NON-ISOTOPIC LEGENDRIAN SUBMANIFOLDS IN R^{2n+1}

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Abstract. The contact homology, rigorously defined in [7], is computed for a number of Legendrian submanifolds in standard contact $(2n+1)$-space. The homology is used to detect infinite families of pairwise non-isotopic Legendrian n-spheres, n-tori, and surfaces which are indistinguishable using previously known invariants.

Contents

1. Introduction

A contact manifold is a $(2n+1)$-manifold N equipped with a completely non-integrable field of hyperplanes ξ. An immersion of an n-manifold into N is *Legendrian* if it is everywhere tangent to the hyperplane field ξ and the image of a Legendrian embedding is a *Legendrian submanifold*. *Standard contact* $(2n+1)$*-space* is Euclidean space $\mathbb{R}^{2n+1}$ equipped with the hyperplane field $\xi = \mathrm{Ker}(\alpha)$, where α is the contact 1-form $\alpha = dz - \sum_{i=1}^{n} y_i\, dx_i$ in Euclidean coordinates $(x_1, y_1, \dots, x_n, y_n, z)$.

Any closed n-manifold M embeds in $\mathbb{R}^{2n+1}$, and it is a consequence of the h-principle for Legendrian immersions [20] that, provided M meets certain homotopy theoretic conditions (which is the case e.g. if M is stably parallelizable), any embedding of M into $\mathbb{R}^{2n+1}$ may be arbitrarily well C^0-approximated by Legendrian embeddings. Thus, Legendrian submanifolds of standard contact $(2n+1)$-space exist in abundance.

Any contact manifold of dimension $2n+1$ is locally contactomorphic (diffeomorphic through a map which takes contact hyperplanes to contact hyperplanes) to standard contact $(2n+1)$-space. In this paper we study local Legendrian knotting phenomena or, in other words, the question: *When are are two Legendrian submanifolds of standard contact* $(2n+1)$*-space isotopic through Legendrian submanifolds?*

For $n = 1$, the question above has been extensively studied, [5, 9, 13, 14, 15]. Here, the *classical invariants* of a Legendrian knot are its topological knot type, its rotation number (the tangential degree of the curve which arises as the projection of the knot into the xy-plane), and its Thurston-Bennequin invariant (the linking number of the knot and a copy of the knot shifted slightly in the z-direction). Many examples of Legendrian non-isotopic knots with the same classical invariants are known. Also, in higher dimensions, when the ambient contact manifold has more topology (for example Legendrian knots in 1-jet spaces of S^n) there are interesting examples of non-trivial Legendrian knots [10].

When $n > 1$, we define in Section 3 two *classical invariants* of an oriented Legendrian submanifold given by an embedding $f\colon L \to \mathbb{R}^{2n+1}$. Following [32], we first define its Thurston-Bennequin invariant (in the same way as in $\mathbb{R}^3$). Second, we note that the h-principle for Legendrian immersions implies that f is determined by certain homotopy theoretic invariants, associated to its differential df, up to regular homotopy through Legendrian immersions, see Section 3.3 for details. We define its *rotation class* as its Legendrian regular homotopy class. The topological embedding invariant in the 3-dimensional case disappears in higher dimensions since, for $n \geq 2$, any two embeddings of an n-manifold into $\mathbb{R}^{2n+1}$ are isotopic [21].

Our results indicate that the theory of Legendrian submanifolds of standard contact $(2n+1)$-space is very rich. For example we show, generalizing the 3-dimensional results mentioned above,

Theorem 1.1. *For any $n > 1$ there is an infinite family of Legendrian embeddings of the n-sphere into $\mathbb{R}^{2n+1}$ that are not Legendrian isotopic even though they have the same classical invariants.*

In Section 4, we prove Theorem 1.1 and a similar theorem for Legendrian surfaces and n-tori. We show that for any $N > 0$ there exists Legendrian isotopy classes of n-spheres and n-tori with fixed Thurston-Bennequin invariants and rotation classes which do not admit a representative having projection into $\mathbb{R}^{2n}$ with less than N double points.

Note that Theorem 1.1 is not known to be true for $n = 1$ and is probably false in this case. It is known that the number of distinct Legendrian knots with the same classical invariants can be arbitrarily large, but in light of recent work of Colin, Giroux and Honda [6] it seems unlikely that there can be infinitely many.

To show that Legendrian submanifolds are not Legendrian isotopic we develop the *contact homology* of a Legendrian submanifold in standard contact $(2n+1)$-space. This theory is sketched in Section 2, with more analytic details appearing in [7]. It is defined using punctured holomorphic disks in $\mathbb{C}^n \approx \mathbb{R}^{2n}$ with boundary on the projection of the Legendrian submanifold, and which limit to double points of the projection at the punctures, see below. This is analogous to the approach taken by Chekanov [5] in dimension 3; however, in that dimension the entire theory can be reduced to combinatorics [15]. Our contact homology, realizes in the language of Symplectic Field Theory [12], Relative Contact Homology of standard contact $(2n+1)$-space and in this framework our main technical theorem can be summarized as

Theorem 1.2. *The contact homology of Legendrian submanifolds in $\mathbb{R}^{2n+1}$ with the standard contact form is well defined. (It is invariant under Legendrian isotopy.)*

More concretely, if $L \subset \mathbb{R}^{2n+1}$ is a Legendrian submanifold we associate to L a differential graded algebra $(\mathcal{A}, \partial)$, freely generated by the double points of the projection of L into $\mathbb{C}^n$. The differential ∂ is defined by counting rigid holomorphic disks with properties as described above. Thus, contact homology is similar to Floer homology of Lagrangian intersections and our proof of its invariance is similar in spirit to Floer's original approach [16, 17] in the following way. We analyze bifurcations of moduli spaces of rigid holomorphic disks under variations of the Legendrian submanifold in a generic 1-parameter family of Legendrian submanifolds and how these bifurcations affect the differential graded algebra. Similar bifurcation analysis is also done in [22, 24, 30, 31]. Our set-up does not seem well suited to the more popular proof of Floer theory invariance which uses an elegant "homotopy of homotopies" argument (see, for example, [18, 29]).

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2. Contact Homology and Differential Graded Algebras

In this section we describe how to associate to a Legendrian submanifold L in standard contact $(2n+1)$-space a differential graded algebra (DGA) $(\mathcal{A}, \partial)$. Up to a certain equivalence relation this DGA is an invariant of the Legendrian isotopy class of L. In Section 2.1 we recall the notion of Lagrangian projection and define the algebra $\mathcal{A}$. The grading on $\mathcal{A}$ is described in Section 2.3 after a review of the Maslov index in Section 2.2.

Sections 2.4 and 2.5 are devoted to the definition of ∂ and Section 2.6 proves the invariance of the homology of $(\mathcal{A}, \partial)$, which we call the contact homology. The main proofs of these three subsections rely on several analytical results which we prove in [7]. Finally, in Section 2.7, we compare contact homology as defined here with the contact homology sketched in [12].

2.1. The algebra $\mathcal{A}$.

Throughout this paper we consider the standard contact structure ξ on $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$ which is the hyperplane field given as the kernel of the contact 1-form

$$\alpha = dz - \sum_{j=1}^{n} y_j dx_j, \tag{2.1}$$

where $x_1, y_1, \dots, x_n, y_n, z$ are Euclidean coordinates on $\mathbb{R}^{2n+1}$. A *Legendrian submanifold* of $\mathbb{R}^{2n+1}$ is an n dimensional submanifold $L \subset \mathbb{R}^{2n+1}$ everywhere tangent to ξ. We also recall that the standard symplectic structure on $\mathbb{C}^n$ is given by

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j,$$

and that an immersion $f\colon L \to \mathbb{C}^n$ of an n-dimensional manifold is *Lagrangian* if $f^*\omega = 0$.

The *Lagrangian projection* projects out the z coordinate:

$$\Pi_{\mathbb{C}}\colon \mathbb{R}^{2n+1} \to \mathbb{C}^n; \quad (x_1, y_1, \dots, x_n, y_n, z) \mapsto (x_1, y_1, \dots, x_n, y_n). \tag{2.2}$$

If $L \subset \mathbb{C}^n \times \mathbb{R}$ is a Legendrian submanifold then $\Pi_{\mathbb{C}}\colon L \to \mathbb{C}^n$ is a Lagrangian immersion. Moreover, for L in an open dense subset of all Legendrian submanifolds (with C^∞ topology), the self intersection of $\Pi_{\mathbb{C}}(L)$ consists of a finite number of transverse double points. We call Legendrian submanifolds with this property *chord generic*.

