

Lectures on Contact Geometry in Low Dimensional Topology

John Etnyre

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1. Introduction

Contact geometry has been a key tool in many recent advances in low-dimensional topology. For example contact geometry was an integral part in the following results:

- (1) Kronheimer and Mrowka's proof that all non-trivial knots satisfy property P [32].
(Recall, a knot satisfies property P if non-trivial surgery on it yields a manifold with non-trivial fundamental group.)
- (2) Ozsváth and Szabó's proof that the unknot, trefoil and figure eight knots are all determined by surgery [37, 38]. That is, if r Dehn surgery on K is the same as r surgery on the unknot, trefoil or figure eight knots then the knot was the unknot, trefoil or figure eight knot. This result, for the unknot, was originally proven by Kronheimer-Mrowka-Ozsváth and Szabó [2].
- (3) Ozsváth and Szabó's proof that Heegaard-Floer invariants detect the Thurston norm of a manifold and the minimal Seifert genus of a knot [37].

There are many other results in which contact geometry has played a key role. For a brief, far from complete list of applications we have: achiral Lefschetz fibrations [19], Harer's conjecture on fibered knots [27], characterizing fibered knots in terms of Heegaard-Floer theory [24, 35], new knot invariants [34] and the existence of Engel structures [44].

We outline how contact geometry shows up in the results highlighted above by sketching the proof that Heegaard-Floer homology detects the Thurston norm (that is the minimal genus of a surface representing a homology class in a 3-manifold). Any unfamiliar terminology will be explained in the sections below.

Georgia Institute of Technology, Atlanta, GA 30332-0160

E-mail address: etnyre@math.gatech.edu

URL: <http://www.math.gatech.edu/~etnyre>

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Start with a closed irreducible oriented 3-manifold M and a surface $\Sigma \subset M$ such that Σ is of minimal genus among surfaces homologous to it. (Assume the genus of Σ is larger than 0.)

- (1) A theorem of Gabai [22], Theorem 4.7 below, gives a taut foliation \mathcal{F} on M that contains Σ as a leaf.
- (2) A theorem of Eliashberg and Thurston [12], Theorem 3.1 below, gives a positive and negative contact structure ξ_{\pm} on M that is C^0 -close to \mathcal{F} .
- (3) They also show, see the proof of Theorem 4.5 below, how to construct a symplectic structure ω on $X = M \times [-\epsilon, \epsilon]$ that weakly fills

$$(M, \xi_+) \cup (-M, \xi_-).$$

- (4) A theorem of Eliashberg [11] and, independently, the author [16], Theorem 6.1 below, shows how to find a closed symplectic manifold (X', ω') into which (X, ω) embeds. This closed manifold is constructed by finding “symplectic caps” to cap off the boundary of X . See Figure 1. There are

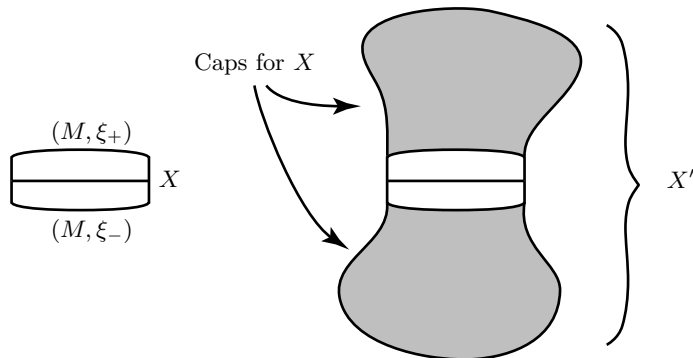


FIGURE 1. On the left is the symplectic manifold $X = M \times [-\epsilon, \epsilon]$. On the right, is the symplectic manifold X' that is constructed by “gluing” caps onto X .

various ways to construct these caps. We discuss the constructions in [16] which uses:

- (a) Giroux’s correspondence between open book decompositions and contact structures [26], Theorem 6.6 below, and
 - (b) Eliashberg [9] and, independently, Weinstein’s [45] ideas of contact surgery and symplectic handle attachment, see Theorem 5.8 below.
- (5) At this point one needs to use something like Seiberg-Witten theory or Heegaard-Floer theory to conclude something about M or Σ based on the existence of (X', ω') with (M, ξ_+) nicely embedded in it. Recall, there are lots of non-vanishing theorems for symplectic manifolds and with non-vanishing invariants you can frequently conclude things about surfaces in homology classes using an adjunction type inequality. Specifically, in our current situation we know that the Heegaard-Floer invariant of X' is non-zero in the spin^c structure associated to the symplectic form and this implies that the Heegaard-Floer homology $HF^+(M, \mathfrak{s}_\xi) \neq 0$, where \mathfrak{s}_ξ is the spin^c structure associated to ξ . Recall that for any spin^c structure \mathfrak{s}

for which $HF^+(M, \mathfrak{s}) \neq 0$ we have the adjunction inequality

$$|\langle c_1(\mathfrak{s}), [\Sigma] \rangle| \leq 2g - 2.$$

But since Σ is a leaf of \mathcal{F} it is easy to see that $\langle c_1(\mathfrak{s}_\xi), [\Sigma] \rangle = 2g - 2$. Thus we see that we can detect the minimal genus of a non-trivial homology class in M by seeing how c_1 of all the spin^c structures on M with non-zero Heegaard-Floer homologies evaluate on the homology class. In other words, Heegaard-Floer homology detects the Thurston norm of homology classes in M .

It is not too hard to adapt this line of argument to see that Heegaard-Floer homology also detects the Seifert genus of a knot in S^3 . The main idea is to perform 0 surgery on the knot and then apply the outline above to the surgered manifold. The other results mentioned above require somewhat different arguments, but the difference is in step (5) and hence the contact geometric input is largely the same.

These lectures are devoted to understanding the contact geometric part of the above outline, that is steps (2) through (4). We begin in Section 2 by defining all our basic concepts like contact structure, foliation, tight, symplectically fillable and so on. In Section 3 we give a detailed sketch of the proof of Eliashberg and Thurston's theorem (step (2) in the above outline). In Section 4 we discuss the relation between various notions associated to foliations and similar notions associated to contact structures. This discussion ends by constructing the symplectic manifold (X, ω) from step (3) of the above outline. The next two sections are devoted to the construction of symplectic caps in step (4). Specifically, we discuss Legendrian surgery and symplectic handle attachment in Section 5. The following section states Giroux's correspondence between contact structures and open book decompositions. We then use this correspondence to outline the construction of symplectic caps. The material in this last section is a bit more sketchy than that in the previous sections. This is because there are already lecture notes devoted to this topic. See [13] for a more complete discussion of Giroux's correspondence and the construction of symplectic caps. Thus you can regard these lectures as primarily concerning the relation between contact structures and foliations as exemplified in steps (2) and (3) above. Step (1) is purely foliation theoretic and would take us far afield of contact geometry. For more details on step (1) see Gabai's original work [22] or the text book [4]. Step (5) is also beyond the scope of these lectures, in that it does not explicitly use contact geometry. For more on this step we refer the reader to the original research articles or the more expository articles [23, 39]. We would like to emphasize that while we are using the applications to low-dimensional topology to motivate and guide our discussion of various topics in contact geometry, each of these topics is quite beautiful in its own right and there are many exciting directions for future research in all of these topics. We hope this beauty and potential is evident in what follows!

2. Contact structures and foliations

Throughout this section M will be a closed oriented 3-manifold. A *plane field* on M is simply a 2-dimensional sub-bundle of the tangent bundle of M . That is, at each $x \in M$ we have a plane ξ_x in the tangent bundle $T_x M$. Locally, one can always find a 1-form α such that

$$\xi_x = \ker \alpha_x,$$

for each $x \in M$.

Exercise 2.1 Show that α may be chosen to be a global 2-form if and only if ξ is orientable.

Hint: It might be helpful to use a Riemannian metric on M .

Example 2.2 Consider \mathbb{R}^3 with coordinates (x, y, z) . Let $\alpha_1 = dz$ and set

$$\xi_1 = \ker \alpha_1 = \text{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}.$$

See Figure 2.

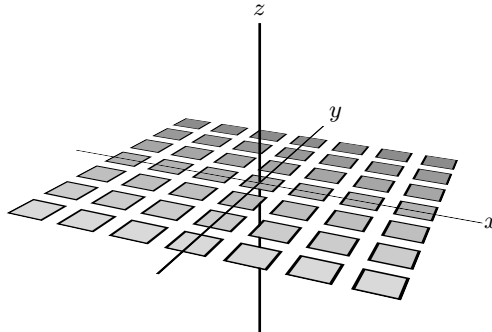


FIGURE 2. The plane field $\xi_1 = \ker dz$.

This is a fairly simple plane field. It is constant and does not change from point to point.

Example 2.3 Again consider \mathbb{R}^3 with coordinates (x, y, z) . Let $\alpha_2 = dz - y dx$ and $\alpha_3 = dz + y dx$. Set

$$\xi_2 = \ker \alpha_2 = \text{span}\left\{\frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right\}$$

and

$$\xi_3 = \ker \alpha_3 = \text{span}\left\{\frac{\partial}{\partial x} - y \frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right\}.$$

See Figure 3. So each ξ_2 and ξ_3 consist of horizontal planes (that is, parallel to the xy -plane) at any point in the xz -plane and as you leave the xz -plane along a ray perpendicular to the xz -plane the planes ξ_2 and ξ_3 are always tangent to this ray and twisting a total of 90° in a counterclockwise, respectively clockwise, manner.

We call a plane field ξ a *foliation* if there is a 1-form α with $\xi = \ker \alpha$ for which

$$\alpha \wedge d\alpha = 0.$$

We call ξ a *positive contact structure*, respectively *negative contact structure*, on M if there is a 1-form α with $\xi = \ker \alpha$ for which

$$\alpha \wedge d\alpha \text{ is a positive multiple of the volume form on } M,$$

respectively

$$\alpha \wedge d\alpha \text{ is a negative multiple of the volume form on } M.$$

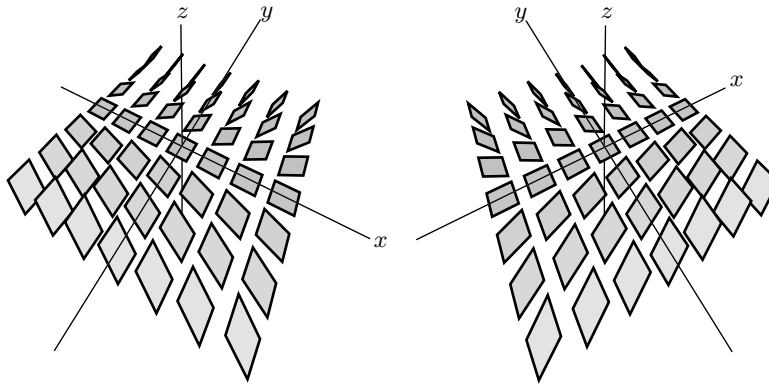


FIGURE 3. On the left is $\xi_2 = \ker(dz - y dx)$ and on the right is $\xi_3 = \ker(dz + y dx)$.

These conditions are usually stated

$$\alpha \wedge d\alpha > 0$$

and

$$\alpha \wedge d\alpha < 0.$$

A plane field ξ is called a *positive confoliation*, respectively *negative confoliation*, if there is a 1-form α with $\xi = \ker \alpha$ for which

$$\alpha \wedge d\alpha \geq 0,$$

respectively

$$\alpha \wedge d\alpha \leq 0.$$

Exercise 2.4 Show that these definitions do not depend on the 1-form α chosen to define ξ .

Exercise 2.5 Show that if a 3-manifold supports a contact structure then it is orientable.

