJv - bv) = 0,$$

where the last equality follows since $v \in \mathcal{C}_\Sigma$. $\qquad \square$

3.3. **Bordered Legendrian open books.**

Let N be a compact manifold with nonempty boundary. A *relative open book* on N is a pair (B, θ) where

- the *binding* B is a nonempty codimension 2 submanifold in the interior of N with trivial normal bundle, and
- $\theta \colon N \setminus B \to S^1$ is a fibration whose fibers are transverse to ∂N, and which coincides in a neighborhood $B \times D^2$ of $B = B \times \{0\}$ with the normal angular coordinate.

Definition 3.6 ([12]). *Let (M, ξ) be a $(2n + 1)$-dimensional contact manifold. A compact $(n + 1)$-dimensional submanifold $N \hookrightarrow M$ with boundary is called a* bordered Legendrian open book *(abbreviated* bLob*), if it has a relative open book (B, θ) such that*

(i) *all fibers of θ are Legendrian, and*
(ii) *the boundary of N is Legendrian.*

If such a submanifold exists then (M, ξ) is called PS-overtwisted.

We notice that the notion of a *plastikstufe* defined in [13] is a special case of a bLob where the fibers of the bLob are of the form $B \times [0,1]$. The term PS-overtwisted originally referred to the existence of a plastikstufe in a contact manifold, but it was generalized in [12]. Although it is not certain if this definition is a definitive generalization of overtwisted to higher dimensional manifolds, it does have some of the properties of 3 dimensional overtwisted contact manifolds. In particular, we have the following result.

Theorem 3.7 (Niederkrüger 2006, [13, 12]). *If (M, ξ) is a PS-overtwisted contact manifold then it cannot be symplectically filled by a semi-positive symplectic manifold. If the dimension of M is less than 7 then it cannot be filled by any symplectic manifold.* □

A $2n$–dimensional symplectic manifold (X, ω) is called semi-positive if every element $A \in \pi_2(X)$ with $\omega(A) > 0$ and $c_1(A) \geq 3 - n$ satisfies $c_1(A) > 0$. Note that all Stein and exact symplectic manifolds are semi-positive.

The presence of a bLob also has dynamical consequences and they will be crucial in our proof of Theorem 1.1. Recall that the the Weinstein conjecture asserts that any Reeb vector field on a closed contact manifold has some closed Reeb orbits. Contractible Reeb orbits do not always exist but the considerations in [12] allows to slightly generalize the main theorem in [1] resulting in the following theorem.

Theorem 3.8 (Albers and Hofer 2009, [1]). *Let (M, ξ) be a closed PS-overtwisted contact manifold. Then every Reeb vector field associated to ξ has a contractible periodic orbit.* □

3.4. **Proof of the tightness radius estimate.** We can now prove Theorem 1.1 which claims that a ball $B(x, r)$ whose radius r is below the convexity radius in a contact metric manifold cannot contain a bLob.

Proof of Theorem 1.1. If we assume the existence of a bLob, then one can start a family of holomorphic disks as in [13]. Because of the Levi form computation of Proposition 3.3, the boundary of a convex ball lifts to a pseudo-convex hypersurface in the symplectzation. The weak maximum principle for elliptic operators then guaranties that holomorphic curves cannot "touch from the inside" this hypersurface. This allows to use the strategy of [1] without any modification and prove the existence of a closed Reeb orbit γ inside the ball B.

However, such an orbit would be a closed geodesic according to Proposition 2.5. Those cannot exist inside B because it would have to be somewhere tangent to a sphere $S(x, r_0)$, for some r_0, with γ lying inside the ball $B(x, r_0)$. Of course r_0 is also below the convexity radius so $\partial B(x, r_0)$ cannot be "touched from the inside" by a geodesic and we get a contradiction.

The part of Theorem 1.1 relating to curvature follows from the above and the estimate on the Hessian of the radial function given in Proposition 3.2. □

Proof of Corollary 1.2. Note that pull-backs of ξ and the metric to any covering space are compatible and the sectional curvature is non-positive. It is well known, by Hadamard Theorem [4, Theorem IV.1.3 page 192], that the universal cover of a manifold with nonpositive curvature is $\mathbb{R}^{2n+1}$ and the space is exhausted by geodesic balls. Moreover, the convexity radius of the universal cover is infinite. Thus Theorem 1.1 says that a ball of any radius is bLob free. The claim follows. □

4. A QUANTITATIVE DARBOUX THEOREM IN ANY DIMENSION

In this section, we establish an estimate on the Darboux radius of a contact manifold with a compatible metric structure. We begin by introducing a number of quantities used throughout this

section that depend on the dimension, the instantaneous rotation θ' of the contact structure and bounds on curvature and injectivity radius. Unless said otherwise, we assume that the sectional curvature of g is between κ and K.

We define

(4.1)
$$r_{\max} = \begin{cases} \min\left(\frac{1}{2}\mathrm{inj}(g), \frac{\pi}{2\sqrt{K}}\right) & \text{if } K > 0 \\ \frac{1}{2}\mathrm{inj}(g) & \text{if } K \leq 0. \end{cases}$$

We also define the quantities

$$A = \frac{4}{3}(2n-1)\,|\sec(g)| \qquad \text{and}$$

$$B = \frac{\theta'}{2} + \sqrt{n\frac{\theta'^2}{4} - \frac{1}{2}\min_M\left(\mathrm{Ric}(R_\alpha)\right)}$$

where $|\sec(g)|$ is the maximum in absolute value sectional curvature over (M, g), and Ric is the Ricci tensor. The square root appearing in B is well defined thanks to Proposition 2.10. (Note than A and B have nothing to do with the tensors appearing in the Levi form computation of the preceding section.)

In addition to the the reference function ct_k defined by Equation (3.1), we will need functions sn_k, also indexed by a real number k

(4.2)
$$\mathrm{sn}_k(r) = \begin{cases} \frac{1}{\sqrt{k}}\sin(\sqrt{k}r), & \text{if } k > 0 \\ r, & \text{if } k = 0, \\ \frac{1}{\sqrt{-k}}\sinh(\sqrt{-k}r), & \text{if } k < 0. \end{cases}$$

These functions combines with A and B and an upper bound K on the sectionnal curvature to define

$$Q(r) := \mathrm{sn}_K^{-1}\left(\left(1 - Br - \frac{1}{2}Ar^2\right)\mathrm{sn}_K(r)\right).$$

and the constants

$$\overline{H}_1 = \begin{cases} \sqrt{2}, & \text{if } \kappa \geq 0, \\ \sqrt{1 + \left(\frac{\mathrm{sn}_\kappa(r_{\max})}{r_{\max}}\right)^2} & \text{if } \kappa < 0, \end{cases}$$

(4.3)
$$\overline{H}_2 = \frac{4}{3}|\sec(g)|\overline{H}_1 r_{\max}, \text{ and}$$

$$\overline{H} = 4(\overline{H}_2 + B\overline{H}_1)\overline{H}_1.$$

Combining all those numbers, and using the fact that B is positive, we define

$$r_\tau = \min\left(\frac{1}{2}\mathrm{inj}(g), \frac{2}{\sqrt{2A + B^2} + B}, \frac{1}{(1 + 2n(n-1))\overline{\overline{H}}}\right)$$

We are now ready to state the refined version of Theorem 1.4, with a refined estimate, which will be proven in this section.

Theorem 4.1. *Given a compatible metric structure (α, g, J) on (M, ξ), with sectional curvature bounded $\kappa \leq \sec(g) \leq K$, the Darboux's radius admits the following bound*

(4.4)
$$\delta(M, \xi) > Q(r_\tau).$$

First we explain how the coarser version announced in the introduction follows from the above result.

Proof of Theorem 1.4. We now discuss how this bound simplifies if we are willing to assume that the sectional curvature of g is in $[-\mathcal{K}, \mathcal{K}]$ for some positive constant $\mathcal{K}$. We set $\rho = \max(\theta', \sqrt{\mathcal{K}})$. In particular, $A \leq 4(2n-1)\rho^2/3$. Using the fact that $\mathrm{Ric}(R_\alpha) \geq -2n\mathcal{K}$ and $\sqrt{(1+x^2)} \leq 1 + x$ for non-negative x we estimate

$$B \leq \frac{\theta'}{2}\left(1 + \sqrt{n}\sqrt{1 + \frac{4\mathcal{K}}{\theta'^2}}\right) \leq \frac{\theta'}{2}\left(1 + \sqrt{n}(1 + \frac{2\sqrt{\mathcal{K}}}{\theta'})\right)$$
$$\leq 2\sqrt{n}\rho.$$

And we compute

$$\overline{H}_1 = \sqrt{1 + \left(\frac{\mathrm{sn}_{-\mathcal{K}}(r_{\max})}{r_{\max}}\right)^2} \leq \sqrt{1 + \left(\frac{\sinh(\sqrt{\mathcal{K}}\frac{\pi}{2\sqrt{\mathcal{K}}})/\sqrt{\mathcal{K}}}{\frac{\pi}{2\sqrt{\mathcal{K}}}}\right)^2}$$
$$= \sqrt{1 + \left(\frac{2}{\pi}\sinh\left(\frac{\pi}{2}\right)\right)^2} < 2.$$

So $\overline{H}_1 \leq 2$ and $\overline{H}_2 \leq \frac{4\pi\sqrt{\mathcal{K}}}{3}$ and $\overline{H} \leq 96\sqrt{n}\rho$.