The Reeb vector field X of a contact form α is uniquely defined by the two equations $\alpha(X) = 1$ and $d\alpha(X, \cdot) = 0$. The *Reeb chords* of a Legendrian submanifold L are segments of flow lines of X starting and ending at points of L. We see from (2.1) that in $\mathbb{R}^{2n+1}$, $X = \frac{\partial}{\partial z}$ and thus $\Pi_{\mathbb{C}}$ defines a bijection between Reeb chords of L and double points of $\Pi_{\mathbb{C}}(L)$. If c is a Reeb chord we write $c^* = \Pi_{\mathbb{C}}(c)$.

Let $\mathcal{C} = \{c_1, \dots, c_m\}$ be the set of Reeb chords of a chord generic Legendrian submanifold $L \subset \mathbb{R}^{2n+1}$. To such an L we associate an algebra $\mathcal{A} = \mathcal{A}(L)$ which is the free associative unital algebra over the group ring $\mathbb{Z}_2[H_1(L)]$ generated by $\mathcal{C}$. We write elements in $\mathcal{A}$ as

$$\sum_i t_1^{n_{1,i}} \dots t_k^{n_{k,i}} \mathbf{c}_i, \tag{2.3}$$

where the t_j's are formal variables corresponding to a basis for $H_1(L)$ thought of multiplicatively and $\mathbf{c}_i = c_{i_1} \dots c_{i_r}$ is a word in the generators. It is also useful to consider the corresponding algebra $\mathcal{A}_{\mathbb{Z}_2}$ over $\mathbb{Z}_2$. The natural map $\mathbb{Z}_2[H_1(L)] \to \mathbb{Z}_2$ induces a reduction of $\mathcal{A}$ to $\mathcal{A}_{\mathbb{Z}_2}$ (set $t_j = 1$, for all j).

2.2. **The Maslov index.** Let Λ_n be the Grassman manifold of Lagrangian subspaces in the symplectic vector space $(\mathbb{C}^n, \omega)$ and recall that $H_1(\Lambda_n) = \pi_1(\Lambda_n) \cong \mathbb{Z}$. There is a standard isomorphism

$$\mu : H_1(\Lambda_n) \to \mathbb{Z},$$

given by intersecting a loop in Λ_n with the Maslov cycle Σ. To describe μ more fully we follow [25] and refer the reader to this paper for proofs of the statements below.

Fix a Lagrangian subspace Λ in $\mathbb{C}^n$ and let $\Sigma_k(\Lambda) \subset \Lambda_n$ be the subset of Lagrangian spaces that intersects Λ in a subspace of k dimensions. The *Maslov cycle* is

$$\Sigma = \overline{\Sigma_1(\Lambda)} = \Sigma_1(\Lambda) \cup \Sigma_2(\Lambda) \cup \dots \cup \Sigma_n(\Lambda).$$

This in an algebraic variety of codimension one in Λ_n. If $\Gamma : [0,1] \to \Lambda_n$ is a loop then $\mu(\Gamma)$ is the intersection number of Γ and Σ. The contribution of an intersection point t' with $\Gamma(t') \in \Sigma$ to $\mu(\Gamma)$ is calculated as follows. Fix a Lagrangian complement W of Λ. Then for each $v \in \Gamma(t') \cap \Lambda$ there exists a vector $w(t) \in W$ such that $v + w(t) \in \Gamma(t)$ for t near t'. Define the quadratic form $Q(v) = \frac{d}{dt}|_{t=t'} \omega(v, w(t))$ on $\Gamma(t') \cap \Lambda$ and observe that it is independent of the complement W chosen. Without loss of generality, Q can be assumed non-singular and the contribution of the intersection point to $\mu(\Gamma)$ is the signature of Q. Given any loop Γ in Λ_n we say $\mu(\Gamma)$ is the *Maslov index* of the loop.

If $f\colon L \to \mathbb{C}^n$ is a Lagrangian immersion then the tangent planes of $f(L)$ along any loop γ in L gives a loop Γ in Λ_n. We define the Maslov index $\mu(\gamma)$ of γ as $\mu(\gamma) = \mu(\Gamma)$ and note that we may view the Maslov index as a map $\mu : H_1(L) \to \mathbb{Z}$. Let $m(f)$ be the smallest non-negative number that is the Maslov index of some non-trivial loop in L. We call $m(f)$ the *Maslov number* of f. When $L \subset \mathbb{C}^n \times \mathbb{R}$ is a Legendrian submanifold we write $m(L)$ for the Maslov number of $\Pi_{\mathbb{C}}\colon L \to \mathbb{C}^n$.

2.3. **The Conley–Zehnder index of a Reeb chord and the grading on $\mathcal{A}$.** Let $L \subset \mathbb{R}^{2n+1}$ be a chord generic Legendrian submanifold and let c be one of its Reeb chords with end points $a, b \in L$, $z(a) > z(b)$. Choose a path $\gamma : [0,1] \to L$ with $\gamma(0) = a$ and $\gamma(1) = b$. (We call such path a *capping path of* c.) Then $\Pi_{\mathbb{C}} \circ \gamma$ is a loop in $\mathbb{C}^n$ and $\Gamma(t) = d\Pi_{\mathbb{C}}(T_{\gamma(t)}L)$, $0 \le t \le 1$ is a path of Lagrangian subspaces of $\mathbb{C}^n$. Since $c^* = \Pi_{\mathbb{C}}(c)$ is a transverse double point of $\Pi_{\mathbb{C}}(L)$, Γ is not a closed loop.

We close Γ in the following way. Let $V_0 = \Gamma(0)$ and $V_1 = \Gamma(1)$. Choose any complex structure I on $\mathbb{C}^n$ which is compatible with ω ($\omega(v, Iv) > 0$ for all v) and with $I(V_1) = V_0$. (Such an I exists since the Lagrangian planes are transverse.) Define the path $\lambda(V_0, V_1)(t) = e^{tI}V_1$, $0 \le t \le \frac{\pi}{2}$. The concatenation, $\Gamma * \lambda(V_0, V_1)$, of Γ and $\lambda(V_0, V_1)$ forms a loop in Λ_n and we define the *Conley–Zehnder index*, $\nu_\gamma(c)$, of c to be the Maslov index $\mu(\Gamma * \lambda(V_0, V_1))$ of this loop. It is easy to check that $\nu_\gamma(c)$ is independent of the choice of I. However, $\nu_\gamma(c)$ might depend on the choice of homotopy class of the path γ. More precisely, if γ_1 and γ_2 are two paths with properties as γ above then

$$\nu_{\gamma_1}(c) - \nu_{\gamma_2}(c) = \mu(\gamma_1 * (-\gamma_2)),$$

where $(-\gamma_2)$ is the path γ_2 traversed in the opposite direction. Thus $\nu_\gamma(c)$ is well defined modulo the Maslov number $m(L)$.

Let $\mathcal{C} = \{c_1, \dots, c_m\}$ be the set of Reeb chords of L. Choose a capping path γ_j for each c_j and define the *grading* of c_j to be

$$|c_j| = \nu_{\gamma_j}(c_j) - 1,$$

and for any $t \in H_1(L)$ define its grading to be $|t| = -\mu(t)$. This makes $\mathcal{A}(L)$ into a graded ring. Note that the grading depends on the choice of capping paths but, as we will see below, this choice will be irrelevant.

The above grading on Reeb chords c_j taken modulo $m(L)$ makes $\mathcal{A}_{\mathbb{Z}_2}$ a graded algebra with grading in $\mathbb{Z}_{m(L)}$. (Note that this grading does not depend on the choice of capping paths.) In addition the map from $\mathcal{A}$ to $\mathcal{A}_{\mathbb{Z}_2}$ preserves gradings modulo $m(L)$.

2.4. **The moduli spaces.** As mentioned in the introduction, the differential of the algebra associated to a Legendrian submanifold is defined using spaces of holomorphic disks. To describe these spaces we need a few preliminary definitions.

Let D_{m+1} be the unit disk in $\mathbb{C}$ with $m+1$ punctures at the points $p_0, \dots p_m$ on the boundary. The orientation of the boundary of the unit disk induces a cyclic ordering of the punctures. Let $\partial \hat{D}_{m+1} = \partial D_{m+1} \setminus \{p_0, \dots, p_m\}$.

Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a Legendrian submanifold with isolated Reeb chords. If c is a Reeb chord of L with end points $a, b \in L$, $z(a) > z(b)$ then there are small neighborhoods $S_a \subset L$ of a and $S_b \subset L$ of b that are mapped injectively to $\mathbb{C}^n$ by $\Pi_{\mathbb{C}}$. We call $\Pi_{\mathbb{C}}(S_a)$ the *upper sheet* of $\Pi_{\mathbb{C}}(L)$ at c^* and $\Pi_{\mathbb{C}}(S_b)$ the *lower sheet.* If $u\colon (D_{m+1}, \partial D_{m+1}) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(L))$ is a continuous map with $u(p_j) = c^*$ then we say p_j is *positive* (respectively *negative*) if u maps points clockwise of p_j on ∂D_{m+1} to the lower (upper) sheet of $\Pi_{\mathbb{C}}(L)$ and points anti-clockwise of p_i on ∂D_{m+1} to the upper (lower) sheet of $\Pi_{\mathbb{C}}(L)$ (see Figure 1).