Theorem 2.6 (Frobenius, see [6]). *If a plane field ξ is closed under Lie brackets (that is, if v, w are sections of ξ then their Lie bracket $[v, w]$ is also a section of ξ), then M is foliated by surfaces tangent to ξ .*

To make the theorem more precise we say a 3-manifold M is *foliated by surfaces* if M can be written as the disjoint union of surfaces such that each point on M is contained in a coordinate chart that maps the intersection of the surfaces with the chart to the constant z -hyperplanes in \mathbb{R}^3 (with coordinates (x, y, z)). The degree of smoothness of the foliation is the degree of smoothness with which the transition functions between charts can be chosen. The surfaces that show up in this theorem are called *leaves* of the foliation. The Frobenius theorem is an “integrability” theorem. That is you can “integrate” information in the tangent space into the manifold: given ξ in the tangent space you find things, surfaces, that are actually in the manifold that induce the plane field.

Exercise 2.7 Show that the condition $\alpha \wedge d\alpha = 0$ is equivalent to $\xi = \ker \alpha$ being closed under Lie bracket.

Hint: Recall the formula for $d\alpha$ in terms of α and Lie brackets.

From this exercise we see that the 1-form conditions in the definitions of foliations and contact structures have to do with the plane field being tangent to surfaces. This integrability condition implies that a contact structure cannot be tangent to any surface along an open set of the surface. (Of course, a surface can be tangent to a contact structure at isolated points and even along lines in the surface, but not along an open set in the surface.)

Example 2.8 Back to the example \mathbb{R}^3 with $\xi_1 = \ker \alpha_1$, where $\alpha_1 = dz$. Clearly $d\alpha_1 = d(dz) = 0$ so $\alpha_1 \wedge d\alpha_1 = 0$ and ξ_1 is a foliation. If we define the surfaces $S_{z_0} = \{(x, y, z) | z = z_0\}$ in \mathbb{R}^3 , then clearly

$$\mathbb{R}^3 = \coprod_{z \in \mathbb{R}} S_z,$$

and $T_{(x,y,z)}S_z = (\xi_1)_{(x,y,z)}$. See Figure 4. This verifies the Frobenius theorem in

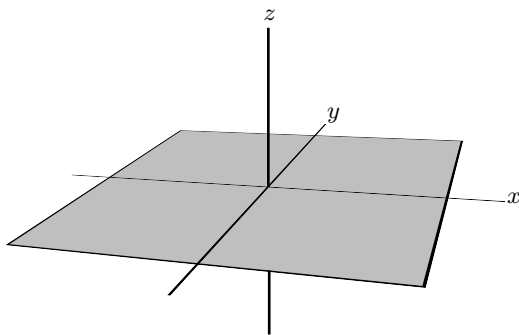


FIGURE 4. Here is the surface to which the planes in Figure 2 are tangent.

this example.

Example 2.9 Let $M = S^1 \times \Sigma$ for some surface Σ . Let θ be the coordinate on the S^1 factor and $\alpha = d\theta$. Clearly the plane field $\xi = \ker \alpha$ is always tangent to $\{pt\} \times \Sigma$ so ξ is a foliation with leaves the surface fibers. More generally, we could let M be a Σ bundle over S^1 and again let ξ be the tangents to the fibers. Note $d\alpha = d(d\theta) = 0$ so $\alpha \wedge d\alpha = 0$ as we expect for a foliation.

Example 2.10 The plane fields ξ_2 and ξ_3 from Example 2.3 are positive and, respectively, negative contact structures since one can easily check that $\alpha_2 \wedge d\alpha_2 > 0$ and $\alpha_3 \wedge d\alpha_3 < 0$.

Example 2.11 We now give a contact structure on a closed 3-manifold. Let S^3 be the unit 3-sphere in $\mathbb{R}^4 = \mathbb{C}^2$. Set $\alpha = r_1^2 d\theta_1 + r_2^2 d\theta_2$ where the coordinates on \mathbb{C}^2 are (z_1, z_2) and $z_j = r_j e^{i\theta_j}$.

Exercise 2.12 Check that when α is restricted to S^3 then $\alpha \wedge d\alpha > 0$. Also show that $\ker \alpha$ is the set of complex tangencies to S^3 , that is ξ is the set of vectors v tangent to S^3 for which iv is also tangent to S^3 , where $i = \sqrt{-1}$.

Fact 2.13. *All oriented 3-manifolds have foliations and positive (negative) contact structures.*

There are many proofs of this fact. We will indicate one proof in Section 6. For another approach see [15].

We would now like to understand plane fields locally. For this we start with the following lemma.

Lemma 2.14. *Given a plane field ξ there are local coordinates so that a 1-form α with $\xi = \ker \alpha$ has the form*

$$\alpha = dz - a(x, y, z) dx$$

Exercise 2.15 Prove this lemma.

Hint: Let $\phi: [-1, 1] \times [-1, 1] \rightarrow M$ be an embedding of a disk such that $\phi_* \frac{\partial}{\partial z}$ is transverse to ξ , where the coordinates on the disk are (x, z) . Now find a vector field v in ξ that is transverse to the image of this embedding. Use the vector field to extend ϕ to an embedding of a 3-ball.

Remark 2.16. Note that your proof (if you followed the hint) of Lemma 2.14 can be extended to the following setting: if $\gamma: N \rightarrow M$ is any embedding, where $N = [0, 1]$ or S^1 , for which $\gamma'(x) \in \xi_{\gamma(x)}$ for all $x \in N$ then we can extend γ to $\Gamma: N \times [-\epsilon, \epsilon] \times [-\epsilon, \epsilon] \rightarrow M$ so that α has the desired form in these coordinates. Also note that if $N = [0, 1]$ then you can also assume the coordinate on γ is y and extend γ to $\Gamma: [-\epsilon, \epsilon] \times N \times [-\epsilon, \epsilon] \rightarrow M$ so that α has the desired form in these coordinates.

To see how this local form for α relates to ξ being a foliation or contact structure, we have the following lemma.

Lemma 2.17. *Given a plane field ξ and a 1-form α as in Lemma 2.14, then*

- (1) ξ is a positive (negative) contact structure if and only if

$$\frac{\partial a}{\partial y} > 0 \quad \left(\frac{\partial a}{\partial y} < 0 \right).$$

- (2) ξ is a foliation if and only if

$$\frac{\partial a}{\partial y} = 0.$$

Exercise 2.18 Verify this lemma.

One can improve the above local results for the 1-form α when $\ker \alpha$ is a contact structure or foliation.

Theorem 2.19 (Darboux/Pfaff Theorem, see [6, 33]). *If ξ is a foliation then there are local coordinates (x, y, z) such that*

$$\xi = \ker dz.$$

If ξ is a positive (negative) contact structure then there are local coordinates (x, y, z) such that

$$\xi = \ker(dz - y dx), \quad (\xi = \ker(dz + y dx)).$$

This theorem says that “locally all foliations (positive/negative contact structures) look the same”. Note this is very different from other “geometries”, like Riemannian metrics, which can look very different locally. Thus foliations and contact structures are in some sense insensitive to local things and thus if they tell us anything about the manifold, it will have to be something global.

This theorem also indicates similarities between foliations and contact structures (*i.e.* they both have local normal forms). We will see many more similarities below, but there are some differences. One major difference is that there are no “non-trivial deformations” of a contact structure where foliations have “non-trivial deformations”. To better understand this last sentence consider the following theorem.

Theorem 2.20 (Gray’s Theorem, [33]). *If $\xi_t, t \in [0, 1]$, is a 1-parameter family of contact structures on M that agree off of a compact subset of M then there is a 1-parameter family of diffeomorphisms $\psi_t: M \rightarrow M$ such that $(\psi_t)_*\xi_0 = \xi_t$.*

This theorem says that isotopies of contact structures (as plane fields) are equivalent to isotopies of the manifold.

If ξ_1 and ξ_2 are two contact structures on M then a diffeomorphism $f: M \rightarrow M$ is called a *contactomorphism* from ξ_1 to ξ_2 if $f_*(\xi_1) = \xi_2$. So Gray’s Theorem says that any family of contact structures are related by a family of contactomorphisms. The situation for foliations is quite different.

Example 2.21 Let \mathcal{F}_s be the foliation of T^2 by lines of slope s . Let $\xi_s = \mathcal{F}_s \times S^1$ be the product foliation on $T^3 = T^2 \times S^1$.

Exercise 2.22 Show there is no family of diffeomorphisms $\phi_s: T^3 \rightarrow T^3$ such that $(\phi_s)_*\xi_0 = \xi_s$.

This last example (and exercise) show that there are deformations of foliations that do not come from diffeomorphisms of the underlying 3-manifold.

To see further similarities between foliations and contact structures we consider special foliations and contact structures on $D^2 \times S^1$.

Example 2.23 In this example we construct a Reeb foliation on the solid torus. We will construct a foliation on \mathbb{R}^3 that is invariant under translations in the z -direction. Thus we can look at $\mathbb{R}^2 \times S^1$ thought of as \mathbb{R}^3 modulo $z \mapsto z + 1$. Then the unit disk in the xy -plane times S^1 will be a solid torus. Consider the function

$$f(x, y, z) = g(x^2 + y^2)e^z$$

where $g: [0, \infty) \rightarrow \mathbb{R}$ is a strictly decreasing function equal to 1 at 0 and 0 at 1 (moreover assume all the higher derivatives of g are 0 at 0). One may check that f has no critical points so the level sets of f give a foliation of \mathbb{R}^3 . Moreover it is easy to check that the foliation is invariant under translation in the z -direction. Thus we get a foliation induced on the unit disk in the xy -plane cross S^1 . The unit circle in the xy -plane times S^1 will be a closed torus leaf in the foliation. The interior of the solid torus is foliated by leaves diffeomorphic to \mathbb{R}^2 .

Exercise 2.24 Picture this foliation.

If we let $D = D^2 \times \{pt\}$ be a meridional disk in the solid torus then intersecting D with ξ will induce a singular foliation on D . This foliation is shown on the left of Figure 5.

A foliation on a manifold M is said to have *Reeb components* if there is a solid torus in M such that the foliation is homeomorphic with the one in the previous example. (Note we are thinking of a foliation here in terms of its leaves and not as

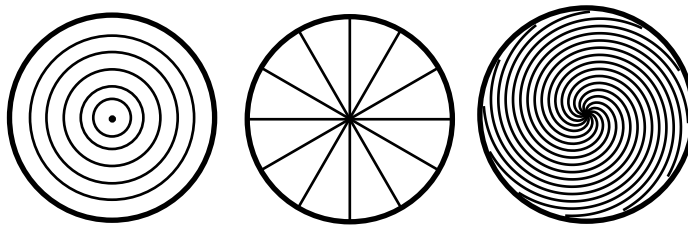


FIGURE 5. The meridional disk in the Reeb torus is shown on the left. The meridional disk in the Lutz tube is shown in the middle and on the right is the meridional disk in the Lutz tube with its interior pushed up slightly.

a plane field. In this way it makes sense to talk about homeomorphic foliations.) A foliation on M that does not have any Reeb components is said to be *Reebless*.

Before we move to the next example let's formalize this idea of an "induced foliation". Let ξ be any plane field on a 3-manifold M . If Σ is a surface embedded in M then at each point $x \in \Sigma$, $\xi_x \cap T_x \Sigma$ is either a line in $T_x \Sigma$ or all of $T_x \Sigma$. The points where $\xi_x = T_x \Sigma$ are called *singular points*. Away from the singular points we have a line field on Σ . It is easy to use the Forbenius Theorem (or in this case just the existence of solutions to ordinary differential equations) to see that we can foliate Σ with 1-manifolds away from the singular points. This is called the *induced singular foliation*, or when ξ is a contact structure it is sometimes called the *characteristic foliation*.