On the other hand, we have:

$$\sqrt{2A + B^2} + B \leq 2B + \sqrt{2A} \leq \left(2\sqrt{n} + \sqrt{\frac{8(2n-1)}{3}}\right)\rho$$
$$\leq 5\sqrt{n}\rho.$$

So, the terms appearing in the definition of r_τ are estimated as follows

$$\frac{2}{\sqrt{2A+B^2}+B} \geq \frac{2}{5\sqrt{n}\rho} \quad \text{and} \quad \frac{1}{(1+2n(n-1))\overline{\overline{H}}} \geq \frac{1}{96\sqrt{n}(1+2n(n-1))\rho}.$$

Setting $d_n := 1/(96(1+2n(n-1))\sqrt{n})$, we see

$$r_\tau \geq \min\left(\frac{1}{2}\mathrm{inj}(g), \frac{d_n}{\rho}\right).$$

Set $r_0 \leq d_n/\rho$, it remains to estimate $Q(r_0)$. Our estimates on A and B give

$$1 - Br_0 - \frac{1}{2}Ar_0^2 \geq 1 - 2\sqrt{n}d_n - \frac{2(2n-1)}{3}d_n^2 \geq \frac{97}{100}.$$

Therefore

$$Q(r_0) \geq \frac{1}{\sqrt{\rho}}\arcsin(\frac{97}{100}\sin(\sqrt{\rho}r_0)) \geq \frac{1}{\sqrt{\rho}}\frac{1}{2}\sqrt{\rho}r_0 = \frac{r_0}{2},$$

which yields the promised Darboux radius estimate. $\qquad\square$

We will now prove Theorem 4.1 modulo a number of propositions which will be proved in subsequent subsections. The goal is to embed a large geodesic ball in our contact manifold into the standard contact $\mathbb{R}^{2n+1}$. The later is the contactization of the standard Liouville structure on $\mathbb{R}^{2n}$ and we will compare it to some contactization of a natural exact symplectic manifold inside our given contact metric manifold M. Recall a *Liouville manifold* is a pair (W, λ) where $d\lambda$ is a symplectic form on W and λ restricted to the boundary of W is a contact form for a positive

contact structure. Also recall that the *contactization* of an exact symplectic manifold (W, β), and in particular a Liouville manifold, is $\mathbb{R} \times W$ equipped with the contact structure $\ker(dt + \beta)$.

Given any point p in M and the contact hyperplane ξ_p at p, the geodesic disk $\mathsf{D}(r)$ centered at p of radius r and tangent to ξ_p is given as the image of the restriction of the exponential map to the disk of radius r in ξ_p, that is

$$\mathsf{D}(r) = \exp_p(D_\xi(r)).$$

where $D_\xi(r) = \left(\{v \in \xi_p;\ |v| < r\} \right)$. Denoting the Reeb flow by $\Phi(t, \mathbf{x}) : \mathbb{R} \times M \longrightarrow M$ we define the map

$$E : \mathbb{R} \times D_\xi(r) \to M : (v, t) \mapsto \Phi(t, \exp_p(v)),$$

and the R_α–invariant "cylindrical" neighborhood $\mathsf{C}(r)$ of $\mathsf{D}(r)$ to be the image of E. Of course $\mathsf{C}(r)$ is not, in general, an embedded submanifold of M, but for r small enough $\mathsf{D}(r)$ will be an embedded disk and R_α will be transverse to $\mathsf{D}(r)$. For such an r, $\mathsf{C}(r)$ will then contain embedded neighborhoods of $\mathsf{D}(r)$, for example $E((-\epsilon, \epsilon) \times D_\xi(r)$, for sufficiently small ϵ. To prove Theorem 4.1 we will proceed in the following steps.

Step I. Find an estimate on the radius r so that R_α is transverse to $\mathsf{D}(r)$.
Step II. Find an estimate on the radius r so that the pull back of the contact structure ξ to $\mathbb{R} \times D_\xi(r)$ via E embeds into the standard contact $\mathbb{R}^{2n+1}$.
Step III. Find an estimate on the size of a geodesic ball about p that embeds in M and is contained in $\mathsf{C}(r)$.

We will first list several propositions, that will be proven in the following subsections, that give the rigorous estimates indicated in the outline above and then assemble them into a proof of Theorem 4.1. The rigorous estimate in Step I is given in the following proposition which is proven in Subsection 4.1.

Proposition 4.2. *Given a compatible metric structure (α, g, J) on the contact manifold (M, ξ), the Reeb vector field R_α is positively transverse to $\mathsf{D}(r)$ if*

$$(4.5) \qquad\qquad r < r_\pitchfork := \min \left\{ \frac{1}{2}\mathrm{inj}(g), \frac{2}{\sqrt{2\,A + B^2} + B} \right\},$$

where the constants are defined at the beginning of this section. Moreover, if n_D is a unit normal vector to $\mathsf{D}(r_0)$, then along any radial geodesic $\gamma = \gamma(r)$

$$(4.6) \qquad\qquad \langle R_\alpha(r), n_\mathsf{D}(r) \rangle \geq 1 - Br - \frac{1}{2}Ar^2,$$

To carry out Step II we first make an observation about contactizations of Liouville domains and exact symplectic manifolds. For the remainder of this section (W, β_0) will be a Liouville domain. Let μ denote the restriction of β_0 to ∂W. By definition μ is a contact form. The completion of W is obtained as usual by adding the cylindrical end $[1, \infty) \times \partial W$ equipped with the Liouville form $t\mu$, where t is the "radial" coordinate on $[1, \infty)$. The resulting manifold will be denoted W_∞ and we will also denote this extended 1-form by β_0. For any constant $a > 1$ we set $W_a = W \cup \big([1, a) \times \partial W\big)$. We say an almost complex structure is *adapted* to β_0 if

    (a) it is tamed by $d\beta_0$,
    (b) it preserves the contact structure $\ker \alpha$ on each, and $\{t\} \times \partial W$
    (c) it sends ∂_t on some positive multiple of the Reeb field R_μ.

Recall that a 2–form ω tames an almost complex structure J if $\omega(u, Ju) > 0$ for any non-zero vector u. Note that ω is then automatically non-degenerate since any u in the kernel of ω would violate the taming condition.

Proposition 4.3. *Suppose β_1 is a 1–form on W_T (for some $T > 0$) such that $d\beta_1$ is a symplectic form on W_T and there is an almost complex structure which is adapted to β_0 and tamed by $d\beta_1$. Then, for any $T_0 \in [1, T)$, the contactization of $(\overline{W}_{T_0}, \beta_1)$ embeds in the contactization of (W_∞, β_0).*

In our situation, we want to apply the above proposition to the almost complex structure on $\mathsf{D}(r)$ obtained by pushing forward, via $\exp_p$, some complex structure on ξ_p tamed by $d\alpha_p$.

Proposition 4.4. *Given a compatible metric structure (α, g, J) on the contact manifold (M, ξ), with sectional curvature bounded $\kappa \leq \sec(g) \leq K$, the almost complex structure $(\exp_p^\xi)_* J_p$ is tamed by the restriction of $d\alpha$ to $\mathsf{D}(r)$ whenever*

$$r < \min\left(r_\text{⋔}, \frac{1}{(1 + 2n(n-1))\overline{\overline{H}}}\right),$$

where the constants are defined at the beginning of this section.

The previous two propositions will guaranty that the pull back of the contact structure on $\mathsf{C}(r)$ via E will be standard, that is embed in the standard contact structure on $\mathbb{R}^{2n+1}$, thus completing Step II. So we are left to complete Step III by estimating the size of a geodesic ball that can be embedded in such a cylinder. We can make such an estimate in a more general context that does not involve anything from the special geometry of compatible metrics except that the Reeb field is geodesic.

Proposition 4.5. *Let (M, g) be a complete Riemannian manifold whose sectional curvature is bounded above by K. Let X be a unit norm geodesic vector field on M and p a point in M. Consider the disk*

$$\mathsf{D}(r_0) := \{\exp_p(v) : v \in X_p^\perp, \|v\| < r_0\}$$

where r_0 is less than the convexity radius of g. We denote by n a unit vector field positively transverse to $\mathsf{D}(r_0)$ and assume we have the following estimate along a radial geodesic γ

(4.7) $$\langle X(\gamma(r)), n(\gamma(r)) \rangle \geq 1 - P(r),$$

where $P = P(r) \geq 0$ depends only on the distance r to p and $P(r) \leq 1$ on $[0, r_0]$. Then the cylinder $\mathsf{C}(r_0) = \Phi\big((-\infty, \infty) \times \mathsf{D}(r_0)\big)$ given by the flow Φ of X contains a geodesic ball of radius

$$\mathrm{sn}_K^{-1}\big(\mathrm{sn}_K(r_0)(1 - P(r_0))\big)$$

about p.

We can now prove Theorem 4.1 estimating the size of a Darboux ball.

Proof of Theorem 4.1. We first set, as in Proposition 4.2,

(4.8) $$r_\text{⋔} = \min\left(\frac{1}{2}\mathrm{inj}(g), \frac{2}{\sqrt{2A + B^2} + B}\right).$$

By Proposition 4.2, if $r < r_\text{⋔}$ then then $\mathsf{D}(r)$ is embedded in M and the Reeb vector field R_α is transverse to $\mathsf{D}(r)$. Since R_α is transverse to $\mathsf{D}(r)$ the restriction of the contact form α to $\mathsf{D}(r)$ is a primitive for an exact symplectic form $d\alpha$ on $\mathsf{D}(r)$.