If a is a Reeb chord of L and if $\mathbf{b} = b_1 \dots b_m$ is an ordered collection (a word) of Reeb chords then let $\mathcal{M}_A(a; \mathbf{b})$ be the space, modulo conformal reparameterization, of maps $u : (D_{m+1}, \partial D_{m+1}) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(L))$ which are continuous on D_{m+1}, holomorphic in the interior of D_{m+1}, and which have the following properties

- p_0 is a positive puncture, $u(p_0) = a^*$,
- p_j are negative punctures for $j > 0$, $u(p_j) = b_j^*$,
- the restriction $u|\partial \hat{D}_{m+1}$ has a continuous lift $\tilde{u}\colon \partial \hat{D}_{m+1} \to L \subset \mathbb{C}^n \times \mathbb{R}$, and
- the homology class of $\tilde{u}(\partial D^*_{m+1}) \cup (\cup_j \gamma_j)$ equals $A \in H_1(L)$,

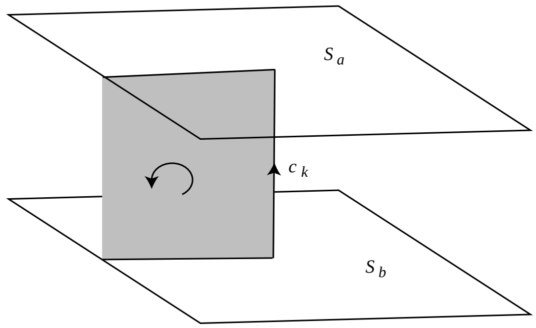

FIGURE 1. Positive puncture lifted to $\mathbb{R}^{2n+1}$. The gray region is the holomorphic disk and the arrows indicate the orientation on the disk and the Reeb chord.

where γ_j is the capping path chosen for c_j, $j = 1, \dots, m$. Elements in $\mathcal{M}_A(a; \mathbf{b})$ will be called *holomorphic disks with boundary on* L or sometimes simply holomorphic disks.

There is a useful fact relating heights of Reeb chords and the area of a holomorphic disk with punctures mapping to the corresponding double points. The *action* (or height) $\mathcal{Z}(c)$ of a Reeb chord c is simply its length and the action of a word of Reeb chords is the sum of the actions of the chords making up the word.

Lemma 2.1. *If* $u \in \mathcal{M}_A(a; \mathbf{b})$ *then*

$$\mathcal{Z}(a) - \mathcal{Z}(\mathbf{b}) = \int_{D_m} u^*\omega = \text{Area}(u) \geq 0. \tag{2.4}$$

Proof. By Stokes theorem, $\int_{D_m} u^*\omega = \int_{\partial D_m} u^*(-\sum_j y_j dx_j) = \int \tilde{u}^*(-dz) = \mathcal{Z}(a) - \mathcal{Z}(\mathbf{b})$. The second equality follows since u is holomorphic and $\omega = \sum_{j=1}^n dx_j \wedge dy_j$. □

Note that the proof of Lemma 2.1 implies that any holomorphic disk with boundary on L must have at least one positive puncture. (In contact homology, only disks with exactly one positive puncture are considered.)

We now proceed to describe the properties of moduli spaces $\mathcal{M}_A(a; \mathbf{b})$ that are needed to define the differential. We prove in [7] that the moduli spaces of holomorphic disks with boundary on a Legendrian submanifold L have these properties provided L is generic among (belongs to a Baire subset of the space of) *admissible* Legendrian submanifolds. (L is admissible if it is chord generic and it is real analytic in a neighborhood of all Reeb chord end points. A more precise definitions of these concepts appears in [7] where it is shown that admissible Legendrian submanifolds are dense in the space of all Legendrian submanifolds.) The moduli spaces $\mathcal{M}_A(a; \mathbf{b})$ can be seen as the 0-sets of certain "$\bar{\partial}$-type" C^1-maps, between infinite-dimensional Banach manifolds. We say a moduli space is *transversely cut out* if 0 is a regular value of the corresponding map.

Proposition 2.2. [7, Corollary 9.15.5] *For a generic admissible Legendrian submanifold* $L \subset \mathbb{C}^n \times \mathbb{R}$ *the moduli space* $\mathcal{M}_A(a; \mathbf{b})$ *is a transversely cut out manifold of dimension*

$$d = \mu(A) + |a| - |\mathbf{b}| - 1, \tag{2.5}$$

provided $d \leq 1$. *(In particular, if* $d < 0$ *then the moduli space is empty.)*

If $u \in \mathcal{M}_A(a; \mathbf{b})$ we say that $d = \mu(A) + |a| - |\mathbf{b}| - 1$ is the *formal dimension of* u, and if v is a transversely cut out disk of formal dimension 0 we say that v is a *rigid disk*.

We mention here two transversality results which will prove useful for our computations in Section 4. Let $\pi_i : \mathbb{C}^n \to \mathbb{C}$, $i = 1, \dots, n$, denote the complex projections.

Proposition 2.3. [7, Corollary 9.22] *Assume $n > 1$. For L in a Baire subset of the space of admissible Legendrian submanifolds, no rigid holomorphic disk passes though the end points of any Reeb chords of L.*

Proposition 2.4. [7, Lemmas 9.24, 9.25] *Assume $n > 1$. Consider a holomorphic disk u with no (or one) negative punctures, and a formal dimension of 0. Assume that $\pi_i \circ u = 0$ for $i = 2, \dots n$ and that the tangent space of the Lagrangian immersion splits along the boundary of u. That is, the one path component of $T_{u(\partial D_1)}\Pi_{\mathbb{C}}(L)$ (or two path components of $T_{u(\partial D_2)}\Pi_{\mathbb{C}}(L)$) is of the form $\gamma \times V$ where $\gamma(t) \subset \mathbb{C}$ is a real line and $V(t) \subset 0 \times \mathbb{C}^n$. Then u is cut out transversely.*

The moduli spaces we consider might not be compact, but their lack of compactness can be understood. It is analogous to "convergence to broken trajectories" in Morse/Floer homology and gives rise to natural compactifications of the moduli spaces, known as Gromov compactness.

A *broken holomorphic curve*, $u = (u^1, \dots, u^N)$, is a union of holomorphic disks, $u^j : (D_{m_j}, \partial D_{m_j}) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(L))$, where each u^j has exactly one positive puncture p^j, with the following property. To each p^j with $j \geq 2$ is associated a negative puncture $q_j^k \in D_{m_k}$ for some $k \neq j$ such that $u^j(p^j) = u^k(q_j^k)$ and $q_{j'}^{k'} \neq q_j^k$ if $j \neq j'$, and such that the quotient space obtained from $D_{m_1} \cup \dots \cup D_{m_N}$ by identifying p^j and q_j^k for each $j \geq 2$ is contractible. The broken curve can be parameterized by a single smooth $v : (D_m, \partial D) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(L))$. A sequence u_α of holomorphic disks *converges* to a broken curve $u = (u^1, \dots, u^N)$ if the following holds:

- For every $j \leq N$, there exists a sequence $\phi_\alpha^j : D_m \to D_m$ of linear fractional transformations and a finite set $X^j \subset D_m$ such that $u_\alpha \circ \phi_\alpha^j$ converges to u^j uniformly with all derivatives on compact subsets of $D_m \setminus X^j$
- There exists a sequence of orientation-preserving diffeomorphisms $f_\alpha : D_m \to D_m$ such that $u_\alpha \circ f_\alpha$ converges in the C^0-topology to a parameterization of u.

Proposition 2.5. *Any sequence u_α in $\mathcal{M}_A(a; \mathbf{b})$ has a subsequence converging to a broken holomorphic curve $u = (u^1, \dots, u^N)$. Moreover, $u^j \in \mathcal{M}_{A_j}(a^j; \mathbf{b}^j)$ with $A = \sum_{j=1}^N A_j$ and*

$$\mu(A) + |a| - |\mathbf{b}| = \sum_{j=1}^{N} \left(\mu(A_j) + |a^j| - |\mathbf{b}^j| \right). \tag{2.6}$$

Heuristically this is the only type of non-compactness we expect to see in $\mathcal{M}_A(a; \mathbf{b})$: since $\pi_2(\mathbb{C}^n) = 0$, no holomorphic spheres can "bubble off" at an interior point of the sequence u_α, and since $\Pi_{\mathbb{C}}(L)$ is exact no disks without positive puncture can form either. Moreover, since $\Pi_{\mathbb{C}}(L)$ is compact, and since $\mathbb{C}^n$ has "finite geometry at infinity", or is "tame at infinity" [3, 7, 11, 30, 31], all holomorphic curves with a uniform bound on area must map to a compact set.