Example 2.25 In this example we construct a Lutz tube, this is a special contact structure on a solid torus. Again we will do this by constructing a contact structure on \mathbb{R}^3 that is invariant under translations in the z -direction. Then if we consider \mathbb{R}^3 / \sim , where $(x, y, z) \sim (x, y, z + 1)$, we get a contact structure on $\mathbb{R}^2 \times S^1$. Choosing a disk in \mathbb{R}^2 will give our desired contact structure on a solid torus. Let $\alpha = \cos r dz + r \sin r d\theta$. This is a 1-form in cylindrical coordinates. Set

$$\xi_{ot} = \ker \alpha.$$

This contact structure is shown in Figure 6. Note ξ_{ot} is radially symmetric and

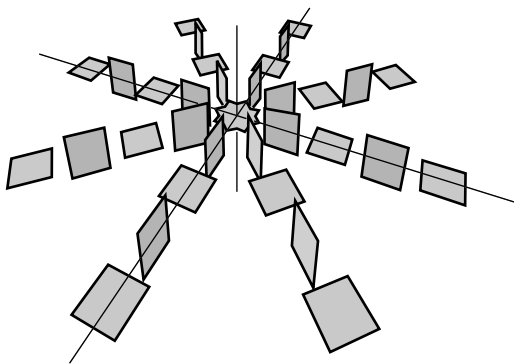


FIGURE 6. The contact structure ξ_{ot} on \mathbb{R}^3 .

invariant under translation in the z -direction. Also note that along a ray perpendicular to the z -axis the contact planes twist in a left handed fashion, and they twist infinitely often. The torus $T = \{(r, \theta, z) | r \leq \pi\}$ in \mathbb{R}^3 / \sim with the contact structure induced from ξ_{ot} is called a *Lutz tube*.

If $D = D^2 \times \{pt\}$ is a meridional disk in a Lutz tube then the characteristic foliation induced on D by the contact structure ξ is shown in the middle of Figure 5. This characteristic foliation is a bit strange in that the boundary of the disk consists entirely of singularities. This is a very “non-generic” phenomenon, that is, usually you don’t see lines of singularities. In particular, try the following exercise.

Exercise 2.26 If the interior of D is pushed up a little then show the characteristic foliation is as shown on the right hand side of Figure 5. By “pushed up a little” we mean let $D' = \{(r, \theta, z) | r \leq \pi, z = \epsilon(-r^2 + \pi^2)\}$, where ϵ is some positive number very very close to zero. Note there is only one singularity in the new characteristic foliation.

The disk shown on the right hand side of Figure 5 is called an *overtwisted disk*. The key feature of this disk is that its boundary is tangent to ξ but the disk itself is transverse to ξ along the boundary. Any contact structure that contains an overtwisted disk is called *overtwisted*, otherwise it is called *tight*.

Remark 2.27. Any overtwisted contact structure contains a Lutz tube (this is not obvious!). So we could have defined a tight contact structure to be a “Lutzless” contact structure in analogy with a Reebless foliation.

It is very easy to construct contact structures with Lutz tubes and foliations with Reeb components. In particular, for contact structures we have the following result.

Theorem 2.28 (Eliashberg 1989, [7]). *On a closed oriented 3-manifold there is a one-to-one correspondence between homotopy classes of plane fields and isotopy classes of overtwisted contact structures.*

Exercise 2.29 Using Eliashberg’s theorem show that any closed oriented 3-manifold has infinitely many distinct overtwisted contact structures.

Exercise 2.30 Try to show that any overtwisted contact structure contains a Lutz tube.

Hint: Try to find a way of modifying a contact structure to introduce a Lutz tube and then try to see if you can do this so as not to change the homotopy class of plain field.

So overtwisted contact structures are fairly well understood and seem to be very flexible. It turns out tight contact structures are much more interesting, but do they exist? The answer is “yes” but not always.

Theorem 2.31 (Etnyre-Honda, [20]). *Let M be the Poincaré homology sphere with its non-standard orientation. (In other words, M can be described as 1 surgery on the right handed trefoil knot.) Then M does not admit a positive tight contact structure. Moreover, $M \# (-M)$ does not admit any tight contact structures (positive or negative). Here $-M$ means M with the opposite orientation.*

This is a little worrisome, but we will see shortly that there are lots of tight contact structures; however, let's first observe an important property of them that indicates they see subtle properties of topology. For this recall that an oriented 2-dimensional bundle, like ξ , has an Euler class $e(\xi) \in H^2(M; \mathbb{Z})$. Moreover, if Σ is not closed but transverse to ξ then choose a vector field $v_x \in \xi_x \cap T_x \Sigma, x \in \partial \Sigma$, along $\partial \Sigma$ that points out of Σ , then there is an Euler class of $\xi|_{\Sigma}$ relative to v .

Theorem 2.32. *Let M be a closed oriented irreducible 3-manifold and ξ a plane field. Let Σ be a surface embedded in M .*

- (1) (Thurston 1986, [42]) *If ξ is a Reebless foliation and Σ is closed, then*

$$|\langle e(\xi), [\Sigma] \rangle| \leq -\chi(\Sigma), \quad \text{if } \Sigma \neq S^2$$

and otherwise

$$|\langle e(\xi), [\Sigma] \rangle| = 0.$$

If Σ has boundary transverse to ξ , then

$$\langle e(\xi), [\Sigma] \rangle \leq -\chi(\Sigma).$$

- (2) (Eliashberg 1992, [10]) *If ξ is a tight positive contact structure and Σ is closed, then*

$$|\langle e(\xi), [\Sigma] \rangle| \leq -\chi(\Sigma), \quad \text{if } \Sigma \neq S^2$$

and otherwise

$$|\langle e(\xi), [\Sigma] \rangle| = 0.$$

If Σ has boundary transverse to ξ , then

$$\langle e(\xi), [\Sigma] \rangle \leq -\chi(\Sigma).$$

Exercise 2.33 Show that this theorem implies there are only finitely many elements in $H^2(M; \mathbb{Z})$ that can be the Euler class of a Reebless foliation or tight contact structure.

The easiest way to prove a contact structure is tight is to find a symplectic filling. Recall a 4-manifold X is a *symplectic manifold* if there is a 2-form ω such that $d\omega = 0$ and $\omega \wedge \omega$ is a never zero 4-form. Note $\omega \wedge \omega$ is a volume form on X . We always assume X is oriented by this form. If $M = \partial X$ (as oriented manifolds) and ξ is a positive contact structure on M then we say that ω *dominates* ξ if $\omega|_{\xi} > 0$, (by this we mean $\omega(v, w) > 0$ for any oriented basis v, w for ξ).

Remark 2.34. It is very important that $M = \partial X$ as oriented manifolds. However, if you are not too interested in the orientation on ξ then the condition that $\omega|_{\xi} > 0$ can be easily arranged if $\omega|_{\xi} \neq 0$ by reversing orientation on ξ .

If (M, ξ) is one component of a contact manifold (M', ξ') and (X, ω) is a compact symplectic manifold for which ω dominates ξ' then we say that (X, ω) is a *weak symplectic semi-filling* of (M, ξ) . If M' is connected (that is $M' = M$) then we say (X, ω) is a *weak symplectic filling* of (M, ξ) . The reason we have brought up weak symplectic fillings is the following theorem.

Theorem 2.35 (Gromov-Eliashberg, [8, 28]). *If (M, ξ) is a weakly symplectically semi-fillable contact structure then ξ is tight.*

Example 2.36 Consider S^3 as the unit sphere in $\mathbb{C}^2 = \mathbb{R}^4$. On \mathbb{C}^2 we use polar coordinates $(r_1, \theta_1, r_2, \theta_2)$. Let $\alpha = r_1^2 d\theta_1 + r_2^2 d\theta_2$. Earlier you checked that α restricted to S^3 is a contact form for $\xi = \ker \alpha$. Clearly,

$$\omega = d\alpha = 2r_1 dr_1 \wedge d\theta_1 + 2r_2 dr_2 \wedge d\theta_2$$

is a symplectic form on \mathbb{C}^2 . So in particular, it is a symplectic form on B^4 . Note

$$\omega|_{\xi} = d\alpha|_{\xi} > 0$$

(since $\alpha \wedge d\alpha > 0$). Thus (B^4, ω) is a weak symplectic filling of (S^3, ξ) , and hence ξ is tight.

Remark 2.37. This result should be compared to the famous result of Novikov that any foliation on S^3 must have Reeb components, [36]. (Recall, tight contact structures have no Lutz tubes and no overtwisted disks.)

We will find many other fillable, and hence tight, contact structures later, but now the obvious question is: Are all tight contact structures fillable? The answer is No.

Theorem 2.38 (Etnyre-Honda 2002, [21]). *There are tight but not weakly semi-fillable contact structures.*

This ends our brief introduction to contact structures and foliations. To learn more about foliations see [3, 4, 29] and to learn more about contact structures see [33]. We now move on to the perturbations of foliations into contact structures.

3. From foliations to contact structures

Consider the interesting foliation on $S^2 \times S^1$ given by

$$\zeta = \ker \alpha,$$

where $\alpha = d\theta$ and θ is the coordinate on the S^1 factor. So ζ is given by the tangents to the two spheres $S^2 \times \{\theta\}$.

Theorem 3.1 (Eliashberg and Thurston, 1998 [12]). *Any oriented C^2 -foliation ξ on an oriented 3-manifold M , other than the foliation ζ of $S^2 \times S^1$, may be C^0 -approximated by a positive and a negative contact structure.*

To make sense of this theorem we need to have a topology on the space of plane fields. To this end recall that we can associate to the tangent bundle of M the bundle of 2-planes. In other words at each point of $p \in M$ we replace $T_p M = \mathbb{R}^3$ with the Grassmann of 2-planes in \mathbb{R}^3 , which we denote $G_{2,3}$. Sections of this new bundle are equivalent to plane fields on M . Thus when we say we have a C^k -plane field that means the corresponding section is C^k -smooth. Moreover, on the space of sections we have the topology of C^k -convergence, so we can talk about C^k -neighborhoods of plane fields.

Exercise 3.2 Reinterpret the space of sections and the topology on them in terms of 1-forms.

We say a foliation ξ can be C^k -deformed into a contact structure if there is a C^k -family $\xi_t, t \in [0, \epsilon]$, so that $\xi_0 = \xi$ and ξ_t is a contact structure for $t > 0$. We also say ξ can be C^k -approximated by a contact structure if in any C^k -neighborhood of ξ there is a contact structure. It is clear that if ξ can be C^k -deformed into a contact structure then it can also be C^k -approximated by one too.

Example 3.3 Consider T^3 thought of as \mathbb{R}^3 , with coordinates (x, y, z) , modulo the action of the lattice \mathbb{Z}^3 . Now consider the 1-form

$$\alpha_n^t = dz + t((\cos 2\pi z) dx + (\sin 2\pi z) dy),$$

where n is any positive integer. When $t = 0$ we get $\alpha_n^0 = dz$ which defines the foliation of T^3 by constant z , T^2 's. When $t > 0$ we get positive contact structures ξ_n^t . Note that by Gray's theorem if we fix n then all the ξ_n^t are isotopic for $t > 0$. So we can unambiguously talk about ξ_n (no t dependence). It is a result of Kanda [31] and Giroux [25] that the ξ_n are all distinct and, up to contactomorphism, give all tight contact structures on T^3 . When $t < 0$ then note the ξ_n^t are negative contact structures on T^3 .

Remark 3.4. We make a few observations about the theorem and this example.

- (1) This last example shows that a fixed foliation can be approximated by (and even deformed into) infinitely many different contact structures! (We are not claiming that any foliation can be approximated by infinitely many different contact structures, just that some can.) This is somewhat surprising, as you might think that if contact structures are sufficiently close then you can deform one to the other through contact structures and hence they would be isotopic/contactomorphic. This example demonstrates that this is not the case.
- (2) The theorem only gives a contact approximation to the foliation not a deformation. It is possible the theorem is true with "approximation" replaced by "deformation".
- (3) We lose smoothness in the theorem. We must start with a C^2 -foliation, but the approximation is only C^0 -close to the original foliation. It is possible that the theorem is true with " C^0 " replaced with " C^2 ".

Before we begin to sketch the proof of Theorem 3.1 we first consider why the foliation ζ on $S^2 \times S^1$ is so special. This is indicated in the following two theorems.

Theorem 3.5 (Reeb stability for confoliations, Eliashberg and Thurston, 1998 [12]). *Suppose a confoliation ξ on M admits an embedded integral 2-sphere S (i.e. for all $x \in S$ we have $T_x S = \xi_x$) then (M, ξ) is diffeomorphic to $(S^2 \times S^1, \zeta)$.*

This theorem is well known, and easier to prove, when ξ is a foliation.