Let β denote the pull back of $\alpha|_{\mathsf{D}(r)}$ to $D_\xi(r)$ by the exponential map. The contactization of $(D_\xi(r), \beta)$ is the contact structure on $\mathbb{R} \times D_\xi(r)$ coming from the contact form $dt + \beta$ and $E :$

$\mathbb{R} \times D_\xi(r) \to M$ is a contact immersion. In fact $E^*\alpha = dt + \beta$ since $dE(\partial_t) = R_\alpha$ and $\mathcal{L}_{R_\alpha}\alpha = 0$, $\alpha(R_\alpha) = 1$, thus $\xi = \ker\alpha$ lifts to $\ker(dt + \beta)$.

The bound of Proposition 4.4 implies that if

$$(4.9) \qquad r < r_\tau := \min\left(r_\pitchfork, \frac{1}{(1 + 2n(n-1))\overline{\overline{H}}}\right).$$

then $d\alpha$ on $\mathsf{D}(r)$ is tamed by $J = (\exp_p^\xi)_* J_p$. Thus $\beta = (\exp_p^\xi)^*\alpha|_{\mathsf{D}(r)}$ on $D_\xi(r) \subset \xi_p$ is tamed by J_p. Of course the standard Liouville form $\lambda = \sum_{i=1}^n y_i\, dx_i$ is adapted to J_p. Thus by Proposition 4.3 the contactization of any open subdomain of $D_\xi(r)$ will contact embed in the contactization of $(\mathbb{R}^{2n}, \lambda)$, that is in the standard contact structure on $\mathbb{R}^{2n+1}$.

We are left to estimate the maximal size of a geodesic ball $B_p(r)$ about p that can be embedded inside the cylinder $\mathsf{C}(r)$ (which of course can then be lifted via E to the contactization of $(D_\xi(r), \beta)$, as above.) Setting $P(r) = Br + \frac{1}{2}Ar^2$ we see from Equation (4.6) that

$$\langle R_\alpha(r), n_\mathsf{D}(r)\rangle \geq 1 - P(r).$$

Thus we can use this $P(r)$ in Proposition 4.5 to conclude that we can embed a geodesic ball of radius $Q(r_\tau)$ inside the cylinder $\mathsf{C}(r_\tau)$ hence this ball is standard. $\qquad\square$

4.1. **Twisting estimates.** Throughout this subsection we will assume that $r \in [0, r_{\max})$ and use the notation established at the beginning Section 4. Our goal is to prove Proposition 4.2 which estimates the angle $\langle R_\alpha, n_\mathsf{D}\rangle$ between the Reeb vector field and the disks $\mathsf{D}(r)$ as

$$\langle R_\alpha, n_\mathsf{D}\rangle \geq 1 - Br - \frac{1}{2}Ar^2,$$

where n_D is the unit normal vector to $\mathsf{D}(r)$ which coincides with R_α at p.

We first show that this estimate implies the transversality result in the proposition. The Reeb field R_α is transverse to $\mathsf{D}(r)$ as long as $\langle R_\alpha, n_\mathsf{D}\rangle$ is positive. Because the roots of $A t^2 + 2B t - 2$ are $(-B \pm \sqrt{2A + B^2})/A$, this is guarantied whenever r is less than $\frac{-B+\sqrt{2A+B^2}}{A} = \frac{2}{\sqrt{2A+B^2}+B} > 0$. Also, note that this term is always smaller than $\pi/2\sqrt{K}$ if we assume that K is a positive upper bound on sectional curvature. Indeed

$$\sqrt{2A + B^2} + B \geq \sqrt{2A} \geq \sqrt{\frac{8(2n-1)}{3}}\sqrt{K} > \frac{4}{\pi}\sqrt{K}.$$

We now establish the estimate on $\langle R_\alpha, n_\mathsf{D}\rangle$. Let q be any point of $\mathsf{D}(r)$ at some distance r_q from p. We denote by γ the radial geodesic between p and q. Estimating along γ we have

$$(4.10) \qquad \begin{aligned} |\langle R_\alpha, n_\mathsf{D}\rangle_q - 1| &= |\langle R_\alpha, n_\mathsf{D}\rangle_q - \langle R_\alpha, n_\mathsf{D}\rangle_p| \\ &= |\int_0^{r_q} \frac{d}{ds}\langle R_\alpha, n_\mathsf{D}\rangle_{\gamma(s)}\, ds| = |\int_0^{r_q} \nabla_{\dot\gamma}\langle R_\alpha, n_\mathsf{D}\rangle\, ds| \\ &\leq \int_0^{r_q} |\langle\nabla_{\dot\gamma}R_\alpha, n_\mathsf{D}\rangle| + |\langle R_\alpha, \nabla_{\dot\gamma}n_\mathsf{D}\rangle|\, ds. \end{aligned}$$

In the first term, $|\langle\nabla_{\dot\gamma}R_\alpha, n_\mathsf{D}\rangle|$ is less than or equal to $|\nabla_{\dot\gamma}R_\alpha|$ and thus Equation (2.23) from Proposition 2.10 gives

$$(4.11) \qquad |\langle\nabla_{\dot\gamma}R_\alpha, n_\mathsf{D}\rangle| \leq B.$$

Hence we are left to estimate $\langle R_\alpha(s), \nabla_{\dot\gamma}n_\mathsf{D}(s)\rangle$ for all $s \in [0, r_q]$. Because $s \leq r_q$ is less than the injectivity radius the Gauss lemma guarantees the differential of the exponential map at p gives us a isomorphism between the orthogonal complement of $\dot\gamma(0)$ in T_pM and the orthogonal

complement of $\dot{\gamma}(s)$ in $T_{\gamma(s)}M$. Using the and the fact that $\mathsf{D}(r)$ is the image of $D_\xi(r)$ under the exponential map one can construct Jacobi fields $J_1, \ldots, J_{2n-1}$ along γ such that

(i) $J_i(0) = 0$ and $J_i'(0)$ is in ξ_p,
(ii) all J_i are tangent to D, and
(iii) $\{n_\mathsf{D}, \dot{\gamma}, J_1, \ldots, J_{2n-1}\}$ is an orthonormal basis at $\gamma(s)$ (but not, *a priori*, anywhere else).

We now derive an estimate on the derivative of the Jacobi fields that will be needed below.

Lemma 4.6. *With the notation as above, let J be any of the J_i Jacobi fields. Then*

(4.12)
$$\langle n_\mathsf{D}, J'(\gamma(s)) \rangle \leq \frac{4}{3} |\sec(g)| \, s.$$

Proof. Integrating the Jacobi equations $J'' + R(J, \gamma')\gamma' = 0$ component-wise in a parallel moving frame along γ, one obtains $J'(s) = J'(0) - \int_0^s R(J(t), \gamma'(t))\gamma'(t)\, dt$ where the integral is again component-wise integration in a parallel frame. Hence, using $\langle n_\mathsf{D}, J'(0) \rangle = 0$ and the bound on $|R| \leq \frac{4}{3}|\sec|$ (see, for example, [4, page 95]) one obtains

(4.13)
$$\langle n_\mathsf{D}, J'(s) \rangle \leq \frac{4}{3} |\sec(g)| \max_t \|J(t)\| \, s.$$

We claim that for $s \leq \min\left(\mathrm{inj}(g), \frac{\pi}{2\sqrt{K}}\right)$ the function $\|J(t)\|$, is increasing on the interval $[0, s]$. Indeed, if r is the distance function from p and $\nabla^2 r$ denotes its Hessian, then a simple computation (or see [17, Lemma III.4.10, p. 109]) coupled with Proposition 3.2 yields

$$\frac{\partial}{\partial t} \|J(t)\|^2 = 2g(\nabla_{\dot{\gamma}(t)} J(t), J(t)) = 2\nabla^2 \mathsf{r}(J(t), J(t))$$
$$\geq (\mathrm{ct}_K \circ \mathsf{r}(\gamma(t)) \, \|J(t)\|^2 = \mathrm{ct}_K(t) \, \|J(t)\|^2,$$

and the length of $J(t)$ must increase until the first zero of $\mathrm{ct}_K(t)$ which occurs at $t = \frac{\pi}{2\sqrt{K}}$, for $K > 0$ and does not exist otherwise. Because $\|J(t)\|^2$ is increasing $\|J(t)\| \leq \|J(s)\| = 1$ and Inequality (4.13) simplifies to the promised Estimate (4.12). $\qquad\square$

At the point $\gamma(s)$, we can decompose the Reeb field as $R_\alpha = R_\gamma \dot{\gamma} + R_n n_\mathsf{D} + \sum R_i J_i$. The first two terms do not contribute to the scalar product with $\nabla_{\dot{\gamma}} n_\mathsf{D}$ since n_D is normal to γ and γ is a geodesic. We can now estimate

$$|\langle R_\alpha, \nabla_{\dot{\gamma}} n_\mathsf{D} \rangle (\gamma(s))| \leq \sum_{i=1}^{2n-1} |R_i \langle J_i(s), \nabla_{\dot{\gamma}} n_\mathsf{D} \rangle| = \sum_{i=1}^{2n-1} |R_i \langle \nabla_{\dot{\gamma}} J_i(s), n_\mathsf{D} \rangle|,$$

where the last equality follows because $\langle J_i(t), n_\mathsf{D} \rangle = 0$ for all t.