Proof. The main step is to prove convergence to some broken curve, which appears as [7, Theorem 11.2]. The statement about the homology classes follows easily from the definition of convergence. Equation (2.6) follows from the definition of broken curves. □

We next show that a broken curve can be glued to form a family of non-broken curves. For this we need some additional notation. Let $\mathbf{c}^1, \dots, \mathbf{c}^r$ be an ordered collection of words of Reeb chords. Let the length of (number of letters in) $\mathbf{c}^j$ be $l(j)$ and let $\mathbf{a} = a_1 \dots a_k$ be a word of Reeb-chords of length $k > 0$. Let $S = \{s_1, \dots, s_r\}$ be r distinct integers in $\{1, \dots, k\}$. Define the word $\mathbf{a}_S(\mathbf{c}^1, \dots, \mathbf{c}^r)$ of Reeb-chords of length $k - r + \sum_{j=1}^r l(j)$ as follows. For each index $s_j \in S$ remove a_{s_j} from the word $\mathbf{a}$ and insert at its place the word $\mathbf{c}^j$.

Proposition 2.6. [7, Proposition 10.1] *Let L be a generic admissible Legendrian submanifold. Let $\mathcal{M}_A(a;\mathbf{b})$ and $\mathcal{M}_B(c;\mathbf{d})$ be 0-dimensional transversely cut out moduli spaces and assume that the j-th Reeb chord in $\mathbf{b}$ is c. Then there exist a $\rho > 0$ and an embedding*

$$G\colon \mathcal{M}_A(a;\mathbf{b}) \times \mathcal{M}_B(c;\mathbf{d}) \times (\rho,\infty) \to \mathcal{M}_{A+B}(a;\mathbf{b}_{\{j\}}(\mathbf{d})).$$

Moreover, if $u \in \mathcal{M}_A(a;\mathbf{b})$ and $u' \in \mathcal{M}_B(c;\mathbf{d})$ then $G(u,u',\rho)$ converges to the broken curve (u,u') as $\rho \to \infty$, and any disk in $\mathcal{M}_A(a;\mathbf{b}_{\{j\}}(\mathbf{d}))$ with image sufficiently close to the image of (u,u') is in the image of G.

2.5. **The differential and contact homology.** Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a generic admissible Legendrian submanifold, let $\mathcal{C}$ be its set of Reeb chords, and let $\mathcal{A}$ denote its algebra. For any generator $a \in \mathcal{C}$ of $\mathcal{A}$ we set

$$\partial a = \sum_{\dim\, \mathcal{M}_A(a;\mathbf{b})=0} (\#\mathcal{M}_A(a;\mathbf{b}))A\mathbf{b}, \tag{2.7}$$

where $\#\mathcal{M}$ is the number of points in $\mathcal{M}$ modulo 2, and where the sum ranges over all words $\mathbf{b}$ in the alphabet $\mathcal{C}$ and $A \in H_1(L)$ for which the above moduli space has dimension 0. We then extend ∂ to a map $\partial : \mathcal{A} \to \mathcal{A}$ by linearity and the Leibniz rule.

Since L is generic admissible, it follows from Propositions 2.5 and 2.6 that the moduli spaces considered in the definition of ∂ are compact 0-manifolds and hence consist of a finite number of points. Thus ∂ is well defined. Moreover,

Lemma 2.7. *The map $\partial : \mathcal{A} \to \mathcal{A}$ is a differential of degree -1. That is, $\partial \circ \partial = 0$ and $|\partial(a)| = |a| - 1$ for any generator a of $\mathcal{A}$.*

Proof. After Propositions 2.5 and 2.6 the standard proof in Morse (or Floer) homology [28] applies. It follows from (2.5) that ∂ lowers degree by 1. □

The *contact homology of L* is

$$HC_*(\mathbb{R}^{2n+1}, L) = \text{Ker }\partial/\text{Im }\partial.$$

It is essential to notice that since ∂ respects the grading on $\mathcal{A}$ the contact homology is a graded algebra.

We note that ∂ also defines a differential of degree -1 on $\mathcal{A}_{\mathbb{Z}_2}(L)$.

2.6. **The invariance of contact homology under Legendrian isotopy.** In this section we show

Proposition 2.8. *If $L_t \subset \mathbb{R}^{2n+1}$, $0 \le t \le 1$ is a Legendrian isotopy between generic admissible Legendrian submanifolds then the contact homologies $HC_*(\mathbb{R}^{2n+1}, L_0)$, and $HC_*(\mathbb{R}^{2n+1}, L_1)$ are isomorphic.*

In fact we show something, that at least appears to be, stronger. Given a graded algebra $\mathcal{A} = \mathbb{Z}_2[G]\langle a_1,\dots,a_n\rangle$, where G is a finitely generated abelian group, a graded automorphism $\phi : \mathcal{A} \to \mathcal{A}$ is called *elementary* if there is some $1 \le j \le n$ such that

$$\phi(a_i) = \begin{cases} A_i a_i, & i \neq j \\ \pm A_j a_j + u, & u \in \mathcal{A}(a_1,\dots,a_{j-1},a_{j+1},\dots,a_n),\ i = j, \end{cases}$$

where the A_i are units in $\mathbb{Z}_2[G]$. The composition of elementary automorphisms is called a *tame* automorphism. An isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ is tame if it is the composition of a tame automorphism with an isomorphism sending the generators of $\mathcal{A}$ to the generators of $\mathcal{A}'$. An isomorphism of DGA's is called *tame* if the isomorphism of the underlying algebras is tame.

Let $(\mathcal{E}_i, \partial_i)$ be a DGA with generators $\{e_1^i, e_2^i\}$, where $|e_1^i| = i, |e_2^i| = i-1$ and $\partial_i e_1^i = e_2^i, \partial_i e_2^i = 0$. Define the *degree i stabilization* $S_i(\mathcal{A},\partial)$ of $(\mathcal{A},\partial)$ to be the graded algebra generated by $\{a_1, \dots, a_n, e_1^i, e_2^i\}$ with grading and differential induced from $\mathcal{A}$ and $\mathcal{E}_i$. Two differential graded algebras are called *stable tame isomorphic* if they become tame isomorphic after each is stabilized a suitable number of times.

Proposition 2.9. *If $L_t \subset \mathbb{R}^{2n+1}$, $0 \le t \le 1$ is a Legendrian isotopy between generic admissible Legendrian submanifolds then the DGA's $(\mathcal{A}(L_0), \partial)$ and $(\mathcal{A}(L_1), \partial)$ are stable tame isomorphic.*

Note that Proposition 2.9 allows us to associate the stable tame isomorphism class of a DGA to a Legendrian isotopy class of Legendrian submanifolds: any Legendrian isotopy class has a generic admissible representative and by Proposition 2.9 the DGA's of any two generic admissible representatives agree.

It is straightforward to show that two stable tame isomorphic DGA's have the same homology, see [5, 15]. Thus Proposition 2.8 follows from Proposition 2.9. The proof of the later given below is, in outline, the same as the proof of invariance of the stable tame isomorphism class of the DGA of a Legendrian 1-knot in [5]. However, the details in our case require considerably more work. In particular we must substitute analytic arguments for the purely combinatorial ones that suffice in dimension three.

A Legendrian isotopy $\phi_t \colon L \to \mathbb{C}^n \times \mathbb{R}$, $0 \le t \le 1$, is *admissible* if $\phi_0(L)$ and $\phi_1(L)$ are admissible Legendrian submanifolds and if there exist a finite number of instants $0 < t_1 < t_2 < \cdots < t_m < 1$ and a $\delta > 0$ such that the intervals $[t_j - \delta, t_j + \delta]$ are disjoint subsets of $(0, 1)$ with the following properties.

(A) For $t \in [0, t_1 - \delta] \cup \left(\bigcup_{j=1}^{m} [t_j + \delta, t_{j+1} - \delta]\right) \cup [t_m + \delta, 1]$, $\phi_t(L)$ is an isotopy through admissible Legendrian submanifolds.

(B) For $t \in [t_j - \delta, t_j + \delta]$, $j = 1, \dots, m$, $\phi_t(L)$ undergoes a *standard self-tangency move*. That is, there exists a point $q \in \mathbb{C}^n$ and neighborhoods $N \subset N'$ of q with the following properties. The intersection $N \cap \Pi_{\mathbb{C}}(\phi_t(L))$ equals $P_1 \cup P_2(t)$ which, up to biholomorphism looks like $P_1 = \gamma_1 \times P_1'$ and $P_2 = \gamma_2(t) \times P_2'$. Here γ_1 and $\gamma_2(t)$ are subarcs around 0 of the curves $y_1 = 0$ and $x_1^2 + (y_1 - 1 \pm t)^2 = 1$ in the z_1-plane, respectively, and P_1' and P_2' are real analytic Lagrangian $(n-1)$-disks in $\mathbb{C}^{n-1} = \{z_1 = 0\}$ intersecting transversely at 0. Outside $N' \times \mathbb{R}$ the isotopy is constant. See Figure 2. (The full definition of a standard self tangency move appears in [7, Definition 5.2]. For simplicity, one technical condition there has been omitted at this point.)