Exercise 3.6 Try to prove this theorem under the assumption that ξ is a foliation. Hint: Try to show the subset of M that is foliated by S^2 's is both open and closed. If you are having trouble maybe read ahead a few pages and come back and try again.

Theorem 3.7 (Eliashberg and Thurston, 1998 [12]). *There is a C^0 -neighborhood of ζ such that any confoliation of $S^2 \times S^1$ in that neighborhood is diffeomorphic to ζ .*

This last theorem really explains why we must have the exceptional case of $(S^2 \times S^1, \zeta)$ in Theorem 3.1. We can also use the Reeb stability theorem to see how not to try to perturb a foliation into a contact structure. Indeed, the next theorem implies that you can not "locally" perturb a foliation into a contact structure.

Theorem 3.8. *Let ξ be a confoliation on the 3-ball B which is standard near ∂B (i.e. near ∂B , ξ is given by $\ker dz$). Then ξ is a foliation and is diffeomorphic to the standard foliation on B .*

PROOF. Given (B, ξ) we can find an embedding of B into $S^2 \times S^1$ so that near ∂B ξ agrees with ζ . Now define ζ' to be ζ on $(S^2 \times S^1) \setminus B$ and ξ on B . This is a confoliation on $S^2 \times S^1$ that has an integral sphere (we can certainly choose our initial embedding of B so that it misses $S^2 \times \{pt\}$ for some pt). Thus the Reeb stability theorem implies that ζ' is diffeomorphic to ζ . This diffeomorphism shows that ξ is a foliation diffeomorphic to ζ restricted to the image of the ball.

Exercise 3.9 Show that ξ is indeed standard the standard foliation on B^3 . □

We can break the proof the main theorem, Theorem 3.1, into two parts:

Part 1 Perturb ξ into a confoliation ξ' such that ξ' is contact on a “sufficiently large” part of M .

Part 2 Perturb ξ' into a contact structure.

These steps seem overly simplistic, but this is a good outline of the strategy. We will actually start with Part 2 so that we can figure out what “sufficiently large” in Part 1 actually means! During the proof of Theorem 3.1 we will find a positive contact structure approximating ξ . The proof for a negative contact structure is similar. From this point on when we say “contact structure” we mean “positive contact structure”.

3.1. Part 2 of the proof of Theorem 3.1

Suppose we are given a confoliation ξ' on a closed oriented manifold M , then set

$$H(\xi') = \{x \in M : \xi'_x \text{ is contact at } x, \text{ (i.e. } (\alpha \wedge d\alpha)_x > 0)\}.$$

The set $H(\xi')$ is called the *hot zone* or *contact region*. (The reason for the terminology “hot zone” will be clear from the discussion below.) Now set

$$G(\xi') = \{x \in M : \text{there is a path } \gamma \text{ from } x \text{ to } y \\ \text{such that } y \in H(\xi') \text{ and } \gamma \text{ is tangent to } \xi'\}.$$

The precise statement of Part 2 of the proof of Theorem 3.1 is contained in the following theorem.

Theorem 3.10. *If $G(\xi') = M$ then ξ' can be C^∞ -deformed into a contact structure.*

There are two proofs of this theorem. An *analytic proof* (due to Altschuler [1]) and a *topological proof* (due to Eliashberg and Thurston [12]). We sketch both these approaches.

The analytic way: Choose a Riemannian metric on M and further choose a 1-form α such that $\xi' = \ker \alpha$ and $|\alpha| = 1$ at all points of M . Now consider the equations

$$\begin{aligned} \frac{\partial}{\partial t} \beta &= *(\alpha \wedge df), \\ \beta_0 &= \alpha, \end{aligned}$$

where $f = *(\alpha \wedge d\beta + \beta \wedge d\alpha)$ and β is a section of $T^*M \times \mathbb{R}^+$. Here \mathbb{R}^+ is the non-negative real numbers and we think of β as a 1-parameter family of 1-forms on M . (We will frequently denote the time dependence of β as a subscript.) In these equations we are given α and we are trying to solve for β .

These equations are a weakly-parabolic system and Altschuler proved that given α as above there is a unique smooth solution for $t \in [0, \infty)$. See [1].

The function f also evolves by a weakly-parabolic equation:

$$\frac{\partial f}{\partial t} = \Delta_\alpha f + \nabla_X f,$$

where X is some time dependent vector field and Δ_α is the ‘‘Laplacian on $\ker \alpha$ ’’. Intuitively Δ_α is the sum of partial derivatives in the directions tangent to ξ . For the precise definition see [1]. A version of the maximum principle for this equation implies the following fundamental property: If q is connected to a point p by a path tangent to ξ' and $f(p, 0) > 0$, then $f(q, t) > 0$ for all $t > 0$. In particular, if $f(p, 0) > 0$ then $f(p, t) > 0$ for all $t > 0$.

As the archetypical parabolic equation is the heat equation we might think of f as representing heat then this last property says that ‘‘Heat flows infinitely fast to all points of M accessible to the hot zone’’. Note that for any point $p \in H(\xi')$ we have $f(p) > 0$. So if $G(\xi') = M$ then $f(p, t) > 0$ for all $t > 0$. This is good because if we set

$$\eta = \alpha + \epsilon \beta_1$$

then

$$d\eta = d\alpha + \epsilon d\beta_1,$$

and

$$\eta \wedge d\eta = \alpha \wedge d\alpha + \epsilon(\alpha \wedge d\beta_1 + \beta_1 \wedge d\alpha) + \epsilon^2 \beta_1 \wedge d\beta_1.$$

Since ξ' is a confoliation we know the first term on the right hand side is greater than or equal to 0. The second term is just $\epsilon(*f_1)$ which is positive everywhere since $G(\xi') = M$. Finally, the last term might have any sign, but by choosing ϵ small enough its magnitude will be smaller than that of the second term. Thus η is a contact form for all ϵ near zero and we have constructed the deformation of ξ' from Theorem 3.10.

The topological way: This proof is due to Eliashberg and Thurston, [12]. Given an arc γ tangent to ξ' with one end in a contact region, we want to show how to extend the contact region to contain a neighborhood of γ . To this end parameterize γ by $[0, 1]$ and we find a neighborhood N of γ of the form $N = [-1, 1] \times [0, 1] \times [-1, 1]$, with coordinates (x, y, z) , such that

- (1) $\gamma = \{x = 0, z = 0\}$,
- (2) $\xi' = \ker \alpha$ where

$$\alpha = dz - a(x, y, z) dx, \text{ and}$$

- (3) ξ' is contact near $\{y = 1\}$.

(A slight modification of your argument in the proof of Lemma 2.14 above will suffice to find this neighborhood.) The main goal now is to prove the following lemma.

Lemma 3.11. *There is a C^∞ -deformation of ξ' supported in N to a plane field that is contact on the interior of N .*

PROOF. Since ξ' is a confoliation we know, from Lemma 2.17, that $\frac{\partial a}{\partial y} \geq 0$ in N and near $\{y = 1\}$, $\frac{\partial a}{\partial y} > 0$. Choosing x_0 and z_0 then $a(x_0, y, z_0)$ is a non-decreasing function that is strictly increasing near $y = 1$. It is easy to choose a new function of y that is strictly increasing for all y in $(0, 1)$ and arbitrarily close to $a(x_0, y, z_0)$. The trick now is to simultaneously perturb $a(x_0, y, z_0)$ for all x_0 and z_0 . To this end, choose δ so that all the $a(x_0, y, z_0)$ are strictly increasing for y in $[1 - \delta, 1]$. Now let

m be the minimum of $\frac{\partial a}{\partial y}(x_0, y, z_0)$ for all $(x_0, y, z_0) \in [-1, 1] \times [1 - \delta, 1] \times [-1, 1]$. Choose a positive function $f(y)$ so that $f(y)$ is strictly increasing on $[0, 1 - \delta]$ and decreasing on $[1 - \delta, 1]$, but with derivative larger than $-m$. Moreover, we can choose f so that $f(y)$ is bounded by a constant that is as small as we like. Choose a cut-off function $g(w)$ such that $g(w)$ is 1 for $w \in [0, 1 - \epsilon]$, for ϵ small, and $g(w)$ and all its derivatives are zero at 1. If we set

$$\tilde{a}_t(x, y, z) = a(x, y, z) + tg(x^2)g(z^2)f(y),$$

then one may easily check that $\tilde{\alpha}_t = dz - \tilde{a}_t(x, y, z) dx$ is the desired deformation of α . \square

We now know we can turn ξ' into a contact structure in neighborhoods of arcs tangent to ξ' . We can find a finite number $\gamma_1, \dots, \gamma_n$ of arcs tangent to ξ' that start in the contact region and so that each has a neighborhood N_i as above and the N_i 's cover $M \setminus H(\xi')$. We would now like to change ξ' into a contact structure in each of the N_i 's, but notice that when we do this for N_1 we have changed ξ' on some of the other N_i 's so the form of the 1-form representing ξ' on these other neighborhoods changes. So we might not be able to continue on the other N_i 's.

Exercise 3.12 Show that if the perturbation on N_1 is sufficiently small then we can slightly modify the N_i 's so that we still have the appropriate form to apply the lemma.

3.2. Part 1 of the proof of Theorem 3.1

Now we know that given a foliation ξ if we can perturb it to through confoliations to ξ' so that $G(\xi') = M$, that is every point in M can be connected to a “contact region” by a path tangent to ξ' , then we can move on to Part 2 and perturb ξ' into a contact structure. So we now want to figure out how to do this first perturbation. Specifically we want to see how to prove the following precise statement of Part (1) of the proof of Theorem 3.1.

Lemma 3.13. *Any C^2 -foliation ξ on an oriented 3-manifold M , other than the foliation ζ on $S^2 \times S^1$, can be C^0 -approximated by a confoliation ξ' such that $G(\xi') = M$.*

Note that in Part 2 of the proof we C^∞ -deformed ξ' into a contact structure. So it is in this part of the proof that we will see the loss of smoothness and the fact that we can only approximate ξ by a contact structure and not necessarily deform ξ into a contact structure.

The main tool we need to introduce contact regions into ξ is *holonomy*. Fix a foliation ξ on M and a closed oriented curve γ in M that is tangent to ξ . We can embed an annulus $A = (-\epsilon, \epsilon) \times S^1$ in M such that

- (1) $\{0\} \times S^1 = \gamma$ (as oriented curves),
- (2) A is transverse to ξ , and
- (3) $(-\epsilon, \epsilon) \times \{p\}$ is transverse to ξ for all $p \in S^1$.