Notice that $R_\gamma^2 + R_n^2 + \sum R_i^2 = \|R_\alpha\|^2 = 1$ so $|R_i| \leq 1$ for all i. Using Equation (4.12) from the Lemma 4.6 we compute

$$|\langle R_\alpha, \nabla_{\dot{\gamma}} n_\mathsf{D} \rangle (q)| \leq (2n - 1)\frac{4}{3} |\sec(g)| s.$$

Using this, Equation (4.11), and Equation (4.10) we see that

$$
\begin{aligned}
|\langle R_\alpha, n_{\mathsf{D}} \rangle_q - 1| &\le \int_0^{r_q} |\langle \nabla_{\dot\gamma} R_\alpha, n_{\mathsf{D}} \rangle| + |\langle R_\alpha, \nabla_{\dot\gamma} n_{\mathsf{D}} \rangle|\, ds \\
&\le B r_q + \left((2n-1)\frac{4}{3}|\sec(g)| \right) r_q^2 \\
&= B r_q - \frac{1}{2} A r_q,
\end{aligned}
$$

from which the main estimate of Proposition 4.2 easily follows.   □

4.2. **Embedding contactizations.** This subsection contains a proof of Proposition 4.3. We begin with a simple lemma about embedding contactizations. Through out this subsection we will be using notation established at the beginning of the section.

Lemma 4.7 (Interpolation lemma). *Let (W, β_0) be a Liouville domain and (W_∞, β_0) its completion. Suppose β_1 is a 1–form on W_T (for some $T > 0$) such that $d\beta_1$ is a symplectic form on W_T and there is an almost complex structure which is adapted to β_0 and tamed by $d\beta_1$. Then there is a positive constant λ and a Liouville form $\widehat\beta$ on W_∞ such that*

  (i) *$\widehat\beta = \lambda\beta_0$ outside W_T,*
 (ii) *$\widehat\beta = \beta_1$ on W_{T_0}, and*
(iii) *$d\widehat\beta$ is tamed by J.*

Proof. We set $\widehat\beta = \rho\beta_1 + (1-\rho)\lambda\beta_0$ where ρ is a function with support in W_T, equals one on a neighborhood of W_{T_0} and which, inside $[1,\infty) \times \partial W$, depends only on t and is non-increasing.

The first two properties of $\widehat\beta$ are obvious from its definition, so we are left to show that λ and ρ can be chosen so that the third property holds. Notice that for this we can restrict our attention to $[1,\infty) \times \partial W$ where we have

$$
d\widehat\beta = (-\rho')\lambda t\, dt \wedge \mu + \rho'\, dt \wedge \beta_1 + \rho\, d\beta_1 + (1-\rho)\lambda\, d\beta_0.
$$

(Recall that μ is the contact form induced on ∂W by λ.) By the hypothesis $-\rho'$ is non-negative. Moreover $dt \wedge \mu(u, Ju)$ is non-negative because J is adapted to β_0. So we are left to prove that, using carefully chosen ρ and λ, for any non zero u the function

$$
G(u) := \big(\rho'\, dt \wedge \beta_1 + \rho\, d\beta_1 + (1-\rho)\lambda\, d\beta_0\big)(u, Ju)
$$

is positive.

Because $d\beta_0$ and $d\beta_1$ tame J, there are positive constants C_0, C_1 and C such that

$$
\frac{1}{\|u\|^2} G(u) \ge -|\rho'|C + \rho C_1 + \lambda(1-\rho)C_0.
$$

We cut the interval $[T_0, T)$ in two halves. On the first interval, ρ will be almost 1 and ρ' will be almost 0 and on the second interval ρ will decrease to zero. More precisely, we choose ρ such that

$$
\rho(t) \ge 1 - \varepsilon \ \text{ and } \ |\rho'(t)| \le \frac{2\varepsilon}{(T - T_0)/2}, \qquad \text{for } t \in [T_0, (T_0 + T)/2],
$$

and

$$
\rho(t) \le 1 - \varepsilon \ \text{ and } \ |\rho'(t)| \le \frac{2(1 - \varepsilon)}{(T - T_0)/2}, \qquad \text{for } t \in [(T_0 + T)/2, T].
$$

On the first subinterval, one has

$$-C|\rho'| + \rho C_1 \geq -C\frac{2\varepsilon}{(T - T_0)/2} + (1 - \varepsilon)C_1$$

which is positive is ε is sufficiently small. On the second interval

$$-C|\rho'| + \lambda(1 - \rho)C_0 \geq -C\frac{2(1 - \varepsilon)}{(T - T_0)/2} + \lambda\varepsilon C_0,$$

which is positive if λ is sufficiently large. $\qquad\square$

We now consider $\widehat{\beta}$ as in the above lemma. The contactization of (W_{T_0}, β_1) embeds into the contactization of $(W_\infty, \widehat{\beta})$. Since the contactizations of β_0 and $\lambda\beta_0$ are isomorphic, the following lemma finishes the proof of Proposition 4.3.

Lemma 4.8. *The contactizations of $\widehat{\beta}$ and $\lambda\beta_0$ are isomorphic.*

Proof. Consider the path of 1–forms $\lambda_s = (1 - s)\lambda\beta_0 + s\widehat{\beta}$. The preceding lemma says that $d\lambda_s$ tame J for all s and thus are all symplectic forms on W_∞. Thus $\alpha_s = t + \lambda_s$ is a path of contact forms. Moser's technique provides an isotopy that connects the corresponding contact structures if the vector fields constructed in Moser's technique can be integrated for a sufficiently long time. It is clear that it can be so integrated since β_0 and $\widehat{\beta}$ coincide outside the compact set $\overline{W_T}$. $\qquad\square$

4.3. **Taming J.** In this subsection we prove Proposition 4.4. We begin with a general lemma that holds in any Riemannian manifold. We will need the auxiliary functions

$$(4.14) \qquad H_1(r) = \sqrt{1 + \left(\frac{\mathrm{sn}_\kappa(r)}{r}\right)^2}, \quad \text{and} \quad H_2(r) = \tfrac{4}{3}|\sec(g)|\left(rH_1(r) + \int_0^r H_1(t)\,dt\right).$$

Lemma 4.9. *Suppose the sectional curvature of (M, g) is bounded below by some constant κ. Let $\gamma : [0, R] \to M$ be a unit speed geodesic without conjugate points and set $p = \gamma(0)$. Let $X(r) = (d\exp_p)_{r\gamma'(0)}v$ be a vector field along γ obtained as the image of a fixed unit vector $v \in T_pM$. (We think of v as a vector in $T_w(T_p(M))$ for every $w \in T_pM$ using the canonical identification of a vector space with its tangent space at any point.) Then, using notations from Equation (4.14) we have*

$$(4.15) \qquad \|X(r)\| \leq H_1(r)$$

and

$$(4.16) \qquad \|\nabla_{\dot\gamma}X\|(r) \leq H_2(r),$$

for $r \in [0, r_{\max})$ where $r_{\max}$ was defined in Equation (4.1).

Proof. Let J be the Jacobi field along γ which satisfies $J(0) = 0$ and $J'(0) = X(0)$. According to [4, Theorem II.7.1 page 88], one has $J(r) = rX(r)$. One can decompose X into $X^\top$ which is parallel to $\dot\gamma$ and $X^\perp$ which is perpendicular to $\dot\gamma$. Then by the Gauss lemma, one has $\|X^\top(r)\| = \|X^\top(0)\|$. The perpendicular part is estimated by Rauch's theorem [4, Theorem IX.2.3 page 390] which gives $\|X^\perp(r)\| \leq \mathrm{sn}_\kappa(r)/r\,\|X^\perp(0)\|$. Thus we have

$$\|X(r)\|^2 = \|X^\top(r)\|^2 + \|X^\perp(r)\|^2$$

$$\leq 1 + \frac{\mathrm{sn}_\kappa(r)^2}{r^2}.$$

We will now establish Equation (4.16). Since $X = r^{-1}J$, we have $X' = r^{-1}(J' - X)$. The Jacobi equation for J reads $J'' + RJ = 0$ where RJ is short hand for $R(J, \gamma')\gamma'$. In a parallel frame along γ, the components of this equation can be integrated component-wise. We can now estimate

$$\|X'(r)\| = r^{-1}\|J'(r) - X(r)\| = r^{-1}\left|\int_0^r -RJ - X'\right| \leq r^{-1}\int_0^r \|RJ\| + \|X'\|$$

$$\leq \frac{4}{3}|\sec(g)| \max \|J\| + r^{-1}\int_0^r \|X'\|$$

$$\leq \frac{4}{3}|\sec(g)| rH_1(r) + \frac{1}{r}\int_0^r \|X'\|.$$

where the second inequality follows from [4, p. 95] just as in the proof of Lemma 4.6. So estimating $\|X'\|$ is now a Grönwall type problem. We set $f(r) = \|X'(r)\|$ and $\alpha(r) = \frac{4}{3}|\sec(g)| rH_1(r)$, so that the above inequality reads

(4.17)
$$f(r) \leq \alpha(r) + \frac{1}{r}\int_0^r f(t)\, dt.$$

Setting $v(r) = \frac{1}{r}\int_0^r f(t)\, dt$ and keeping in mind that f is smooth and $f(0) = 0$, we have

$$v'(r) = \frac{1}{r}\left(f - \frac{1}{r}\int_0^r f(t)\, dt\right) \leq \alpha(r)/r.$$

Since $v(0) = 0$, we see that $v(r) \leq \int_0^r \frac{\alpha(u)}{u}\, du$, which can be substituted into Equation (4.17) to obtain the announced estimate. $\qquad\square$

Lemma 4.10. *Given a radial geodesic γ in $\mathsf{D}(r_0)$ starting at $p = \gamma(0)$, consider two vector fields X and Y along $\gamma = \gamma(r)$ which are the images of unit vectors in ξ_p under the differential of $\exp_p$ as described in Lemma 4.9. Then the function $F(r) = 1/\theta'\, d\alpha(X, Y)$ is bounded by the constant $\overline{H}$ defined in (4.3), i.e.*

(4.18)
$$|F'(r)| \leq \overline{H}, \qquad for \quad r \in [0, r_{\max}],$$

$r_{\max}$ is defined in (4.1).