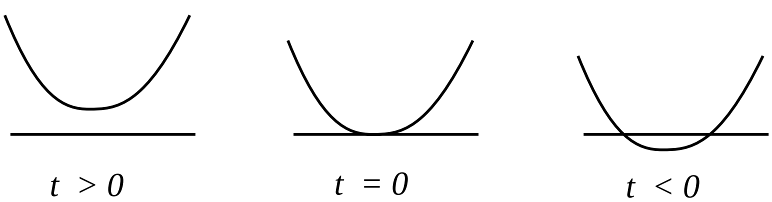

FIGURE 2. Type B double point move.

Lemma 2.10. [7, Lemma 5.6] *Any two admissible Legendrian submanifolds of dimension $n > 1$ which are Legendrian isotopic are isotopic through a an admissible Legendrian isotopy.*

This result does not hold when $n = 1$; one must allow also a "triple point move" see [5, 15].

To prove Proposition 2.9, we need to check that the differential graded algebra changes only by stable tame isomorphisms under Legendrian isotopies of type (A) and (B). We start with type (A) isotopies.

Lemma 2.11. *Let $L_t, t \in [0,1]$ be a type (A) isotopy between generic admissible Legendrian submanifolds. Then the DGA's associated to L_0 and L_1 are tame isomorphic.*

To prove this we use a parameterized version of Proposition 2.2. If $L_t, t \in I = [0,1]$ is a type (A) isotopy then the double points of $\Pi_{\mathbb{C}}(L_t)$ trace out continuous curves. Thus, when we refer to a Reeb chord c of $L_{t'}$ for some $t' \in [0,1]$ this unambiguously specifies a Reeb chord for all L_t. For any t we let $\mathcal{M}^t_A(a;\mathbf{b})$ denote the moduli space $\mathcal{M}_A(a;\mathbf{b})$ for L_t and define

$$\mathcal{M}^I_A(a;\mathbf{b}) = \{(u,t) | u \in \mathcal{M}^t_A(a;\mathbf{b})\}. \tag{2.8}$$

As above "generic" refers to a member of a Baire subset, see [7] for a more precise formulation of this term for 1-parameter families.

Proposition 2.12. [7, Corollary 9.16.5] *For a generic type (A) isotopy $L_t, t \in I = [0,1]$ the following holds. If $a, \mathbf{b}, A$ are such that $\mu(A) + |a| - |\mathbf{b}| = d \le 1$ then the moduli space $\mathcal{M}^I_A(a;\mathbf{b})$ is a transversely cut out d-manifold. If X is the union of all these transversely cut out manifolds which are 0-dimensional then the components of X are of the form $\mathcal{M}^{t_j}_{A_j}(a_j,\mathbf{b}_j)$, where $\mu(A_j) + |a_j| - |\mathbf{b}_j| = 0$, for a finite number of distinct instances $t_1, \dots, t_r \in [0,1]$. Furthermore, $t_1, \dots, t_r$ are such that $\mathcal{M}^{t_j}_B(c;\mathbf{d})$ is a transversely cut out 0-manifold for every $c, \mathbf{d}, B$ with $\mu(B) + |c| - |\mathbf{d}| = 1$.*

At an instant $t = t_j$ in the above proposition we say a *handle slide* occurs, and an element in $\mathcal{M}^{t_j}_{A_j}(a_j,\mathbf{b}_j)$ will be called a *handle slide disk.* (The term handle slide comes form the analogous situation in Morse theory.)

The proof of Lemma 2.11 depends, just as the proof of Lemma 2.7, on one compactness- and one gluing result which we describe next.

Proposition 2.13. *Any sequence u_α in $\mathcal{M}^I_A(a;\mathbf{b})$ has a subsequence that converges to a broken holomorphic curve with the same properties as in Proposition 2.5.*

The proof of this proposition is identical to that of Proposition 2.5.

Proposition 2.14. [7, Theorems 10.2 and 10.3] *Let $\delta > 0$ and let $L_t, t \in I = [-\delta,\delta]$ be a small neighborhood of a handle slide at $t = 0$ in a generic type (A) isotopy. Then for δ sufficiently small, $L_{\pm\delta}$ are generic admissible and, with $u \in \mathcal{M}^0_A(a;\mathbf{b})$ denoting the handle slide disk, the following holds.*

(1) *Assume that c is the j-th letter in $\mathbf{b}$. Let $\mathcal{M}^0_B(c;\mathbf{d})$ be a moduli space of rigid holomorphic disks. Then there exist $\rho_0 > 0$ and an embedding*

$$G\colon \mathcal{M}^0_B(c;\mathbf{d}) \times [\rho_0,\infty) \to \mathcal{M}^I_{A+B}(a;\mathbf{b}_{\{j\}}(\mathbf{d})).$$

Given $v \in \mathcal{M}^0_B(c;\mathbf{d})$, $G(v,\rho)$ converges to the broken curve (v,u) as $\rho \to \infty$. Moreover, any curve in $\mathcal{M}^I_{A+B}(a;\mathbf{b}_{\{j\}}(\mathbf{d}))$ with image sufficiently close to the image of (v,u) is in the image of G.

(2) *Let $\mathcal{M}^0_B(c;\mathbf{d})$ be a moduli space of rigid holomorphic disks, where $S = \{s_1,\dots,s_r\}$, and $\mathbf{d}$ has a at every position of an element in S. Then there exist $\rho_0 > 0$, and an embedding*

$$G'\colon \mathcal{M}^0_B(c;\mathbf{d}) \times [\rho_0,\infty) \to \mathcal{M}^I_{B+r\cdot A}(c;\mathbf{d}_S(\mathbf{b},\dots,\mathbf{b})).$$

Given $v \in \mathcal{M}^0_B(c;\mathbf{d})$, $G'(v,\rho)$ converges to the broken curve $(v,u,\dots,u)$. Moreover, any curve in $\mathcal{M}^I_{B+r\cdot A}(c;\mathbf{d}_S(\mathbf{b},\dots,\mathbf{b}))$ with image sufficiently close to the image of $(v,u,\dots,u)$ is in the image of G'.

Proof. We show here why the above are the only kind of broken curves to consider gluing. If the broken curve lives in the compactification of the one-dimensional $\mathcal{M}^I_B(c_0;\mathbf{c})$, then by (2.6) at least one of its pieces must have negative formal dimension. Since the handle slide disk u is the only disk with negative formal dimension, all but one of the pieces of the broken disk must be u. The requirement that our disks have just one positive puncture and Lemma 2.1 reduce all possible configurations of the broken curve to the ones considered above. □

We now prove Lemma 2.11 in two steps. First consider type (A) isotopies without handle slides.

Lemma 2.15. *Let $L_t, t \in [0,1]$ be a generic type (A) isotopy of Legendrian submanifolds for which no handle slides occur. Then the boundary maps ∂_0 and ∂_1 on $\mathcal{A} = \mathcal{A}(L_0) = \mathcal{A}(L_1)$ satisfies $\partial_0 = \partial_1$.*

Proof. Propositions 2.13 and 2.14 imply that $\mathcal{M}^I_A(a;\mathbf{B})$ is compact when its dimension is one. Since if a sequence in this space converged to a broken curve $(u^1,\dots,u^N)$ then at least one u^j would have negative formal dimension. This contradicts the assumptions that no handle slide occurs and that the type (A) isotopy is generic. Thus the corresponding 0 dimensional moduli spaces $\mathcal{M}^0_A$ and $\mathcal{M}^1_A$ used in the definitions of ∂_0 and ∂_1, respectively, form the boundary of a compact 1-manifold. Hence their modulo 2 counts are equal. □

We consider what happens around a handle slide instant. Let L_t, $t \in [-\delta,\delta]$ and $\mathcal{M}^0_A(a;\mathbf{b})$ be as in Lemma 2.14. Let ∂_- denote the differential on $\mathcal{A} = \mathcal{A}(L_{-\delta})$, and ∂_+ the one on $\mathcal{A} = \mathcal{A}(L_\delta)$. For generators c in $\mathcal{A}$ define

$$\phi_a(c) = \begin{cases} c & \text{if } c \neq a, \\ a + A\mathbf{b} & \text{if } c = a. \end{cases}$$

and extend ϕ_a to a tame algebra automorphism of $\mathcal{A}$.