Thus ξ induces a foliation on A , which we denote A_ξ . Clearly γ is a leaf in this foliation. If we restrict the projection map $\pi : A \rightarrow S^1$ to a leaf l of A_ξ then π will be a local diffeomorphism. Thus if we choose a point x on a l and let C be a path in S^1 based at $\pi(x)$ then we may try to lift C to a path in l . Just like when you prove path lifting for covering spaces, it is easy to see you can lift C as long as the lift does not “run off of A ”. For example if C just parameterizes γ , and we

lift C starting at a point $x \in \{0\} \times S^1$ then the lifted curve C just parameterizes $\{0\} \times S^1$. Similarly if x is a point very near $\{0\} \times S^1$ then C will lift to a closed leaf of A_ξ or an arc contained in a non-closed leaf. If we fix a point p in S^1 and set $I = (-\epsilon, \epsilon) \times \{p\}$ then we have just seen for x near 0 in I , we can lift C to a curve C_x tangent to a leaf of A_ξ . If x is close enough to 0 then the curve C_x will intersect I in either one point $\{x\}$ (if C_x parameterizes a closed leaf) or two points $\{x, y\}$. See Figure 7. So we can define a map

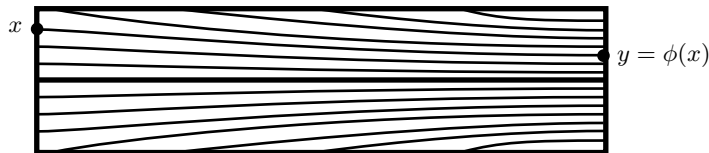


FIGURE 7. The foliation induced on the annulus A and the map ϕ . The heavier line in the center is γ .

$$\phi_\gamma: I' \rightarrow I$$

where I' is a sub-interval of I containing 0. This map sends $x \in I'$ to x if C_x is a closed curve and y otherwise. Note $\phi_\gamma(0) = 0$. This map is called the *holonomy along γ* . Actually, to precisely define the holonomy along γ takes more work, in particular we need to make ϕ_γ independent of A . While it is not so important for us here, let's consider some of the subtleties. First, if we change A a little then the map ϕ_γ may be conjugated by a diffeomorphism of I . Secondly, when we change A the interval I' on which the map ϕ_γ is well-defined might change. So it is really the conjugacy class of the germ of ϕ_γ that is the holonomy of ξ along γ . While these subtleties are important we will largely ignore them. The careful reader should make sure we don't miss anything by doing this. Also note that given a leaf l of ξ we get a map

$$\Phi: \pi_1(l) \rightarrow \{\text{germs of maps of } I \text{ to itself at } 0\},$$

by sending $\gamma \in \pi_1(l)$ to ϕ_γ . The (conjugacy class) of the image of Φ is called the holonomy group of l .

We will call the holonomy along a curve γ , *non-trivial* if $\phi_\gamma \neq id_I$ and *linearly non-trivial* if $\phi'_\gamma(0) \neq 1$. The holonomy will be said to be *attracting (repelling)* if $|\phi_\gamma(x)| < |x|$ ($|\phi_\gamma(x)| > |x|$) for all x , near 0. Finally, we say the holonomy is *sometimes attracting (repelling)* if $|\phi_\gamma(x)| < |x|$ ($|\phi_\gamma(x)| > |x|$) for a sequence of x approaching 0 from both sides (this is the same as saying for x on intervals arbitrarily close to, but not necessarily including, 0).

So why are we interested in holonomy? It helps us create contact regions!

Theorem 3.14. *Let (M, ξ) be a C^k -foliated manifold.*

- (1) *If γ is a curve tangent to ξ and has non-trivial linear holonomy, then there are neighborhoods N and N' of γ such that $N \subset N'$ and ξ can be C^k -deformed through confoliations so that it is a positive contact structure in N , unchanged outside of N' and diffeomorphic to ξ outside N .*
- (2) *If γ is a curve tangent to ξ and has sometimes attracting (repelling) holonomy, then there are neighborhoods N and N' of γ such that $N \subset N'$ and ξ can be C^0 -approximated by a confoliation so that it is a positive contact*

structure in N , unchanged outside of N' and diffeomorphic to ξ outside N .

PROOF. Part (1) of the theorem can be proved in two ways. We illustrate both ways, since the first is simpler and the second generalizes to provide a proof of part (2). For the first proof let $U = \gamma \times [-1, 1] \times [-1, 1]$ be a neighborhood of γ with coordinates (x, y, z) corresponding to the three factors in the product. From Lemmas 2.14 and 2.17 these coordinates can be chosen so that there is a 1-form α satisfying $\xi = \ker \alpha$ and

$$\alpha = dz - a(x, z) dx.$$

We in addition claim that, because γ has linear holonomy, the coordinates can be chosen so that

$$-\frac{\partial a}{\partial z} \geq C,$$

for some positive constant $C > 0$.

Exercise 3.15 Justify this claim.

Hint: Construct a model situation satisfying the inequality and then make sure this can be inserted into the given situation. More specifically, on $\gamma \times [-1, 1]$ (where we think of $[-1, 1]$ as the z -direction) try to abstractly construct an $a(z)$ with the desired property and inducing the same holonomy map as in the given situation. Then use the ideas in the proof of Lemma 2.14 to construct the neighborhood U above.

Now let $h: [0, 1] \rightarrow \mathbb{R}$ be a function so that $h(0) = 1, h(1) = 0$ and h is decreasing. Set

$$\beta = h(y^2 + z^2) dy.$$

Note that

$$\alpha \wedge d\beta + \beta \wedge d\alpha = (-a_z h + 2zah') dx \wedge dy \wedge dz.$$

Since a is strictly decreasing with respect to z and equal to 0 at $z = 0$ we know that $(-a_z h + 2zah') > 0$. Thus if $\alpha_\epsilon = \alpha + \epsilon\beta$ then one may easily check that α_ϵ is a contact form on $N = \gamma \times \{(y, z) | y^2 + z^2 < 1\}$ for $\epsilon > 0$ and $\alpha_\epsilon = \alpha$ outside of N . Thus we have completed the proof of part (1) of the theorem.

Now for the second proof of (1), that will generalize to a proof of (2). Again consider a neighborhood $U = \gamma \times [-1, 1] \times [-1, 1]$ as above. Let

$$A_{y_0} = \{(x, y, z) \in U | y = y_0\}.$$

Note each annulus A_{y_0} has a foliation induced on it by ξ . Moreover ξ at any point in U is spanned by a vector in the foliation on A_{y_0} and $\frac{\partial}{\partial y}$. Thus the foliations induced on the A_{y_0} 's essentially determine ξ . We will describe a way to change the foliations on the annuli A_{y_0} to get a contact structure.

The idea will be to construct a diffeomorphism $F: U \rightarrow U$ such that

- (1) F is the identity on ∂U (and C^k -tangent to the identity on ∂U),
- (2) F preserves the annuli A_{y_0} and
- (3) the slope of the foliation on A_{y_0} , for $y_0 \in (-1, 1)$, after pushing the foliation forward by the diffeomorphism is always less than the slope before the diffeomorphism.

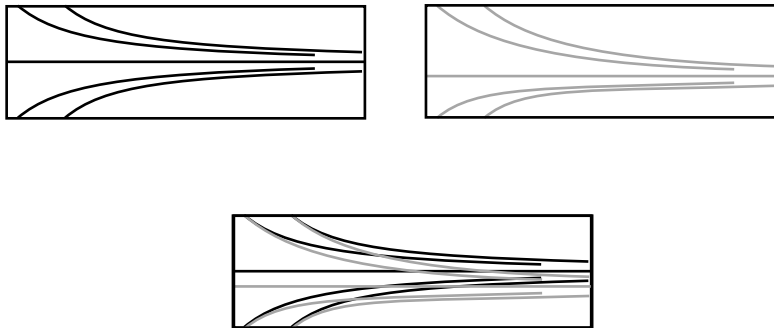


FIGURE 8. The foliation on A_y induced by ξ on the left and the foliation on A_y after applying the diffeomorphisms on the right. On the bottom are the two foliations superimposed on each other. Here one can see that the slope of the foliation at any point (away from the boundary) after the diffeomorphism is less than the slope of the original foliation.

See Figure 8. Now we have two foliations on U . The first is the one we started with ξ and the second is $\xi' = F_*(\xi)$. Note that both ξ and ξ' contain $\frac{\partial}{\partial y}$ so they are determined by the foliations induced on the A_y 's. Also note that $\xi = \xi'$ on ∂U . Thus we could replace $\xi|_U$ by ξ' and we would have another foliation on M . Instead we construct $\tilde{\xi}$ on U . Let U_{y_0} denote the set $\{(x, y, z) | y \leq y_0\}$. We define $\tilde{\xi}$ to be equal to ξ on $U_{-\frac{1}{2}}$ and equal to ξ' on $U \setminus U_{\frac{1}{2}}$. On the remaining region we will interpolate between these foliations. To define $\tilde{\xi}$ in this region fix a point (x_0, z_0) and consider the line segment $l_{(x_0, z_0)} = \{(x, y, z) | x = x_0, y \in [-\frac{1}{2}, \frac{1}{2}], z = z_0\}$. The slope $s_{-\frac{1}{2}}$ of $\tilde{\xi}$ on $A_{-\frac{1}{2}}$ at (x_0, z_0) is greater than the slope $s_{\frac{1}{2}}$ of ξ on $A_{\frac{1}{2}}$ at (x_0, z_0) . Thus we can define $\tilde{\xi}$ to be the plane field that is always tangent to $l_{(x_0, z_0)}$ and the slope of the intersection with $\tilde{\xi}$ with A_y uniformly decreases from $s_{-\frac{1}{2}}$ to $s_{\frac{1}{2}}$ as y goes from $-\frac{1}{2}$ to $\frac{1}{2}$. From Lemma 2.17 it should be clear that $\tilde{\xi}$ is a positive contact structure in $U_{\frac{1}{2}} \setminus U_{-\frac{1}{2}}$.

Exercise 3.16 Write down forms for ξ and ξ' . Using the properties of F listed above write down a form for $\tilde{\xi}$ and show that this is a contact form in the region $U_{\frac{1}{2}} \setminus U_{-\frac{1}{2}}$.

So we are left to construct the diffeomorphism F . To this end we need a lemma.

Lemma 3.17. *Let v_x be a family of smooth functions on $[-1, 1]$ such that $v_x(0) = 0$ and v_x are monotonically increasing for all x . Then there exists a diffeomorphism $f: [-1, 1] \rightarrow [-1, 1]$, C^∞ -close to the identity and C^∞ -tangent to the identity at $\{-1, 1\}$ and satisfying*

$$f'(z)v_x(z) > v_x(f(z)),$$

for all $z \in (-1, 1)$ and x .

Exercise 3.18 Prove this lemma.

Now in the case when γ has linear holonomy we can choose our coordinates as in the first proof of this theorem above. Thus if we set $v_x(z) = -a(x, z)$ then

v_x satisfies the conditions of the lemma and we thus get the function f from the lemma. Let $f_s, s \in [-1, -\frac{1}{2}]$ be a linear homotopy between f_{-1} the identity on $[-1, 1]$ and $f_{-\frac{1}{2}} = f$. If we choose this isotopy to be tangent to the identity at the end points we can extend it to be the constant isotopy for $s \in [-\frac{1}{2}, 1]$. Finally we can set $F(x, y, z) = (x, y, f_y(z))$.

Exercise 3.19 Check that F has the properties listed above.

Thus we have finished our second proof of the theorem in the case of linear holonomy along γ . For the case of sometimes attracting (repelling) holonomy we just need to see how to construct the diffeomorphism F . For this we need the following lemma.

Lemma 3.20. *Let v_x be a smooth family of C^1 -functions on $(-1, 1)$ such that $v_x(0) = 0$ and there is a sequence of points z_n decreasing to 0 and points z'_n increasing to 0 such that $v_x(z_n) > 0$ and $v_x(z'_n) < 0$ for all x and n . Then for any sufficiently small $\epsilon > 0$ there exists a diffeomorphism $f: (-1, 1) \rightarrow (-1, 1)$ which is fixed outside the interval $(-\epsilon, \epsilon)$ and satisfies*

$$f'(z)v_x(z) > v_x(f(z)),$$

for all $z \in (-\epsilon, \epsilon)$ and x .

Exercise 3.21 Prove this lemma.

Exercise 3.22 Find a suitable modification of the argument in the linear holonomy case to handle this more general case.

□

OK, so now we know that we can use holonomy to create regions of contact in a foliation. We are now ready to prove Lemma 3.13.

PROOF OF LEMMA 3.13. We need perturb ξ to a confoliation ξ' so that any point in M can be connected to a region where ξ' is contact by a path tangent to ξ' . From Theorem 3.14 this is the same as proving that we can perturb ξ into another foliation so that every leaf in the new foliation is arbitrarily close to a leaf with a curve having holonomy. To this end we study minimal sets. A *minimal set* in a foliation ξ is a non-empty closed union of leaves that contains no smaller such set.

Exercise 3.23 Show that a set is minimal if it is a non-empty closed union of leaves and every leaf in the set is dense in the set.

Exercise 3.24 Show that every leaf in ξ limits to some minimal set.