Proof. By Proposition 2.2 the metric g can be expressed as

$$g(X, Y) = \frac{1}{\theta'}d\alpha(X, \phi(Y)) + \alpha(X)\alpha(Y),$$

therefore

$$\frac{1}{\theta'}d\alpha(X, Y) = -\langle X, \phi(Y)\rangle + \langle R_\alpha, X\rangle\langle R_\alpha, \phi(Y)\rangle,$$

Using the fact that $\nabla_r(\phi(X)) = (\nabla_r\phi)X + \phi(\nabla_r X)$ and $\phi^2 = -\mathrm{Id} + \alpha(\,\cdot\,)R_\alpha$, from Remark 2.4, one easily computes

$$\frac{d}{dr}F(r) = -\langle \nabla_r X, \phi(Y)\rangle - \langle X, (\nabla_r\phi)(Y)\rangle + \langle \phi X, \nabla_r Y\rangle$$

$$+ (\langle \nabla_r R_\alpha, X\rangle + \langle R_\alpha, \nabla_r X\rangle)\langle R_\alpha, Y\rangle + \langle R_\alpha, Y\rangle(\langle \nabla_r R_\alpha, Y\rangle + \langle R_\alpha, \nabla_r Y\rangle).$$

Since $\|R_\alpha\| = 1$, and $\|\phi\| = 1$, Lemma 4.9, and estimates (2.24), (2.23), yield

$$\left|\frac{d}{dr}F(r)\right| \leq H_2(r)\|\phi Y\| + 2B\|X\|\|Y\| + H_2(r)\|\phi X\|$$
$$+ \left(B\|X\| + H_2(r)\|R_\alpha\|\right)\|R_\alpha\|\|Y\|$$
$$+ \left(B\|Y\| + H_2(r)\|R_\alpha\|\right)\|R_\alpha\|\|X\|$$
$$\leq 4\left(H_2(r) + BH_1(r)\right)H_1(r).$$

Note that $H_1(r)$ is increasing when $\kappa < 0$, constant (equal $\sqrt{2}$) for $\kappa = 0$, and decreasing on $[0, \frac{\pi}{2\sqrt{\kappa}}]$ to 1 at $\frac{\pi}{2\sqrt{\kappa}}$ if $\kappa > 0$. Also the function $H_2(r)$ is increasing and vanishing at $r = 0$. Thus, on the interval (4.1), $[0, r_{\max})$ where H_1 and H_2 are defined we have the following simple estimates

(4.19)
$$H_1(r) \leq \overline{H}_1 := \max_{[0,r_{\max})} H_1(r) = \begin{cases} H_1(0), & \text{if } \kappa \geq 0, \\ H_1(r_{\max}) & \text{if } \kappa < 0. \end{cases}$$
$$H_2(r) \leq \overline{H}_2 = \frac{4}{3}|\sec(g)|\overline{H}_1 r_{\max}.$$

Therefore $|F'(r)| \leq \overline{H}$, for $\overline{H} = 4(\overline{H}_2 + B\overline{H}_1)\overline{H}_1$ as claimed. $\qquad\square$

Proof of Proposition 4.4. Given any nonzero vector u tangent to $\mathsf{D}(r)$ as some point q let γ be the geodesic from p to q. Choose a symplectic basis $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ of ξ at p with $Y_i = JX_i$ and $|X_i| = 1$ and use the same notation for the vector fields along γ obtained using $\exp_p$ as in Lemma 4.9. Notice that the first paragraph of the proof of Lemma 4.9 implies that the X_i and Y_i are proportional to Jacobi fields J with the property that $J(0) = 0$ and $J'(0) \in \xi_p$. Since $\mathsf{D}(r)$ is the image of $D_\xi(r)$ under the exponential map it is easy to see that such a J is always tangent to $\mathsf{D}(r)$ and thus so are the X_i and Y_i. Hence there are constants a_i and b_i such that $u = \sum_i a_i X_i + b_i Y_i$. Then

$$d\alpha(u, Ju) = \sum_{1\leq i,j\leq n} \left(-a_i b_j d\alpha(X_i, X_j) + a_i a_j d\alpha(X_i, Y_j) - b_i b_j d\alpha(Y_i, X_j) + a_j b_i d\alpha(Y_i, Y_j)\right)$$
$$= \sum_{i=1}^{n}(a_i^2 + b_i^2)d\alpha(X_i, Y_i) + \sum_{1\leq i\neq j\leq n} \left(-a_i b_j d\alpha(X_i, X_j) + a_i a_j d\alpha(X_i, Y_j)\right.$$
$$\left. - b_i b_j d\alpha(Y_i, X_j) + a_j b_i d\alpha(Y_i, Y_j)\right).$$

At $r = 0$, $1/\theta' d\alpha(X_i, Y_i) = 1$, and $d\alpha(Z, W) = 0$ if Z and W are any pair of vectors appearing in the second part of the above sum. From the derivative estimate in Equation (4.18) we see that

$$\frac{1}{\theta'}d\alpha(X_i, Y_i) \geq 1 - \overline{H}r \qquad \text{and} \qquad \frac{1}{\theta'}|d\alpha(Z, W)|(r) \leq \overline{H}r.$$

To continue the computation, extend the X_i, Y_i and n_D to a neighborhood of γ and use them to define an auxiliary metric on the neighborhood so that they from an orthonormal basis. In this metric the norm squared of u, which we denote by $N(u)$, is $\sum_{i=1}^{n} a_i^2 + b_i^2$. We now have

$$\frac{1}{\theta'}d\alpha(u, Ju) \geq (1 - \overline{H}r)N(u) - r\overline{H}\sum_{1\leq i\neq j\leq n} \left(|a_i||b_j| + |a_i||a_j| + |b_i||b_j| + |a_j||b_i|\right)$$

Since there are $2\binom{n}{2} = n(n-1)$ terms in the sum and $\frac{1}{2}N(u) \geq |a_*||b_*|$ (or $|a_*||a_*|$ or $|b_*||b_*|$), we see that

$$\frac{1}{\theta'}d\alpha(u, Ju) \geq \left(1 - (1 + 2n(n-1)\overline{H}r)\right)N(u).$$

Hence, for $u \neq 0$ we have $d\alpha(u, Ju) > 0$ if $r < \left((1 + 2n(n-1))\overline{H}\right)^{-1}$. $\qquad \square$

4.4. **From cylinders to balls.** The goal of this subsection is to prove Proposition 4.5 which guaranties that Reeb flow cylinders contain embedded geodesic balls of a certain radius.

Proof of Proposition 4.5. Recalling the hypothesis of the proposition we have a complete Riemannian manifold (M, g) whose sectional curvatures are bounded above by K and a number r_0 less than the convexity radius of g. The vector field X is a unit speed geodesic vector field on M and p is some point in M. We denote by $D_{X_p}(r_0) = \{v \in X_p^\perp, \|v\| < r_0\}$ and

$$\mathsf{D}(r_0) = \{\exp_p(v) : v \in D_{X_p}(r_0)\}.$$

Moreover, we set

$$E : \mathbb{R} \times D_{X_p}(r_0) \to M : (t, v) \mapsto \Phi(t, \exp_p(v)),$$

and $\mathsf{C}(r_0) = \mathrm{image}(E)$.

Denote by $\mathsf{B}^{\mathrm{conv}}$ the convex ball at p of radius r_0, and $\mathsf{C}^{\mathrm{conv}}(r_0) = \mathsf{C}(r_0) \cap \mathsf{B}^{\mathrm{conv}}$ the portion of $\mathsf{C}(r_0)$ in $\mathsf{B}^{\mathrm{conv}}$. Let B be the connected component of $E^{-1}(\mathsf{C}^{\mathrm{conv}}(r_0))$ that contains $(0, 0)$. Since $r_0 < \mathrm{conv}(g)$ we observe that E restricted to the B is an embedding. To see this we need to study the structure of B. We first notice that $B \cap \xi_p$ is the ball of radius r_0 in ξ_p, $D_\xi(r_0)$. Now for each $x \in D_\xi(r_0)$ let $I_x = (\mathbb{R} \times \{x\}) \cap B$. So $B = \cup_{x \in D_\xi(r_0)} I_x$. Each I_x clearly contains the origin (i.e. $(0, x)$.) We moreover claim that it is a connected interval for all x. If not, then there is some x_0 for which it is not connected. Let γ be the geodesic in $\mathsf{D}(r_0)$ from p to x_0. Let J_x be the smallest interval containing I_x for each $x \in \gamma$ and K_x the image of x under the flow of X for times in J_x. Finally the union of all K_x for $x \in \gamma$ is a disk A. By hypothesis A is not contained in $\mathsf{B}^{\mathrm{conv}}$. If x' is the first point on γ such that $K_{x'}$ is not contained in the interior of $\mathsf{B}^{\mathrm{conv}}$, then $K_{x'}$ is a geodesic which has an interior tangency with $\partial \mathsf{B}^{\mathrm{conv}}$. This contradicts the convexity of $\mathsf{B}^{\mathrm{conv}}$. Thus the I_x are all connected. This implies that the restriction of E to B is injective because, since we are below the injectivity radius, the ball $\mathsf{B}^{\mathrm{conv}}$ is divided by two connected components by $\mathsf{D}(r_0)$ and, in particular, there is no trajectory of X leaving $\mathsf{D}(r_0)$ and returning to it without leaving $\mathsf{B}^{\mathrm{conv}}$.