Lemma 2.16. *Let c be a generator of $\mathcal{A}$ then*

$$\partial_+ c = \begin{cases} \phi_a(\partial_- c) & \textit{if } c \neq a, \\ \partial_-(\phi_a(c)) & \textit{if } c = a. \end{cases}$$

Proof. Any $\alpha \in \mathcal{A}$ can be expressed in a unique way as a $\mathbb{Z}_2$-linear combination of elements $C\mathbf{w}$, where $C \in H_1(L)$ and $\mathbf{w}$ is a word in the generators of $\mathcal{A}$, see (2.3). Let $\langle \alpha, C\mathbf{w}\rangle$ denote the coefficient (0 or 1) in such an expansion. It follows from Proposition 2.14 that for any generator $c \neq a$

$$\langle (\partial_+ - \partial_-)c, B\mathbf{w}_1\mathbf{b}\mathbf{w}_2\rangle = \langle \partial_- c, (BA^{-1})\mathbf{w}_1 a\mathbf{w}_2\rangle.$$

From this, the formula for $\partial_+ c$ follows when $c \neq a$. The formula when $c = a$ follows similarly. □

Lemma 2.17. *The map $\phi_a : \mathcal{A} \to \mathcal{A}$ is a tame isomorphism from $(\mathcal{A},\partial_-)$ to $(\mathcal{A},\partial_+)$.*

Proof. As ϕ_a is clearly a tame isomorphism of algebras we only need to check that it is also a chain map. If $c \neq a$ is a generator then $\phi_a\partial_- c = \partial_+ c = \partial_+\phi_a c$. It follows from Lemma 2.1 that $\partial_+ a$ contains no terms which contain an a and that the word $\mathbf{b}$ does not contain the letter a. Thus $\partial_+ A\mathbf{b} = \partial_- A\mathbf{b}$ and hence

$$\phi_a\partial_- a = \phi_a\partial_-(\phi_a(a + A\mathbf{b})) = \phi_a(\partial_+ a + \partial_+ A\mathbf{b}) = \partial_+(a + A\mathbf{b}) = \partial_+\phi_a a.$$

□

Proof of Lemma 2.11. The lemma follows from Lemmas 2.15, 2.16, and 2.17. □

We consider elementary isotopies of type (B). Let $L_t, t \in I = [-\delta, \delta]$ be an isotopy of type (B) where two Reeb chords $\{a, b\}$ are born as t passes through 0. Let o be the degenerate Reeb chord (double point) at $t = 0$ and let $\mathcal{C}' = \{a_1, \dots, a_l, b_1, \dots, b_m\}$ be the other Reeb chords. Again we note that $c_i \in \mathcal{C}'$ unambiguously defines a Reeb chord for all L_t and a and b unambiguously define two Reeb chords for all L_t when $t > 0$. It is easy to see that (with the appropriate choice of capping paths) the grading on a and b differ by 1 so let $|a| = j$ and $|b| = j - 1$. Let $(\mathcal{A}_-, \partial_-)$ and $(\mathcal{A}_+, \partial_+)$ be the DGA's associated to $L_{-\delta}$ and L_δ, respectively.

Lemma 2.18. *The stabilized algebra $S_j(\mathcal{A}_-, \partial_-)$ is tame isomorphic to $(\mathcal{A}_+, \partial_+)$.*

Proof of Proposition 2.9 and 2.8 . The first proposition follows from Lemmas 2.11 and 2.18 and implies in its turn the second. □

We prove Lemma 2.18 in several steps below. Label the Reeb chords of L_t so that

$$\mathcal{Z}(b_m) \leq \ldots \leq \mathcal{Z}(b_1) \leq \mathcal{Z}(b) < \mathcal{Z}(a) \leq \mathcal{Z}(a_1) \leq \ldots \leq \mathcal{Z}(a_l),$$

let $\mathcal{B} = \mathbb{Z}_2[H_1(L)]\langle b_1, \dots, b_m \rangle$ and note that $\mathcal{B}$ is a subalgebra of both $\mathcal{A}_-$ and $\mathcal{A}_+$. Then

Lemma 2.19. *For $\delta > 0$ small enough*

$$\partial_+ a = b + v,$$

where $v \in \mathcal{B}$.

Proof. Let $\mathbf{0} \in H_1(L)$ denote the zero element. In the model for the type (B) isotopy there is an obvious disk in $\mathcal{M}^t_{\mathbf{0}}(a; b)$ for $t > 0$ small which is contained in the z_1-plane. We argue that this is the only point in the moduli space. We restrict attention to the neighborhood N of o^* that is biholomorphic to the origin in $\mathbb{C}^n$ as in the description of a type (B) move. Let $\pi_i : \mathbb{C}^n \to \mathbb{C}$ be the projection onto the i^{th} coordinate. If $u : D \to \mathbb{C}^n$ is a holomorphic map in $\mathcal{M}^t_{\mathbf{0}}(a; b)$ then $\pi_i \circ u$ will either be constant or not. If $\pi_i \circ u$ is non-constant for $i > 1$ then the image of $\pi_1 \circ u$ intersected with N has boundary on two transverse Lagrangian submanifolds. As such it will have a certain area A_i. Since $\mathcal{Z}(a) - \mathcal{Z}(b) \to 0$ as $t \to 0+$ we can choose t small enough so that $\mathcal{Z}(a) - \mathcal{Z}(b) < A_i$, for all $i > 1$. Then $\pi_i \circ u$ must be a point for all $i > 1$ and for $i = 1$, it can only be the obvious disk. Proposition 2.4 shows that $\mathcal{M}^t_{\mathbf{0}}(a; b)$ is transversely cut out and thus contributes to $\partial_+ a$. If $u \in \mathcal{M}^t_A(a; b)$, where $A \neq \mathbf{0}$ then the image of u must leave N. Thus, the above argument shows that $\mathcal{M}^t_A(a; b) = \emptyset$ for t small enough. Also, for $t > 0$ sufficiently small $\mathcal{Z}(a) - \mathcal{Z}(b) < \mathcal{Z}(b_m)$. Hence by Lemma 2.1, $v \in \mathcal{B}$. □

Define the elementary isomorphism $\Phi_0 : \mathcal{A}_+ \to S_j(\mathcal{A}_-)$ (on generators) by

$$\Phi_0(c) = \begin{cases} e_1^j & \text{if } c = a, \\ e_2^j + v & \text{if } c = b \\ c & \text{otherwise.} \end{cases}$$

The map Φ_0 fails to be a tame isomorphism since it is not a chain map. However, we use it as the first step in an inductive construction of a tame isomorphism $\Phi_l \colon \mathcal{A}_+ \to S_j(\mathcal{A}_-)$. To this end, for $0 \leq i \leq l$, let $\mathcal{A}_i$ be the subalgebra of $\mathcal{A}_+$ generated by $\{a_1, \dots, a_i, a, b, b_1, \dots, b_m\}$ (note that $\mathcal{A}_l = \mathcal{A}_+$). Then, with $\tau : S_j(\mathcal{A}_-) \to \mathcal{A}_-$ denoting the natural projection and with ∂^s_- denoting the differential induced on $S_j(\mathcal{A}_-)$, we have

Lemma 2.20.

$$\Phi_0 \circ \partial_+ w = \partial^s_- \circ \Phi_0 w \tag{2.9}$$

for $w \in \mathcal{A}_0$ and

$$\tau \circ \Phi_0 \circ \partial_+ = \tau \circ \partial^s_- \circ \Phi_0. \tag{2.10}$$

Before proving this lemma, we show how to use it in the inductive construction which completes the proof of Lemma 2.18.

Proof of Lemma 2.18. The proof is similar to the proof of Lemmas 6.3 and 6.4 in [15] (*cf* [5]). Define the map $H : S_j(\mathcal{A}_-) \to S_j(\mathcal{A}_-)$ on words $\mathbf{w}$ in the generators by

$$H(\mathbf{w}) = \begin{cases} 0 & \text{if } \mathbf{w} \in \mathcal{A}_-, \\ 0 & \text{if } \mathbf{w} = \alpha e_1^j \beta \text{ and } \alpha \in \mathcal{A}_- \\ \alpha e_1^j \beta & \text{if } \mathbf{w} = \alpha e_2^j \beta \text{ and } \alpha \in \mathcal{A}_-, \end{cases}$$

and extend it linearly. Assume inductively that we have defined a graded isomorphism $\Phi_{i-1} : \mathcal{A}_+ \to S_j(\mathcal{A}_-)$ so that it is a chain map when restricted to $\mathcal{A}_{i-1}$ and so that $\Phi_{i-1}(a_k) = a_k$, for $k > i-1$. (Note that Φ_0 has these properties by Lemma 2.20.)