Thus we just need to see that we can perturb ξ so that every minimal set has a curve with holonomy. Minimal sets can be quite complicated for arbitrary foliations, but if we restrict to C^2 foliations or better we have the following classification of minimal sets.

Theorem 3.25. *In a C^2 -foliation every minimal set is either*

- (1) *all of M (in which case the foliation is called minimal),*
- (2) *a closed compact leaf, or*
- (3) *an exceptional minimal set.*

For more discussion of this theorem and exceptional minimal sets see [29]. For our purposes an exceptional minimal set will mean a minimal set that is not of type (1) or (2). With this definition the above theorem is not too hard to prove! However, there is a lot of structure to exceptional minimal sets. We will need

Theorem 3.26 (Sachsteder, 1965 [40]). *Exceptional minimal sets contain leaves with linear holonomy.*

Thus from our perspective all we need to know is that we can always find a curve with linear holonomy in an exceptional minimal set. We have just two other types of minimal sets to consider. If ξ is minimal, that is M is the only minimal set of ξ then there are two cases to consider. The first is when there is some holonomy in ξ . In this case a result of Ghys (see [12]) says that ξ has linear holonomy as well. Thus we are done in this case. If ξ has no holonomy then a theorem of Tischler [43] implies that ξ may be C^0 approximated by a fibration over the circle. That is M is a fibration over the circle and ξ is C^0 -close to the foliation of M by the fibers of this fibration. Note, by assumption, the fiber of this fibration cannot be S^2 . We claim we can now C^∞ -approximate this foliation by surfaces by a C^0 -foliation with only two closed leaves and linear holonomy along the closed leaves. We can then use the linear holonomy to create regions of contact and then finally perturb this confoliation into a contact structure, so the fact that at one point the foliation was only C^0 is irrelevant (since the contact condition is an open condition on 1-forms, we can always perturb a contact 1-form to be as smooth as we like). To perturb our fibration we can think of M as a mapping torus

$$\Sigma \times [0, 1] / \sim,$$

where $(0, x) \sim (1, f(x))$, for some diffeomorphism f of Σ . We will perturb the foliation tangent to the fibers of this fibration so that $\Sigma \times \{0\}$ and $\Sigma \times \{\frac{1}{2}\}$ are the only two closed leaves. To this end choose a separating curve γ in Σ and cut $\Sigma \times [0, \frac{1}{2}]$ open along $\gamma \times [0, \frac{1}{2}]$. We can now shear all the leaves in the foliation as shown in Figure 9 and reglue. If you would prefer a more rigorous description of this perturbation then consider $U = S^1 \times [-1, 1] \times [0, \frac{1}{2}]$ a neighborhood of $\gamma \times [0, \frac{1}{2}]$. The original foliation is by $S^1 \times [-1, 1] \times \{pt\}$'s. We can replace this foliation by $\ker \alpha$ where $\alpha = dz + g(y)h(z)dy$ and $h: [0, \frac{1}{2}] \rightarrow \mathbb{R}$ is 0 at 0 and $\frac{1}{2}$ and negative on $(0, \frac{1}{2})$ and $g: [-1, 1] \rightarrow \mathbb{R}$ vanishes to high order at -1 and 1 and is positive on $(-1, 1)$. Note we have not changed $\Sigma \times [0, \frac{1}{2}]$ but we have changed the foliation on it. In particular, if γ' is a closed curve in $\Sigma \times \{\frac{1}{2}\}$ that intersects γ in one point, then there will be one-sided linear holonomy along γ' . We can now do the same thing with $\gamma \times [\frac{1}{2}, 1]$ on $\Sigma \times [\frac{1}{2}, 1]$. After this note γ' in $\Sigma \times \{\frac{1}{2}\}$ now has linear holonomy. If the monodromy f of the surface bundle fixed γ then we would have a nice smooth foliation with holonomy along γ' in $\Sigma \times \{0\}$ too. In general γ will not be fixed by f this leads to the lack of smoothness in our foliation, but it is not too hard to find a curve in $\Sigma \times \{0\}$ that has linear holonomy.

Exercise 3.27 Find this curve.

Hint: Find a γ'' for $f(\gamma)$ that acts as γ' did for γ . Then add γ'' to γ' .

We note that with a great deal more care this idea can be significantly generalized to produce a C^2 -foliation by performing this shearing on infinitely many curves γ . See [12].

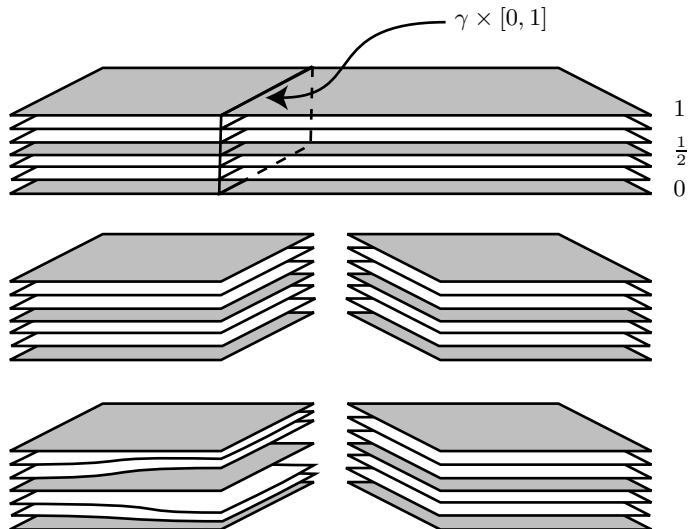


FIGURE 9. The top figure is a neighborhood of $\gamma \times [0, 1]$ in the foliation. In the middle figure the foliation has been cut open along $\gamma \times [0, 1]$. At the bottom we shear the leaves on one side of the cut open foliation. The gray leaves are $\Sigma \times \{0, \frac{1}{2}, 1\}$.

So we are done if ξ is a minimal foliation. If ξ is not minimal then we just need to worry about exceptional minimal sets and closed leaves. It turns out there are always finitely many exceptional minimal sets, but this does not have to be the case for closed leaves. However we do have the following theorem.

Theorem 3.28. *The foliation ξ can be C^0 -perturbed so that it has only finitely many closed leaves.*

The idea to prove this theorem is very similar to the idea above. If you have a bunch of closed leaves you can try to shear as we did above. For all the details see [12]. From now on we assume there are only finitely many closed leaves.

If our foliation has a closed leaf Σ there are three cases to consider (1) no holonomy, (2) linear or weakly attracting/repelling holonomy and (3) holonomy but not weakly attracting/repelling. In case (1) we have the Reeb stability theorem [29] that says there is a neighborhood of the closed leaf foliated by closed leaves. This contradicts the fact that we arranged to have only finitely many closed leaves! Thus case (1) cannot happen. Case (2) is the good case where we can introduce regions of contact. In case (3) we know we have holonomy, but it is not weakly attracting/repelling. If no curve on Σ has holonomy on one side then a version of Reeb stability implies that Σ has a one sided neighborhood that is foliated by closed surfaces. Since we are assuming only finitely many surfaces this cannot happen. Thus we can assume there is a curve γ on the leaf such that the holonomy is weakly attracting on one side and weakly repelling on the other side. In this case we can split the manifold open along the leaf, glue in a product neighborhood $\Sigma \times [-\epsilon, \epsilon]$ and extend the foliation over this product just to be $\Sigma \times \{pt\}$. Now use the shearing trick above to make sure that $\Sigma \times \{-\epsilon\}$ has weakly attracting holonomy along γ and $\Sigma \times \{\epsilon\}$ has weakly repelling holonomy. See Figure 10.

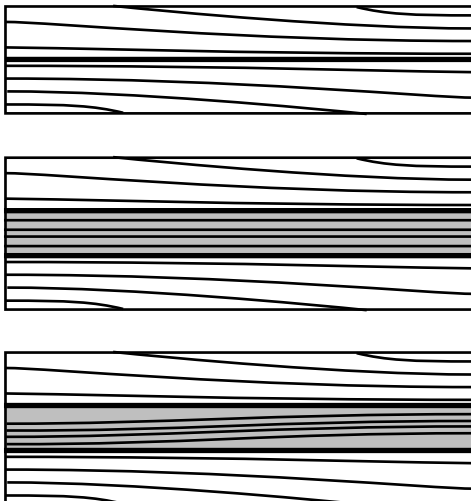


FIGURE 10. The top figure is a transverse annulus about a curve γ whose holonomy is attracting on one side and repelling on the other side. (The curve γ is the thicker line in the center.) In the middle figure is the foliation on the transverse annulus after the foliation has been cut open and the trivial foliation on $\Sigma \times [-\epsilon, \epsilon]$ is added. The bottom figure shows the final foliation after shearing has been done to create two copies of γ , one with attracting holonomy and the other with repelling holonomy.

Putting this all together we see that after, possibly, perturbing our foliation we can assume there are finitely many exceptional minimal sets and finitely many closed leaves and all these minimal sets contain leaves with linear or weakly attracting/repelling holonomy. Thus we can use Theorem 3.14 to perturb the foliation into the desired confoliation. \square

4. Taut foliations and symplectic fillings

From Theorem 3.1 we know that we can perturb a foliation ξ into contact structures ξ' (assuming of course that the foliation is not the trivial one on $S^1 \times S^2$), but what can we say about ξ' ? Is it tight? Is it fillable? Is it overtwisted?

To answer this recall that Thurston showed that if a foliation ξ on M is Reebless then for all surfaces $\Sigma \neq S^2$ embedded in M we have

$$|\langle e(\xi), [\Sigma] \rangle| \leq -\chi(\Sigma).$$

Thus if ξ' is a contact structure C^0 -approximating ξ then $e(\xi')$ will satisfy the same inequality. This might lead one to believe that ξ is tight, but it does not constitute a proof that ξ' is tight. If we could arrange that $e(\xi')$ of the contact structure satisfied the inequality in Theorem 2.32 involving transverse knots this would be sufficient to conclude that ξ' is tight. However, curves transverse to ξ are different from curves transverse to ξ' so we cannot conclude that $e(\xi')$ satisfies this inequality just because $e(\xi)$ does. None the less our intuition on this matter is indeed correct.

Theorem 4.1. *If ξ' is a positive contact structure C^0 -close to a Reebless foliation, then ξ' is tight. Moreover, ξ' is still tight when pulled back to the universal cover of M .*

This result was originally stated in Eliashberg and Thurston's book [12], but an error in the proof was discovered in [5] which was corrected in [18].

It is interesting to note that there are many foliation that have Reeb components but still perturb to a tight contact structure. For example, recall that S^3 can be thought of as the union of two solid tori. We can foliate each of these tori with Reeb foliations. Topologically there are two ways to do this (depending on the direction of the spiraling of the leaves on both sides of the unique torus leaf). It turns out that one of these Reeb foliations on S^3 perturbs to an overtwisted contact structure. The other Reeb foliation of S^3 perturbs to a tight contact structure.

Exercise 4.2 Determine which foliation perturbs to an overtwisted contact structure. You might try to prove the other foliation perturbs to a tight contact structure, but this is much harder.

It is very common for contact structures to be perturbations, even deformations, of foliations with Reeb components. In particular the following is known.

Theorem 4.3 (Etnyre 2006, [17]). *Every positive and negative contact structure on a closed oriented 3-manifold is a C^∞ -deformation of a C^∞ -foliation. Moreover, the foliation has Reeb components.*

It would be very interesting to determine which contact structures were perturbations of Reebless foliations or taut (see below) foliations.

A foliation ξ on M is called *taut* if each leaf of ξ intersects a closed transversal curve. Equivalently, a foliation is taut if there is a vector field ξ that is transverse to ξ and that preserves a volume form Ω on M . From the first definition it is easy to see that a taut foliation is Reebless since the torus leaf in a Reeb component separates M and a transverse curve that passes through this leaf cannot close up (it's stuck on one side of the surface). More generally, a taut foliation cannot contain a compact separating leaf. With this observation it is not hard to find Reebless foliations that are not taut. So tautness is a strictly stronger notion than Reebless.