Next, we estimate a radius $\bar{r}$ of a convex ball centered at p and contained in $\mathsf{C}^{\mathrm{conv}}(r_0)$, a maximal such ball $B_p(\bar{r})$ will either be of radius $\geq r_0$ or have a tangency with $(\partial \mathsf{C}(r_0)) \cap \mathsf{B}^{\mathrm{conv}}$. Let q be a point of the tangency and v be the point of intersection of the orbit of X through q and the disk $\mathsf{D}(r_0)$. Clearly $v \in (\partial \mathsf{C}(r_0)) \cap \mathsf{B}^{\mathrm{conv}}$ and thus $d(p, v) = r_0$. Moreover we know that $d(p, q) = \bar{r}$. Consider the geodesic triangle $T(p, q, v)$ consisting of the unique geodesics connecting these points. So both sides (p, q) and (p, v) of $T(p, q, v)$ are radial geodesics emanating from p, and (q, v) is a piece of the orbit of X.

We choose $\gamma(s)$ to be the radial geodesic parameterizing (p, v), *i.e.* $\gamma(0) = p$ and $\gamma(r_0) = v$. Denote by $0 \leq \phi \leq \frac{\pi}{2}$ the angle between $X(\gamma(r_0))$ and the normal $n(\gamma(r_0))$, and by $0 \leq \varphi \leq \frac{\pi}{2}$ the angle between $X(\gamma(r_0))$ and the line spanned by $\gamma'(r_0)$. Observe that $\frac{\pi}{2} - \phi \leq \varphi$, thus using Estimate (4.7) we have

$$(4.20) \qquad \sin(\varphi) \geq \cos(\phi) \geq 1 - P(r_0).$$

Comparing, with the geodesic triangle $T(p', q', v')$ in the space form M_K, $(\sec(g) \leq K)$ based at vertex v' with angle $\varphi' = \varphi$ at v', $d(p', v') = d(p, v) = r_0$, and $d(q', v') = d(q, v)$, Toponogov's

theorem [4, p. 400] shows that
$$d(p', q') \leq d(p, q) = \bar{r}.$$
Let θ' be the angle at the vertex q' in the reference triangle. The law of sines [4, Note II.5 page 103 and references therein] applied to the triangle $T(p', q', v')$ yields
$$\frac{\sin \theta'}{\mathrm{sn}_K(r_0)} = \frac{\sin \varphi}{\mathrm{sn}_K(d(p', q'))}$$
This combines with $\sin(\theta') \leq 1$ and Inequality (4.20) to give
$$\mathrm{sn}_K(\bar{r}) \geq \mathrm{sn}_K(r_0)(1 - P(r_0)).$$
hence the annouced bound since the functions sn_K are increasing. $\qquad\square$

5. Size of standard neighborhoods in dimension 3

In this section we show how to use a geometric methods similar to the one in Section 4 to provide a stronger estimate on the Darboux radius in dimension 3. We also show how to use this idea to construct standard neighborhoods of closed geodesics with an estimated size.

In Subsection 5.1 we state two propositions that essentially say as long as one can control the twisting of the Reeb vector field with respect to the disks $\mathsf{D}(r)$ and then prove our main results in dimension 3 using previously derived geometric estimates of Section 4. We begin by discussing characteristic foliations and their use in the proof these propositions.

5.1. **From geometric control to topology.** The geometric setup used throughout this section is the following. We consider (α, g, J) a compatible metric structure to (M, ξ), and denote its Reeb vector field R_α. Let $\zeta = \varphi(p, [-r, r])$ be a portion of Reeb orbit traced by the flow φ of R_α and $\nu\zeta$ the normal bundle to ζ with respect to g. A radius r_0 is such that the normal exponential map $\exp_\nu$ restricted to the bundle of disks of radius r_0 in $\nu\zeta$ is an embedding. For each z in ζ and $r < r_0$ we consider the disk
$$\mathsf{D}_z := \exp_\nu\left(\{v \in \nu_z\zeta \; ; \; |v| \leq r\}\right)$$
and we denote by n_{D} the unit vector field orthogonal to all D_z which coincides with R_α along ζ. For each radius $r \leq r_0$ we denote by
$$\mathsf{T}(r) = \exp_\nu\left(\{v \in \nu\zeta \; ; \; |v| \leq r\}\right)$$
the tube of radius r around ζ. It is either a solid torus or a thickened disk depending on whether ζ is a circle or an interval. The following lemma, which will be proved at the end of this section, gives a sufficient condition to have T embedded.

Lemma 5.1. *If*
$$r < \min\left(\frac{1}{2}\mathrm{inj}(g), \frac{\pi}{2\sqrt{K}}\right),$$
then $\mathsf{T}(r)$ is embedded.

Proposition 5.2. *If ζ is a closed Reeb orbit such that $\mathsf{T}(r)$ is embedded and ξ is transverse to n_{D} then $(\mathsf{T}(r), \xi)$ is contactomorphic to a domain in $(S^1 \times \mathbb{R}^2, \ker(d\phi + \rho^2\, d\theta))$, where ϕ is the angular coordinate on S^1 and (ρ, θ) are polar coordinates on $\mathbb{R}^2$. In particular, $(\mathsf{T}(r), \xi)$ is universally tight.*

In the case where ζ is an interval in a Reeb orbit, we will impose the stronger condition that the Reeb vector field R_α stays transverse to all D_z. This will allow us to embed $\mathsf{T}(r)$ in a solid torus (not living in M) with a contact structure to which we will apply Lemma 5.4:

Proposition 5.3. *In the above setup and under the hypothesis of Lemma 5.1, if ζ is an interval and the Reeb vector field R_α is transverse to the disks D_z, then $(\mathsf{T}(r), \xi)$ is contactomorphic to a domain in $(\mathbb{R}^3, \xi_{std})$.*

Now, Theorems 1.6 and 1.7 follow at once.

Proof of Theorem 1.6. By the hypothesis of the theorem and Proposition 5.2 the cylinder $\mathsf{T}(r)$ is embedded and the contact structure on $\mathsf{T}(r)$ is contactomorphic to a domain in $(\mathbb{R}^3, \xi_{std})$. Thus any geodesic ball embedded in $\mathsf{T}(r)$ is standard. We will now prove that the geodesic ball of radius r is contained in $\mathsf{T}(r)$ and thus the theorem will follow.

Since $\mathsf{T}(r)$ contains very small balls around p, there is a positive first radius r_0 such that the sphere $S(p, r_0)$ intersects $\partial\mathsf{T}(r)$. We denote by $p_\pm = \zeta(\pm r)$ the two extremities of ζ. We denote by $D_\pm$ the geodesic disks normal to ζ centered at $p_\pm$. The boundary of $\mathsf{T}(r)$ is made of $D_\pm$ which we will call horizontal together with the vertical part made of points at distance r from ζ. One can see that the intersection between $S(p, r_0)$ and $\partial\mathsf{T}(r)$ cannot meet the intersection between the horizontal and vertical parts. At a point in the interior of the vertical part, the plane tangent to $\partial\mathsf{T}(r)$ is the orthogonal to a geodesic which is normal to ζ (by the generalized Gauss Lemma see e.g. [9, Lemma 2.11 page 26]). Likewise, the tangent space of $S(p, r_0)$ at any point is the orthogonal to a geodesic normal to $S(p, r_0)$ and going through p. So, at a point where $S(p, r_0)$ is tangent to the vertical part of $\partial\mathsf{T}(r)$, those geodesics coincide and we see that the relevant point of ζ is p. Hence $r_0 = r$ in this case.

Suppose now that the intersection between $S(p, r_0)$ and $\partial\mathsf{T}(r)$ is in D_+ or D_-. Let q be a tangency point in one of those disks, say D_+. Let γ be the geodesic from p_+ to q (recall we are below the convexity radius). Note that γ is normal to ζ. Let γ' be the geodesic between p and q. Since $S(p, r_0)$ is tangent to D_+ at q the Gauss lemma implies that γ' is normal to $\gamma \subset D_+$. Now there are two cases to consider. If $q = p_+$ then $r_0 = r$ and we have established our claim. If $q \neq p_+$ then we contradict Lemma 5.1, applied to γ, because the tube around γ with radius r is not embedded. Indeed we have two geodesics ζ and γ' normal to γ with length at most r which intersect. $\qquad\square$

Proof of Theorem 1.7. By the hypothesis of the theorem and Proposition 4.2, the geodesic tube $\mathsf{T}(r)$ about the geodesic γ is embedded and the Reeb vector field stays transverse to all the disks D_z. Since the contact planes are perpendicular to the Reeb vector field, this implies that they are transverse to the vector field n_D normal to the disks. The theorem now follows from Proposition 5.2. $\quad\square$

The next sections are devoted to the proof of Propositions 5.2 and 5.3 but first we come back to Lemma 5.1.