Define the elementary isomorphism $g_i : S_j(\mathcal{A}_-) \to S_j(\mathcal{A}_-)$ on generators by

$$g_i(c) = \begin{cases} c & \text{if } c \neq a_i, \\ a_i + H \circ \Phi_{i-1} \circ \partial_+(a_i) & \text{if } c = a_i \end{cases}$$

and set $\Phi_i = g_i \circ \Phi_{i-1}$. Then Φ_i is a graded isomorphism. To see that Φ_i is a chain map when restricted to $\mathcal{A}_i$ observe the following facts: $\tau \circ H = 0, \tau \circ g_i = \tau$, and $\tau \circ \Phi_i = \tau \circ \Phi_0$ for all i. Moreover, $\partial_+ a_i \in \mathcal{A}_{i-1}$ and $\tau - \mathrm{id}_{S_j(\mathcal{A}_-)} = \partial_-^s \circ H + H \circ \partial_-^s$, where in the last equation we think of $\tau\colon S_j(\mathcal{A}_-) \to S_j(\mathcal{A}_-)$ as $\tau\colon S_j(\mathcal{A}_-) \to \mathcal{A}_-$ composed with the natural inclusion.

Using these facts we compute

$$\begin{aligned} \partial_-^s g_i(a_i) =& \partial_-^s(a_i) + (\partial_-^s H)\Phi_{i-1}\partial_+(a_i) = \partial_-^s(a_i) + (H\partial_-^s + \tau + \mathrm{id})\Phi_{i-1}\partial_+(a_i) \\ =& \partial_-^s(a_i) + \tau\Phi_0\partial_+(a_i) + \Phi_{i-1}\partial_+(a_i) = \Phi_{i-1}\partial_+(a_i). \end{aligned}$$

Thus $\Phi_i \circ \partial_+(a_i) = \partial_-^s \circ g_i(a_i) = \partial_-^s \circ \Phi_i(a_i)$. Since Φ_i and Φ_{i-1} agree on $\mathcal{A}_{i-1}$ it follows that Φ_i is a chain map on $\mathcal{A}_i$. Continuing we eventually get a tame chain isomorphism $\Phi_l : \mathcal{A}_+ \to S_j(\mathcal{A}_-)$. □

The proof of Lemma 2.20 depends on the following two propositions.

Proposition 2.21. [7, Theorem 10.4] *Let $L_t, t \in I = [-\delta, \delta]$ be a generic Legendrian isotopy of type (B) with notation as above (that is, o is the degenerate Reeb chord of L_0 and the Reeb chords a and b are born as t increases past 0).*

(1) *Let $\mathcal{M}_A^0(o, \mathbf{c})$ be a moduli space of rigid holomorphic disks. Then there exist $\rho > 0$ and a local homeomorphism*

$$S\colon \mathcal{M}_A^0(o; \mathbf{c}) \times [\rho, \infty) \to \mathcal{M}_A^{(0,\delta]}(a; \mathbf{c}),$$

with the following property. If $u \in \mathcal{M}_A^0(o; \mathbf{c})$ then any disk in $\mathcal{M}_A^{(0,\delta]}(a; \mathbf{c})$ sufficiently close to the image of u is in the image of S.

(2) *Let $\mathcal{M}_A^0(c, \mathbf{d})$ be a moduli space of rigid holomorphic disks. Let $S \subset \{1, \dots, m\}$ be the subset of positions of $\mathbf{d}$ where the Reeb chord o appears (to avoid trivialities, assume $S \neq \emptyset$). Then there exists $\rho > 0$ and a local homeomorphism*

$$S'\colon \mathcal{M}_A^0(c, \mathbf{d}) \times [\rho, \infty) \to \mathcal{M}_A^{(0,\delta]}(c, \mathbf{d}_S(b)),$$

with the following property. If $u \in \mathcal{M}_0(c; \mathbf{d})$ then any disk in $\mathcal{M}_A^{(0,\delta]}(c; \mathbf{d}_S(b))$ sufficiently close to the image of u is in the image of S'.

Proposition 2.22. [7, Theorem 10.5] *Let $L_t, t \in I = [-\delta, \delta]$ be a generic isotopy of type (B). Let $\mathcal{M}_{A_1}^0(o; \mathbf{c}^1)$, ..., $\mathcal{M}_{A_r}^0(o; \mathbf{c}^r)$, and $\mathcal{M}_B^0(c; \mathbf{d})$ be moduli spaces of rigid holomorphic* *disks. Let $S \subset \{1, \dots, m\}$ be the subset of positions in* $\mathbf{d}$ *where the Reeb chord o appears and assume that S contains r elements. Then there exists $\rho > 0$ and an embedding*

$$G\colon \mathcal{M}^0_B(c;\mathbf{d}) \times \Pi_{j=1}^r \mathcal{M}^0_{A_j}(o;\mathbf{d}^j) \times [\rho,\infty) \to \mathcal{M}^{[-\delta,0)}_{B+\sum A_j}(c;\mathbf{d}_S(\mathbf{c}^1,\dots,\mathbf{c}^r)),$$

with the following property. If $v \in \mathcal{M}_0(c;\mathbf{d})$ and $u_j \in \mathcal{M}_0(o;\mathbf{c}^j)$, $j = 1,\dots,r$ then any disk in $\mathcal{M}^{[-\delta,0)}_{B+\sum A_j}(c;\mathbf{d}_S(\mathbf{c}^1,\dots,\mathbf{c}^r))$ sufficiently close to the image of $(v, u_1,\dots,u_r)$ is in the image of G.

Proof of Lemma 2.20. Equation (2.9) follows from arguments similar to those in Lemma 2.11. Specifically, one can use these arguments to show that $\partial_+ b_i = \partial_- b_i$. Then since $\partial_+ b_i \in \mathcal{B}$ and since Φ_0 is the identity on $\mathcal{B}$,

$$\Phi_0\partial_+ b_i = \partial_+ b_i = \partial_- b_i = \partial^s_- \Phi_0 b_i.$$

We also compute

$$\Phi_0\partial_+ a = \Phi_0(b + v) = e^j_2 + v + v = e^j_2 = \partial^s_- \Phi_0 a,$$

and, since $\partial_+ b$ and $\partial_+ v$ both lie in $\mathcal{B}$,

$$\Phi_0\partial_+ b = \partial_+ b, \quad \partial^s_- \Phi_0 b = \partial^s_-(e^j_1 + v) = \partial_- v = \partial_+ v.$$

Since $0 = \partial_+\partial_+ a = \partial_+ b + \partial_+ v$, we conclude that (2.9) holds.

To check (2.10), we write $\partial_+ a_i = W_1 + W_2 + W_3$, where W_1 lies in the subalgebra generated by $\{a_1,\dots,a_l,b_1,\dots,b_m\}$, where W_2 lies in the ideal generated by a and where W_3 lies in the ideal generated by b in the subalgebra generated by $\{a_1,\dots,a_l,b,b_1,\dots,b_m\}$.

Let u_t, be a family of holomorphic disks with boundary on L_t. As $t \to 0$, u_t converges to a broken disk $(u^1,\dots,u^N)$ with boundary on L_0. This together with the genericity of the type (B) isotopy implies that for $t \neq 0$ small enough there are no disks of negative formal dimension with boundary on L_t since a broken curve which is a limit of a sequence of such disks would have at least one component u^j with negative formal dimension.

Let $u_s\colon D \to \mathbb{C}^n$, $s \neq 0$ be rigid disks with boundary on L_s. If, the image $u_{-t}(\partial D)$ stays a positive distance away from o^* as $t \to 0+$ then the argument above implies that u_{-t} converges to a non-broken curve. Hence $\partial_- a_i = W_1 + W_4$ where for each rigid disk $u_{-t}\colon D \to \mathbb{C}^n$ contributing to a word in W_4 there exists points $q_{-t} \in \partial D$ such that $u_{-t}(q_{-t}) \to o^*$ as $t \to 0+$. The genericity assumption on the type (B) isotopy implies that no rigid disk with boundary on L_0 maps any boundary point to o^*, see Proposition 2.3. Hence u_{-t} must converge to a broken curve $(u^1,\dots,u^N)$ which breaks at o^*. Moreover, by genericity and (2.6), every component u^j of the broken curve must be a rigid disk with boundary on L_0. Proposition 2.22 shows that any such broken curve may be glued and Proposition 2.21 determines the pieces which we may glue. It follows that $W_4 = \hat{W}_2$ where $\hat{W}_2$ is obtained from W_2 by replacing each occurrence of b with v. Therefore,

$$\tau\Phi_0\partial_+(a_i) = \tau\Phi_0(W_1 + W_2 + W_3) = W_1 + \hat{W}_2 = \partial_-(a_i) = \tau\partial^s_-\Phi_0(a_i).$$

□

2.7. **Relations with the relative contact homology of** [12]**.** Our description of contact homology is a direct generalization of Chekanov's ideas in [5]. We now show how the above theory fits into the more general, though still developing, relative contact homology of [12].