Exercise 4.4 Try to prove the two definitions of taut are equivalent.

Hint: Poincaré recurrence is useful to prove the second definition implies the first.

Theorem 4.5 (Eliashberg-Thurston 1998, [12]). *If ξ' is a positive contact structure that is C^0 -close to a taut foliation ξ then ξ' is weakly symplectically semi-fillable.*

PROOF. Let $X = M \times [-1, 1]$. Use t as the coordinate on $[-1, 1]$. Let α be a 1-form such that $\xi = \ker \alpha$ and set $\tilde{\omega} = \iota_v \Omega$ where v and Ω are the vector field and volume form from the definition of taut. Note that $\tilde{\omega}|_\xi > 0$ and

$$d\tilde{\omega} = d\iota_v \Omega = \iota_v d\Omega + \mathcal{L}_v \Omega = 0$$

(where \mathcal{L} is the Lie derivative). Thus $\omega = \tilde{\omega} + \epsilon d(t\alpha)$ is a symplectic form on X .

Exercise 4.6 Show that $\omega \wedge \omega > 0$ on X .

Note also that $\omega|_{\xi \times \{\pm 1\}} > 0$. Thus if ξ' is C^0 -close to ξ then $\omega|_{\xi' \times \{1\}} > 0$. Let ξ'' be a negative contact structure on M that is C^0 close to ξ . Clearly $\omega|_{\xi'' \times \{-1\}} > 0$ so (X, ω) weakly symplectically fills $(M, \xi') \amalg (-M, \xi'')$. \square

We can now construct lots of tight contact structures using a theorem of Gabai.

Theorem 4.7 (Gabai 1983, [22]). *Let M be an irreducible 3-manifold and Σ an oriented surface realizing a non-trivial homology class in M and of minimal genus among representatives of its homology class. Then there is a taut foliation ξ on M with Σ as a leaf.*

Corollary 4.8. *With M and Σ as above, there is a fillable contact structure ξ' on M such that $\langle e(\xi'), [\Sigma] \rangle = \pm\chi(\Sigma)$.*

Actually, one needs to be a little careful proving this corollary. The foliation from Gabai's theorem is C^2 if the genus of Σ is larger than 1, but when the genus is 1 the foliation is not C^2 . However the only part of the foliation that is not C^2 is along the surface Σ and there is holonomy along Σ . Thus looking back at the proof of Theorem 3.1 we see we can still perturb this foliation into a contact structure.

5. Symplectic handle attachment and Legendrian surgery

We now want to discuss how to build symplectic manifolds. We will do this by starting with simple symplectic pieces and “gluing” them together. In order to accomplish this gluing we need a stronger notion of symplectic filling. To this end, let (X, ω) be a symplectic manifold, then we call a vector field v *symplectically dilating* if

$$\mathcal{L}_v\omega = \omega.$$

Suppose v is transverse to $M = \partial X$ and is pointing out of X along M , then set $\alpha = (\iota_v\omega)|_M$. We compute

$$d\alpha = d\iota_v\omega = d\iota_v\omega + \iota_v d\omega = \mathcal{L}_v\omega = \omega,$$

so

$$\alpha \wedge d\alpha = (\iota_v\omega) \wedge \omega = \frac{1}{2}\iota_v(\omega \wedge \omega).$$

Since $\omega \wedge \omega$ is a volume form on X it is clear that $\alpha \wedge d\alpha$ is a volume form on M . In other words, α is a contact form on M . We say a contact manifold (M, ξ) is *strongly filled* by a compact symplectic manifold (X, ω) if $\partial X = M$ and there is a dilating vector field v for ω defined near the boundary of X that is transversely pointing out of the boundary of X and such that $\iota_v\omega$ is a contact form for ξ . We also say that (X, ω) is a *strong convex filling* of (M, ξ) . If the vector field v points into X then we say (X, ω) is a *strong concave filling* of (M, ξ) .

Exercise 5.1 Prove that a strong symplectic filling of a contact manifold (M, ξ) is also a weak symplectic filling.

It is not true that a weak filling of a contact manifold is a strong filling, but we can sometimes create a strong filling from a weak filling.

Theorem 5.2. *If M is a homology sphere and (X, ω) is a weak symplectic filling of a contact structure ξ on M then ω can be altered to ω' so that (X, ω') is a strong symplectic filling of (M, ξ) .*

Exercise 5.3 Try to prove this theorem.

The key reason we have brought up the notion of strong fillings is to “glue” symplectic manifolds together. Specifically we have the following theorem.

Theorem 5.4. *If (X_1, ω_1) is a strong symplectic filling of (M, ξ) and (X_2, ω_2) is a strong concave filling of (M, ξ) then $X = X_1 \cup X_2$ has a symplectic structure ω such that $\omega|_{X_1} = \omega_1$ and $\omega|_{X_2 \setminus N} = c\omega_2$ where N is a neighborhood of ∂X_2 in X_2 and $c > 0$ is a constant. See Figure 11.*

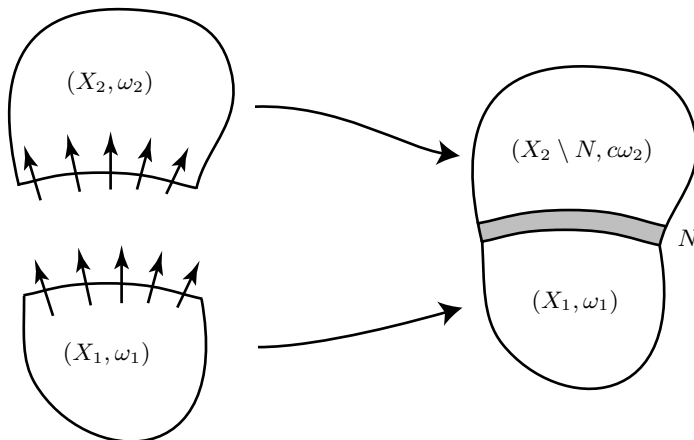


FIGURE 11. Gluing two symplectic manifolds.

Thus, after possibly rescaling the symplectic form on one of the pieces, a symplectic manifold with strongly convex boundary and one with strongly concave boundary can be glued together to get a symplectic manifold if the contact structures induced on their boundaries are the same. Such a gluing result is definitely not true for weak concavity/covexity. For more on this and all the various forms of convexity see [14].

We will use this gluing theorem repeatedly. Our first application of it will be to symplectic handle attachment. To this end we recall a little 4-dimensional topology. Suppose we are given a 4-manifold X with boundary. Then a 1-handle is $h^1 = D^1 \times D^3$ (where D^n is the unit disk in \mathbb{R}^n) and when we attach a 1-handle to X we glue h^1 to X along $A^1 = (\partial D^1) \times D^3 = S^0 \times D^3 = \{2 \text{ points}\} \times D^3$. The set A^1 is called the attaching region of h^1 . If we identify two disjoint 3-balls in ∂X there will be a unique way to glue h^1 to ∂X so that A goes to these two 3-balls. Since we cannot draw this picture we illustrate it one dimension lower in Figure 12.

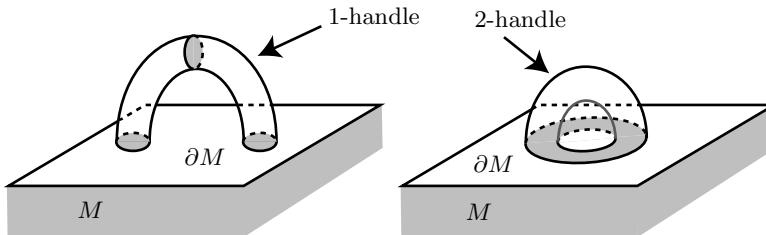


FIGURE 12. A 3-dimensional 1-handle, left, and 2-handle, right.

A 2-handle is $h^2 = D^2 \times D^2$ and when we attach a 2-handle to D we glue h^2 to ∂X along $A^2 = (\partial D^2) \times D^2 = S^1 \times D^2$. Thus to attach a 2-handle we must identify a knot K in ∂X , this is where $S^1 \times \{pt\}$ will be glued, and we must also fix a framing on K so that we will know how to glue A^2 to a neighborhood of K . Recall, there is an integers worth of trivializations of $D^2 \times S^1$ and any such trivialization is called a framing of the core of the solid torus.

Exercise 5.5 Show that an isotopy class of curves in $\partial(D^2 \times S^1)$ that represents a generator in $\pi_1(D^2 \times S^1)$ determines and is determined by a framing.

Again we illustrate, in Figure 12, a 2-handle attachment in dimension three (here there is a unique way to attach a 2-handle once a curve in the boundary 3-manifold is specified).

Suppose X' is X with a 2-handle attached along a knot K with framing \mathcal{F} . Then

$$\partial X' = [\partial X \setminus (S^1 \times D^2)] \cup (D^2 \times S^1)$$

where the $S^1 \times D^2$ is a neighborhood of K . It is removed from ∂X because when the handle is attached it becomes part of the interior of X' . Thus $\partial X'$ is obtained from ∂X by removing a neighborhood of a knot K and replacing it with another solid torus. This is clearly just a Dehn surgery on ∂X . (If you have not studied Dehn surgery, then take this for the definition.)

Exercise 5.6 Check that this Dehn surgery is, in fact, a surgery with framing \mathcal{F} . By this we mean that the new solid torus is glued in so that $\partial D^2 \times \{pt\}$ is glued to a curve determined by \mathcal{F} .

Now to relate this to contact geometry we say a knot K in a contact manifold (M, ξ) is *Legendrian* if K is always tangent to ξ :

$$T_x K \subset \xi_x, \quad x \in K.$$

Note that since K is tangent to ξ , the contact structure defines a trivialization of the normal bundle of K . That is, there is a natural *contact framing* on a Legendrian knot.

Exercise 5.7 Convince yourself of this.

Hint: Take a vector field in ξ along K and use it to push off a copy of K .

We can now discuss symplectic handle attachment.

Theorem 5.8 (Eliashberg, 1990 [9] and Weinstein, 1991 [45]). *If (X, ω) is a symplectic manifold with strongly/weakly convex boundary and X' is obtained from X by attaching a 1-handle to X or attaching a 2-handle to X along a Legendrian knot in ∂X with framing one less than the contact framing, then ω extends to a symplectic form ω' on X' in such a way that $\partial X'$ has a strongly/weakly convex boundary.*

If X' is obtained from X by a symplectic 2-handle attachment along a Legendrian knot K as in the theorem, then we say that the contact manifold $\partial X'$ is obtained from ∂X by *Legendrian surgery* along K .

SKETCHY PROOF OF THE THEOREM. We consider 1-handle attachment. The basic idea is to consider the model 1-handle in \mathbb{C}^2 illustrated in Figure 13. Here we give \mathbb{C}^2 coordinates $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$. In this model we can construct

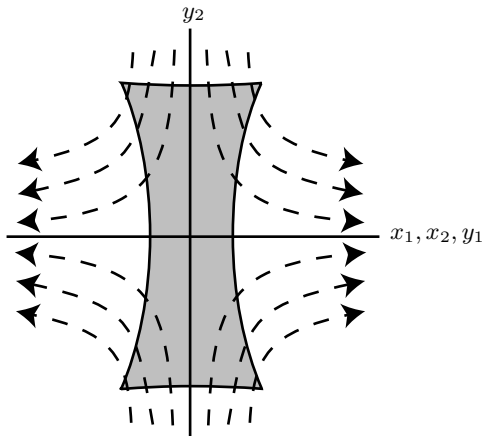


FIGURE 13. A model symplectic 1-handle.

an expanding vector field v' that is transversely pointing into the handle along the attaching region A^1 and transversely pointing out of the handle along the other boundary component. Now if v is the vector field implicated in the definition of the convexity of ∂X then we can use v' and v to glue the standard symplectic structure on \mathbb{C}^2 to ω when we attach h^1 to X . You can do this by carrying out the following exercises.

Exercise 5.9 Show that the contact structures induced on the attaching region $A^1 \subset h^1$ and on the two 3-balls to which A^1 is glued are the same. Moreover, show that we can choose our gluing map so that the contact forms are the same.