Proof of Lemma 5.1. Thus the restriction of $\exp_\nu$ to $\{z\} \times D(r)$ for every fixed z is injective, since $\exp_\nu$ agrees with the "full" exponential map $\exp : B(r) \longrightarrow M$ on $\{z\} \times D(r)$ which is injective on the ball of radius r. Thus injectivity of $\exp_\nu$ can fail only when there is a geodesic triangle $T(p_0, p_1, p_2)$ in $(-r, r) \times D(r)$ formed by a piece of the geodesic ζ between $p_0 = \zeta(z_0)$ and $p_1 = \zeta(z_1)$ in $\mathsf{T}(r)$ and two "radial" geodesics γ_0 in D_{z_0}, and γ_1 in D_{z_1}, such that $\gamma_0(0) = p_0$, $\gamma_1(0) = p_1$ and $\gamma_0(t_0) = \gamma_1(t_1) = p_2$ for some $t_0, t_1 \leq r$. Let us denote by γ_{ij}, $i < j$ the sides of the triangle T, *i.e.* the geodesics between p_i and p_j is denoted γ_{ij}. The length of γ_{ij} will be denoted l_{ij}. Since ζ is orthogonal to D_{z_0} and D_{z_1}, the triangle T also has two right angles at vertices p_0 and p_1. Without loss of generality we assume

$$l_{12} \leq l_{02} < \frac{\pi}{2\sqrt{K}},$$

otherwise we may switch the labels of the vertices p_0 and p_1 (the only feature of the vertex needed is that the angle at that vertex is $\frac{\pi}{2}$). Since, the geodesics γ_{ij} are minimizing we may compare

the right geodesic triangle $T(\gamma_{01}, \gamma_{02}, \gamma_{12})$ to the right triangle $\bar{T}(\bar{\gamma}_{01}, \bar{\gamma}_{02}, \bar{\gamma}_{12})$ in the 2-sphere of constant curvature K, assuming that at $\bar{p}_0$ the angle between $\dot{\bar{\gamma}}_{01}$ and $\dot{\bar{\gamma}}_{02}$ is $\frac{\pi}{2}$ (which is the same as between $\dot{\gamma}_{01}$ and $\dot{\gamma}_{02}$) and $l_{01} = \bar{l}_{01}$, $l_{02} = \bar{l}_{02}$. For such a right triangle on the 2-sphere the law of cosines implies

$$\bar{l}_{02} < \bar{l}_{12},$$

and we obtain a contradiction since

$$l_{02} = \bar{l}_{02} < \bar{l}_{12} \leq l_{12} \leq l_{02},$$

where the second inequality is a consequence of Toponogov's Theorem [4, p. 400]. $\qquad\square$

5.2. **Background on characteristic foliations.** Recall that an oriented singular (this adjective will be implicit in the following) foliation on an oriented surface S is an equivalence class of 1–forms where $\alpha \sim \beta$ if there is a positive function f such that $\alpha = f\beta$. Let α be a representative for a singular foliation $\mathscr{F}$. A singularity of $\mathscr{F}$ is a point where α vanishes. The singularity p is said to have non-zero divergence if $(d\alpha)_p$ is an area form on $T_p S$. If ω is an area form on S (compatible with the chosen orientation) then to each singular point p we attach the sign of the unique real number μ such that $(d\alpha)_p = \mu\omega_p$. One can easily check that singular points and their signs do not depend on the choice of α in its equivalence class or of ω if we keep the same orientation.

Let S be an oriented surface in a contact manifold (M, ξ) with $\xi = \ker \alpha$, co-oriented by α. The characteristic foliation ξS of S is the equivalence class of the restriction of α to S. The contact condition ensures that all singularities of characteristic foliations have non-zero divergence and hence have non-zero sign. Singularities of ξS correspond to points where S is tangent to ξ and they are positive or negative according as the orientation of ξ and S match or do not match. We also notice that α provides a co-orientation, and hence if S is oriented by an area form ω the orientation of the line field ξS is given by the vector field X which satisfies $\iota_X \omega = \alpha|_S$. One may dually think of the characteristic foliation on S as coming from the singular line field on S given by $T_p S \cap \xi_p$ for each $p \in S$.

5.3. **Characteristic foliations on tori and contact embeddings.** In the next section, we will need to show, informally speaking, how a contact structure which is transverse to the core of the solid torus and "does not rotate more than half a turn between the core and the boundary" embeds inside the standard contact structure on $\mathbb{R}^3$. In the following we make this statement precise and provide a proof of it. We will denote by T a torus and by T_t the torus $T \times \{t\}$ in the toric annulus $T \times [0, 1]$.

We first recall some notions about suspensions on tori. A non-singular foliation $\mathcal{F}$ on T is called a suspension if there is a simple closed curve intersecting all leaves transversely. The name comes from the fact that $\mathcal{F}$ can be reconstructed by suspending of the Poincaré first return map on the transversal curve. To such a foliation one can associate a line in $H_1(T; \mathbb{R})$. This line $d(\mathcal{F})$ is called the asymptotic direction of $\mathcal{F}$. We briefly sketch the construction. Pick any point x in T, follow the leaf of $\mathcal{F}$ through x for a length T, and create a closed curve $O(x, T)$ using a geodesic (for some auxiliary metric). Then the limit homology class $\lim_{T \to \infty} \frac{1}{T}[O(x, T)]$ exists for every x and it defines a line in $H_1(T; \mathbb{R})$ that does not depend on x or T. The limit is called the *asymptotic direction*. Two easy examples are when $\mathcal{F}$ is linear (we recover the intuitive notion of direction) and when there is a closed leaf (its asymptotic direction is spanned by the homology class of this leaf).

Let $T \times [0, 1]$ be a toric annulus. If the characteristic foliations on all the tori T_t induced by some contact structure ξ are suspensions then the contact condition forces the asymptotic directions

$d(\xi T_t)$ to always rotate continuously in the same direction (which is determined by the orientations of the manifold, the contact structure and the tori). This direction can be constant along some sub-intervals[1] but it cannot be constant in a neighborhood of t if ξT_t is linear.

If a contact structure ξ on a solid torus W is transverse to a core curve K of W then it lifts to a contact structure on the toric annulus $T \times [0, 1]$ obtained by blowing[2] up K. The lifted contact structure induces a linear foliation on the boundary component which projects to K, say T_0. The direction of this foliation is spanned by the meridian class, i.e. the class in $H_1(T)$ which spans the kernel of the map from $H_1(T \times [0, 1])$ to $H_1(W)$ induced by the projection.

The following lemma gives a precise formulation of the idea described informally at the beginning of this subsection.

Lemma 5.4. *Let ξ' be a contact structure on a solid torus W transverse to a core curve K of W. Let ξ be the contact structure on $T \times [0, 1]$ lifted from W as described above. If all characteristic foliations ξT_t are suspensions whose asymptotic directions never contains the meridian homology class for $t > 0$ then (W, ξ') is contactomorphic to a domain in $(S^1 \times \mathbb{R}^2, \xi_{rot} = \ker(d\phi + r^2\, d\theta))$.*

This lemma is an easy consequence of the following result of Giroux.

Theorem 5.5 (Giroux 2000, [8, Theorem 3.3]). *Suppose two contact structures ξ_0 and ξ_1 on $T \times [0, 1]$ induce suspensions on each torus T_t which agree on T_0 and T_1. If the two paths of asymptotic directions $t \mapsto d(\xi_i T_t)$ are non constant and homotopic relative to their common end-points then ξ_0 and ξ_1 are isotopic relative to the boundary $T_0 \cup T_1$.* $\square$

Proof of Lemma 5.4. Because of the asymptotic direction assumption, we can choose a longitudinal curve L for W whose homology class does not belong to the asymptotic direction of any of the foliations ξT_t (pick any longitude and add a sufficiently large multiple of the meridian). In particular we can choose L to intersect transversely all leaves of $\xi \partial W$ and such that the corresponding Poincaré map rotates all points clockwise. So there is an identification of $(W, \partial W)$ with $(D^2 \times S^1, S^1 \times S^1)$ sending L to some $\{*\} \times S^1$ and $\xi \partial W$ to a foliation directed by $v = \frac{\partial}{\partial\theta} + F(\theta, \phi)\frac{\partial}{\partial\phi}$ where we use coordinates (θ, ϕ) on $S^1 \times S^1$ and $F(\theta, \phi) \leq 0$.

We now consider an embedding φ of $D^2 \times S^1$ into $\mathbb{R}^2 \times S^1$ which is the identity along $\{0\} \times S^1$ and agrees with $(1, \theta, \phi) \mapsto ((-F(\theta, \phi))^{\frac{1}{2}}, \theta, \phi)$ along $S^1 \times S^1$. An immediate computation reveals that φ sends $\xi \partial W$ to $\xi_{rot}\varphi(\partial W)$. Using the standard neighborhood theorem for curves transverse to a contact structure, it can also be easily arranged that φ sends ξ to ξ_{rot} along K. One may now blow up K and apply Giroux theorem to further isotope φ to a contact embedding. The homotopy hypothesis is guaranteed because the homology class of $[L]$ belongs to no asymptotic direction of a ξT_t. $\square$

Proof of Proposition 5.2. The vector field n_D is tangent to each torus $T_r = \partial \mathsf{T}(r)$, $0 < r \leq r_0$ and ξ is transverse to n_D so each characteristic foliation ξT_r is non-singular and transverse to n_D. In addition, n_D is transverse to the foliation by meridian circles of T_r coming from the disks D_z so it is a suspension. So there is a simple closed curve $\mathcal{C}$ in T_r which is not a meridian and transversely intersects all leaves of ξT_r (the homology class of $\mathcal{C}$ belongs to a rational approximation of the asymptotic cycle of n_D). For brevity, we denote the asymptotic directions $d(\xi T_r)$ and $d(n_D)$ by X and N, respectively, and by M the line spanned by the meridian homology class. One can see X and N as points on the circle $P(H_1(T^2; \mathbb{R}))$ continuously moving as r increases. Note that

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[1]This obviously happens around each t such that ξT_t is structurally stable.

[2]The blow up map from $[0, 1] \times S^1 \times S^1$ to $D^2 \times S^1$ is simply $(r, \theta, \varphi) \mapsto (re^{i\theta}, \varphi)$.

X initially equals M and moves clockwise in a monotone way whereas N moves in some way but never hits M or X. This easily implies that X cannot become meridional and one can apply Lemma 5.4 to obtain the desired conclusion. □

In order to prove Proposition 5.3, we first need to understand neighborhoods of disks transverse to Reeb vector fields.

Lemma 5.6. *Let M be a 3–manifold with a contact form α and D a closed disk embedded in M. If the Reeb vector field of α is transverse to D then there is a neighborhood V of D in M and a embbeding φ of V in $\mathbb{R}^3$ such that $\varphi(D)$ is transverse to vertical lines in $\mathbb{R}^3$ and $\varphi_*\alpha = dz + r^2 d\theta$.*

Proof. Let D' be a disk containing D and still transverse to the Reeb vector field R. Let ε be a positive number such that the flow of R_α embeds $D' \times [-\varepsilon, \varepsilon]$ into M. If r and θ are polar coordinates on D' and z is the coordinate in $[-\varepsilon, \varepsilon]$ the pull-back of α is $dz + f(r, \theta)\, dr + g(r, \theta)\, d\theta$ for some functions f and g. This contact form extends trivially to $D' \times \mathbb{R}$ as the contactization of the exact symplectic manifold (D', β) where $\beta = f\, dr + g\, d\theta$. We now think of D' as the unit disk in $\mathbb{R}^2$. The following claim will be proved below.

Claim: There is an extension of β to $\mathbb{R}^2$ such that $d\beta$ is symplectic everywhere and $\beta = r^2\, d\theta$ outside some large disk.

Since the symplectic condition is convex in dimension 2, $\alpha_t = dz + (1-t)\beta + tr^2\, d\theta, t \in [0,1]$, is a family of contact forms on $\mathbb{R}^2 \times \mathbb{R}$. We now use a general fact: if β_t is a family of 1–forms on a surface S such that each $d\beta_t$ is symplectic and all β_t agree outside some compact set, then there is an isotopy φ_t of the contactization $S \times \mathbb{R}$ sending surfaces $S \times \{z\}$ to surfaces transverse to lines $\{s\} \times \mathbb{R}$ and such that $\varphi_t^*\alpha_t = \alpha_0$. The isotopy is constructed using Moser's technique as the flow of a vector field $X_t = Y_t + \lambda_t \partial_z$ with Y_t the vector field on the surface defined by $d\beta_t(Y_t, \cdot) = -\dot\beta_t$ and $\lambda_t = -\beta_t(Y_t)$. The transversality condition comes from the commutation of X_t and ∂_z.

We now prove the claim. First we fix the symplectic form $\omega = r\, dr d\theta$ on $\mathbb{R}^2$ so that the problem is reformulated in terms of vector fields ω-dual to the forms we consider. We have a vector field Y with positive divergence on D' and we want to extend it to $\mathbb{R}^2$ such that it coincides with $Y_0 = r\partial_r$ outside some large disk. We denote by D_r the disk of radius r around the origin so $D' = D_1$. We first extend Y arbitrarily to a neighborhood of D_1 so its divergence stays positive in some $D'' = D_{1+\delta}$. Let h be a smooth function from $[0, \infty)$ to $\mathbb{R}$ with h and h' vanishing on $[0, 1]$ and everywhere non negative. If h grows sufficiently fast between 1 and $1 + \delta$, the vector field $Y' = Y + h\partial_r$ has positive divergence and is transverse to the boundary of D''. It is then easy to extend $Y'|_{D''}$ to a vector field which is transverse to all circles ∂D_r for $r \geq 1 + \delta$ and has the same orbits as Y_0 outside the disk $D_{1+2\delta}$. We can then rescale it in the region between D'' and some large disk D_r (here we do not control r) so that it still has positive divergence and coincides with Y_0 outside D_r. □

Proof of Proposition 5.3. Since any ball inside $(S^1 \times \mathbb{R}^2, \ker(d\phi + \rho^2\, d\theta))$ easily embeds inside $(\mathbb{R}^3, \xi_{\text{std}})$ we only need to construct an embedding into the former model. The proposition will follow from the construction of an embedding of $\mathsf{T}(r)$ into a solid torus satisfying the hypothesis of Lemma 5.4. We construct this embedding in several steps.

We first introduce a technical definition. Let Y be either a closed interval or S^1. Let D^2 be the unit disk in $\mathbb{R}^2$ and D_r denotes the disk of radius r. A contact structure on a $D^2 \times Y$ is under control if there is a vector field tangent to $\{0\} \times Y$ and to $\partial D_r \times Y$ for all r which is transverse to both ξ and the obvious foliation by disks. A contact structure on a domain C is under control if there is a diffeomorphism from C to $D^2 \times Y$ sending ξ to a contact structure under control. The inverse

images of the objects involved in the above discussion are then said to control ξ. In particular, in our geometric setup, ξ is under control on $\mathsf{T}(r)$ and Lemma 5.4 applies to any contact structure which is under control on a solid torus.

We want to construct thickened disks T_t and T_b that can be glued to the top and bottom, respectively, of $\mathsf{T}(r)$ so that the characteristic foliation on the top and bottom of $U = \mathsf{T}_t \cup \mathsf{T}(r) \cup \mathsf{T}_b$ is "standard" and ξ stays under control on this larger thicken disk. (Here top and bottom refer to $\mathsf{T}(r)$ seen with ζ vertical and oriented from bottom to top. Also "standard" means that the singular foliation has a single elliptic singularity and the rest of the leaves are radial.) Thus we will be able to glue the top and bottom of U together to obtain a solid torus S with a contact structure under control into which $\mathsf{T}(r)$ embeds and Lemma 5.4 will finish the proof.

We discuss the construction of T_t, the construction of T_b being analogous. Let z_t be the top extremity of ζ and $\mathsf{D} := \mathsf{D}_{z_t}$. The previous lemma gives a contact embedding φ of a neighborhood of D in $\mathbb{R}^2 \times \mathbb{R}$ such that the image of D is the graph of some function f over some (deformed) disk Ω in $\mathbb{R}^2$.

Let K be a constant such that $f(\rho, \theta) < K$ for all $(\rho, \theta) \in \Omega$. Set $C = \{(\rho, \theta, z) | (\rho, \theta) \in \Omega, f(\rho, \theta) \leq z \leq K\}$, this will (almost) be the bottom part of T_t. It is foliated by the graphs G_s of functions $(1 - s)f + sK$, $s \in [0, 1]$ and can also be seen as the union of the Reeb orbit $\varphi(z_t) \times [f(z_t), K]$ and vertical annuli over the images c_r of the $\partial \mathsf{T}(r) \cap \mathsf{D}$. Together with the vector field ∂_z, these objects show that ξ_{std} is under control on C. We now smooth $\varphi(\mathsf{T}(r) \cap Dom(\varphi)) \cup C$ to get a thicken disk C' extending $\varphi(\mathsf{T}(r) \cap Dom(\varphi))$ above $\varphi(\mathsf{D})$ which coincides with C when z is close to K and such that ξ_{std} is under control on C'.

Then we notice that there is a large number R such that Ω is contained in the disk of radius R about the origin and there is an isotopy $\Psi_t : \Omega \to \mathbb{R}^2, t \in [0, 1]$, such that $\Psi_0 = id_\Omega$ and $\Psi_1(\Omega)$ is a disk of radius R centered about the origin. Consider the embedding $\Psi : \Omega \times [0, 1] \to \mathbb{R}^3$ defined by $\Psi(p, t) = (\Psi_t(p), K + K't)$, where K' is a large positive constant to be determined soon. For each radius r, the characteristic foliations on $\Psi(c_r \times [0, 1])$ is given as the kernel of $\Psi|^*_{c_r \times [0,1]}(dz + \rho^2 \, d\theta) = K' \, dt + \beta$, where β is independent of K'. Thus for K' large enough $\Psi_* \frac{\partial}{\partial t}$ is never tangent to the characteristic foliation of $\Psi(c_r \times [0, 1])$ and ξ_{std} is under control on the image of Ψ. We then choose T_t to a smoothed version of $C' \cup \Psi(\Omega \times [0, 1])$. It can be glued to $\mathsf{T}(r)$ using φ. After doing the same thing for the bottom of $\mathsf{T}(r)$ we get T_b and we can do the construction of T_b and T_t with the same large radius R so that the top and bottom of $\mathsf{T}_b \cup \mathsf{T}(r) \cup \mathsf{T}_t$ can be glued to get a solid torus with a contact structure under control. $\qquad\square$

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