Exercise 5.10 Use the previous exercise and the flow of v and v' to show that a neighborhood of A^1 in $\mathbb{C}^2 \setminus h^1$ is symplectomorphic to a neighborhood of the two 3-balls in X . Use this symplectomorphism to glue h^1 to X and extend ω over the handle.

Gluing a 2-handle is a little more tricky, but the main idea is the same.

Exercise 5.11 Try to extend ω over a 2-handle attached as in the statement of the theorem. Or at least try to figure out why you have the framing condition in the theorem.

The observant reader will have noticed that we seem to be assuming that X has strongly convex boundary, but the theorem works for weakly convex boundary as well. This follows from (a relative version of) Theorem 5.2. \square

6. Open book decompositions and symplectic caps

Our main goal in this section is to sketch the proof of the following theorem.

Theorem 6.1 (Eliashberg and Etnyre, 2004 [11, 16]). *If (X, ω) is a compact symplectic manifold with weakly convex boundary then there is a closed symplectic manifold (X', ω') into which (X, ω) symplectically embeds.*

Recall from the Introduction that this completes the contact geometric input into the various advance in low-dimensional topology discussed there. We will only sketch the ideas in the proof of this theorem here, for a more complete discussion see the original papers [11, 16] or the survey article [13].

The last ingredient we need to prove this theorem is open book decompositions. Let Σ be a compact oriented surface with boundary and $\phi : \Sigma \rightarrow \Sigma$ an orientation preserving diffeomorphism of Σ that is equal to the identity near $\partial\Sigma$. The *mapping torus* of ϕ is

$$T_\phi = \Sigma \times [0, 1] / \sim,$$

where $(x, 1) \sim (\phi(x), 0)$. For each boundary component of Σ we have a boundary component of T_ϕ . This boundary component is a torus, with two canonical circles $\mu = \{pt\} \times [0, 1] / \sim$ and $\lambda = \partial\Sigma \times \{pt\}$. Let

$$M_{(\Sigma, \phi)} = \left(T_\phi \cup \coprod_{|\partial\Sigma|} S^1 \times D^2 \right) / \sim,$$

where \sim glues a solid torus $S^1 \times D^2$ to T_ϕ so that $\{pt\} \times \partial D^2$ is glued to μ in ∂T_ϕ and $S^1 \times \{pt\}$ is glued to λ .

Exercise 6.2 If L is the union of the cores of all the $S^1 \times D^2$'s in $M_{(\Sigma, \phi)}$ then show that $M \setminus L$ fibers over the circle with fiber diffeomorphic to Σ . We call L the *binding* of the open book and Σ the *page*.

An *open book decomposition*, or just open book, of a closed oriented 3-manifold M is an identification of M with $M_{(\Sigma, \phi)}$ for some (Σ, ϕ) as above. This is not the best definition of open book as it only defines an open book up to diffeomorphism, but it will suffice for our purposes. For a better definition see [13].

Fact 6.3. *All closed oriented 3-manifolds have open book decompositions*

Exercise 6.4 Prove this fact!

Hint: You might find it useful to recall that all 3-manifold are branched covers of S^3 branched over some link and that links can be braided about the unknot.

An open book (Σ, ϕ) for M is said to *support* or be *compatible with* a contact structure ξ on M if there is a 1-form α such that $\xi = \ker \alpha$, $\alpha(v) > 0$ for any $v \in TL$ that agrees with the orientation on the binding L , and $d\alpha|_{page} \neq 0$ and induces the correct orientation on the page. Thurston and Winkelnkemper [41] have shown that every open book supports a contact structure and Giroux [26] observed this contact structure is unique.

Given an open book (Σ, ϕ) supporting a contact structure ξ on M then the *positive stabilization* of (Σ, ϕ) is the open book (Σ', ϕ') with

$$\Sigma' = \Sigma \cup (1\text{-handle})$$

and $\phi' = \phi \circ D_\gamma$ where D_γ is a right handed Dehn twist along a curve γ that runs over the new 1-handle in Σ' exactly once.

Exercise 6.5 Show that (Σ', ϕ') is still an open book for M and still supports ξ .

Theorem 6.6 (Giroux 2002, [26]). *There is a one-to-one correspondence between*
 $\{\text{oriented contact structures on } M \text{ up to isotopy}\}$

and

$\{\text{open book decompositions of } M \text{ up to positive stabilization}\}.$

To use this theorem to prove Theorem 6.1 we need to see how Legendrian surgery interacts with open books. To this end suppose (Σ, ϕ) supports ξ on M . We begin by forgetting about the contact structure ξ and concentrating on M . Let γ be a simple closed curve contained in a page of the open book. Note that γ gets a framing \mathcal{F} from the page. Let M' be the manifold obtained from M by $\mathcal{F} \pm 1$ surgery on γ .

Exercise 6.7 Show that an open book for M' is $(\Sigma, \phi \circ D_\gamma^\mp)$.

Hint: If you cut $M \setminus L$ open along the page containing γ and reglue by D_γ^\pm then the resulting manifold will differ from M in a neighborhood of γ . The key now is to see that this difference is a Dehn surgery with the appropriate framing.

Fact 6.8 (Legendrian realization principle [30]). *If γ is a non-separating curve on a page of the open book then we can isotop the open book slightly so that γ is Legendrian and the contact framing agrees with the page framing.*

Actually this is not exactly the Legendrian realization principle, but this fact easily follows from it.

Fact 6.9. *Let γ be a Legendrian knot in a page of the open book (Σ, ϕ) supporting the contact manifold (M, ξ) . If (M', ξ') is obtained from (M, ξ) by Legendrian surgery on γ , then (M', ξ') is supported by $(\Sigma, \phi \circ D_\gamma)$.*

So, summarizing, given a symplectic filling (X, ω) of (M, ξ) and a non-separating curve γ on a page of an open book (Σ, ϕ) supporting ξ then we can attach a symplectic 2-handle to (X, ω) , as in Theorem 5.8, to get (X', ω') and $\partial X' = M'$ and ξ' is filled by ω' where a supporting open book for ξ' is $(\Sigma, \phi \circ D_\gamma)$.

Now we need some facts about the mapping class group of a surface.

Fact 6.10. *If Σ is a surface with one boundary component then any diffeomorphism of Σ that is fixed on the boundary can be written as*

$$\phi = D_c^m \circ D_{\gamma_1}^{-1} \circ \dots \circ D_{\gamma_n}^{-1},$$

where the γ_i are separating curves on Σ and c is a curve parallel to $\partial\Sigma$.

We are now ready to begin our sketch of the proof of Theorem 6.1. We start with a weak symplectic filling (X, ω) of the contact manifold (M, ξ) . Let (Σ, ϕ) be an open book for M supporting ξ . By positively stabilizing if necessary we can assume that Σ has only one boundary component. We can further assume

$$\phi = D_c^m \circ D_{\gamma_1}^{-1} \circ \dots \circ D_{\gamma_n}^{-1},$$

as in Fact 6.10 above.

We would now like to simplify the monodromy of this open book. Using Theorem 5.8 we can attach 2-handles to X along Legendrian knots in the pages of the open book for (M, ξ) and extend the symplectic structure of X over the 2-handles. Then since the upper boundary of the new 4-manifold is obtained from (M, ξ) by Legendrian surgery Fact 6.9 tells us that the monodromy of this new boundary will have an extra right handed Dehn twists along each of the attaching curves of the 2-handles. Thus if we can attach n symplectic 2-handles to X along the γ_i 's to get

a symplectic manifold (X', ω') with $\partial(X', \omega') = (M', \xi')$ and ξ' supported by the open book (Σ, ϕ') where

$$\phi' = D_c^m.$$

Now attach $2g$ more symplectic 2-handles to get (X'', ω'') having contact boundary (M'', ξ'') supported by the open book (Σ, ϕ'') where

$$\phi'' = D_c^m \circ D_{\gamma_1} \circ \dots \circ D_{\gamma_{2g}},$$

where the curves γ_i are shown in Figure 14.

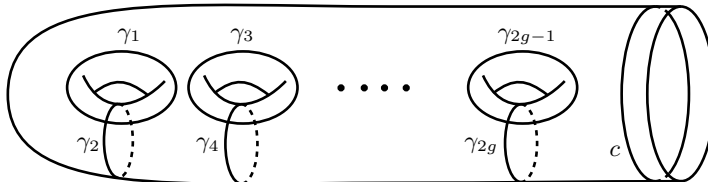


FIGURE 14. The curves γ_i and c on Σ .

Exercise 6.11 Show that if Σ has genus g then M'' is the manifold depicted in Figure 15. In particular, M'' is a homology 3-sphere.

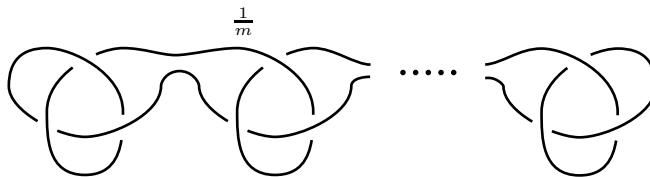


FIGURE 15. Topological description of M'' .

Thus according to Theorem 5.2 we can slightly perturb ω'' so that (X'', ω'') is a strong symplectic filling of (M'', ξ'') . Also note that since (X'', ω'') was built by attaching symplectic 2-handles to (X, ω) , we can symplectically embed (X, ω) into (X'', ω'') . We can now hope to glue a concave filling of (M'', ξ'') to (X'', ω'') to get a closed symplectic manifold. Before we do this, we will attach a few more symplectic 2-handles to (X'', ω'') to further normalize its boundary.

Recall if $\gamma_1, \dots, \gamma_{2g}$ is a collection of simple closed curves in Σ that satisfy $\gamma_i \cdot \gamma_j$ is 1 if $|i - j| = 1$ and is 0 otherwise, where \cdot means geometric intersection, then Σ is a neighborhood of the union of the γ_i 's. Moreover we have the so called chain relation in the mapping class group of Σ :

$$(D_{\gamma_1} \circ \dots \circ D_{\gamma_{2g}})^{4g+2} = D_c.$$

Using the chain relation it is easy to see that we can attach $8g^2 + 3g$ more symplectic 2-handles to (X'', ω'') to obtain $(X^{(3)}, \omega^{(3)})$ which strongly fills $(M^{(3)}, \xi^{(3)})$ where $\xi^{(3)}$ is supported by $(\Sigma, \phi^{(3)})$ where

$$\phi^{(3)} = D_c^k,$$

for some k .

Finally, we can stabilize the open book $(\Sigma, \phi^{(3)})$ for $(M^{(3)}, \xi^{(3)})$ so that we can attach more symplectic 2-handles to $(X^{(3)}, \omega^{(3)})$ to get a symplectic manifold $(X^{(4)}, \omega^{(4)})$ strongly symplectically filling $(M^{(4)}, \xi^{(4)})$ where $\xi^{(4)}$ is supported by $(\Sigma', \phi^{(4)})$ where

$$\phi^{(4)} = D_{c'},$$

and c' is again a curve parallel to the boundary of Σ' .

Exercise 6.12 Prove this last assertion.

Exercise 6.13 Show that $M^{(4)}$ is an S^1 -bundle over Σ'' with Euler number -1 where Σ'' is the closed surface obtained from Σ' by attaching a disk to $\partial\Sigma'$.

Exercise 6.14 If Y is the D^2 bundle over Σ'' with Euler number 1, then Y admits a symplectic structure with concave boundary and $\partial Y = -M^{(4)}$.

Hint: Think about a connection on the circle bundle ∂Y .

Exercise 6.15 Show the contact structure induced on ∂Y from the strong concave filling is contactomorphic to $\xi^{(4)}$.

Hint: Both contact structures are transverse to the fibers of the S^1 -bundle.

Now we can use Theorem 5.8 to glue Y with the above constructed symplectic structure to $(X^{(4)}, \omega^{(4)})$ to get a closed symplectic manifold into which (X, ω) embeds, thus completing the proof of Theorem 6.1.

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