We start with a Legendrian submanifold L in a contact manifold (M,ξ) and try to build an invariant for L. To this end, let α be a contact form for ξ and X_α its Reeb vector field. We let $\mathcal{C}$ be the set of all Reeb chords, which under certain non-degeneracy assumptions is discrete. Let $\mathcal{A}$ be the free associative non-commutative unital algebra over $\mathbb{Z}_2[H_1(L)]$ generated by $\mathcal{C}$. The algebra $\mathcal{A}$ can be given a grading using the Conley-Zehnder index (see [12]). To do this we must choose capping paths γ in L for each $c \in \mathcal{C}$ which connects its end points. Note that $c \in \mathcal{C}$, being a piece of a flow line of a vector field, comes equipped with a parameterization $c : [0,T] \to M$. For later convenience, we reparameterize c by precomposing it with $\times T \colon [0,1] \to [0,T]$.

We next wish to define a differential on $\mathcal{A}$. This is done by counting holomorphic curves in the symplectization of (M,ξ). Recall the *symplectization* of (M,ξ) is the manifold $W = M \times \mathbb{R}$ with the symplectic form $\omega = d(e^w \alpha)$ where w is the coordinate in $\mathbb{R}$. Now choose an almost complex structure J on W that is compatible with ω ($\omega(v,Jv) > 0$ if $v \neq 0$), leaves ξ invariant and exchanges X_α and $\frac{\partial}{\partial w}$. Note that $\overline{L} = L \times \mathbb{R}$ is a Lagrangian (and hence totally real) submanifold of (W,ω). Thus we may study holomorphic curves in (W,ω,J) with boundary on $\overline{L}$. Such curves must have punctures. When the Reeb field has no periodic orbits (as in our case) there can be no internal punctures, so all the punctures occur on the boundary. To describe the behavior near the punctures let $u : (D_m, \partial D_m) \to (W, \overline{L})$ be a holomorphic curve where D_m is as before. Each boundary puncture has a neighborhood that is conformal to a strip $(0,\infty) \times [0,1]$ with coordinates (s,t) such that approaching ∞ in the strip is the same as approaching p_i in the disk. If we write u using these conformal strip coordinates near p_i then we say u *tends asymptotically to a Reeb chord* $c(t)$ at $\pm\infty$ if the component of $u(s,t)$ lying in M limits to $c(t)$ as $s \to \infty$ and the component of $u(s,t)$ lying in $\mathbb{R}$ limits to $\pm\infty$ as $s \to \infty$. The map u must tend asymptotically to a Reeb chord at each boundary puncture. Some cases of this asymptotic analysis were done in [1]. For $\{a, b_1, \dots, b_m\} \subset \mathcal{C}$ we consider the moduli spaces $\mathcal{M}^s_A(a; b_1, \dots b_m)$ of holomorphic maps u as above such that: (1) at p_0, u tends asymptotically to a at $+\infty$; (2) at p_i, u tends asymptotically to b_i at $-\infty$; (3) and $\Pi_M(u(\partial D_*)) \cup_i \gamma_i$ represents the homology class A. Here the map $\Pi_M : W \to M$ is projection onto the M factor of W. We may now define a boundary map ∂ on the generators c_i of $\mathcal{A}$ (and hence on all of $\mathcal{A}$) by

$$\partial c_i = \sum \#(\mathcal{M}^s_A(c_i; b_1, \dots, b_m)) A b_1 \dots b_m,$$

where the sum is taken over all one dimensional moduli spaces and $\#$ means the modulo two count of the points in $\mathcal{M}^s_A/\mathbb{R}$. Here the $\mathbb{R}$-action is induced by a translation in the w-direction.

Though this picture of contact homology has been known for some time now, the analysis needed to rigorously define it has yet to appear. Moreover, there have been no attempts to make computations in dimensions above three. Above, by specializing to a nice – though still rich – situation, we gave a rigorous definition of contact homology for Legendrian submanifolds in $\mathbb{R}^{2n+1}$.

Recall that in our setting $(M,\alpha) = (\mathbb{R}^{2n+1}, dz - \sum_{j=1}^n y_j dx_j)$, the set of Reeb chords is naturally bijective with the double points of $\Pi_{\mathbb{C}}(L)$. Thus, clearly the algebra of this subsection is identical to the one described in Section 2.1.

We now compare the differentials. We pick the complex structure on the symplectization of $\mathbb{R}^{2n+1}$ as follows. The projection $\Pi_{\mathbb{C}} : \mathbb{R}^{2n+1} \to \mathbb{C}^n$ gives an isomorphism $d\Pi_{\mathbb{C}}$ from $\xi_x \subset T_x\mathbb{R}^{2n+1}$ to $T_{\Pi_{\mathbb{C}}(x)}\mathbb{C}^n$ and thus, via $\Pi_{\mathbb{C}}$, the standard complex structure on $\mathbb{C}^n$ induces a complex structure $E : \xi \to \xi$ on ξ. Define the complex structure J on the symplectization $\mathbb{R}^{2n+1} \times \mathbb{R}$ by $J(v) = E(v)$ if $v \in \xi$ and $J(\frac{\partial}{\partial w}) = X$. Then J is compatible with $\omega = d(e^w\alpha)$. Our moduli spaces and the ones used in the standard definition of contact homology are related as follows. If u in $\mathcal{M}^s_A(a; b_1, \dots, b_m)$ then define $p(u)$ to be the map in $\mathcal{M}_A(a; b_1, \dots, b_m)$ as $p(u) = \Pi_{\mathbb{C}} \circ \Pi_M \circ u$ and $\widetilde{p(u)} = \Pi_M \circ u | \partial \hat{D}$, where $\Pi_M : \mathbb{R}^{2n+1} \times \mathbb{R} \to \mathbb{R}^{2n+1}$, the projection from the symplectization back to the original contact manifold.

Lemma 2.23. *The map* $p : \mathcal{M}^s(a; b_1, \dots, b_m)/\mathbb{R} \to \mathcal{M}(a; b_1, \dots, b_m)$ *is a homeomorphism.*

Proof. This was proven in the three dimensional case in [15] and the proof here is similar. (For details we refer the reader to that paper but we outline the main steps.) It is clear from the definitions that p is a map between the appropriate spaces (we mod out by the $\mathbb{R}$ in $\mathcal{M}^s$ since the complex structure on the symplectization is $\mathbb{R}$-invariant and any two curves that differ by translation in $\mathbb{R}$ will clearly project to the same curve in $\mathbb{C}^n$). The only non-trivial part of this lemma is that p is invertible. To see this let $u \in \mathcal{M}^s$ be written $u = (u', z, \tau) : D_m \to \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}$. The fact that u is holomorphic for our chosen complex structure implies that z is harmonic and hence determined by its boundary data. Moreover, the holomorphicity of u also implies that τ is determined, up to translation in w-direction, by u' and z. Thus if we are given a map $u' \in \mathcal{M}$ then we can construct a z and τ for which $u = (u', z, \tau)$ will be a holomorphic map $u : D_m \to \mathbb{R}^{2n+1} \times \mathbb{R}$. If it has the appropriate behavior near the punctures then $u \in \mathcal{M}^s$. The asymptotic behavior near punctures was studied in [26]. □

3. Legendrian submanifolds

In this section, we review the Lagrangian projection and introduce the front projection, both of which are useful for the calculations of Section 4. In Sections 3.3 and 3.4, we discuss the Thurston-Bennequin invariant and the rotation class, which were the only invariants before contact homology which could distinguish Legendrian isotopy classes. Finally, we construct in Section 3.5 a useful technique for calculating the Conley-Zehnder index of Reeb chords.

3.1. **The Lagrangian projection.** Recall that for a Legendrian submanifold $L \subset \mathbb{C}^n \times \mathbb{R}$, $\Pi_{\mathbb{C}} \colon L \to \mathbb{C}^n$ is a Lagrangian immersion. Note that $L \subset \mathbb{C}^n \times \mathbb{R}$ can be recovered, up to rigid translation in the z direction, from $\Pi_{\mathbb{C}}(L)$: pick a point $p \in \Pi_{\mathbb{C}}(L)$ and choose any z coordinate for p; the z coordinate of any other point $p' \in L$ is then determined by

$$\sum_{j=1}^{n} \int_{\gamma} y_j dx_j, \tag{3.1}$$

where $\gamma = \Pi_{\mathbb{C}} \circ \Gamma$ and Γ is any path in L from p to p'. Furthermore, given any Lagrangian immersion f into $\mathbb{C}^n$ with isolated double points, if the integral in (3.1) is independent of the path $\gamma = f \circ \Gamma$ then we obtain a Legendrian immersion $\tilde{f}$ into $\mathbb{R}^{2n+1}$ which is an embedding provided the integral is not zero for paths connecting double points.

A Lagrangian immersion $f : L \to \mathbb{C}^n$ is *exact* if $f^*(\sum_{j=1}^n y_j dx_j)$ is exact and, in this case, (3.1) is independent of γ. In particular, if $H^1(L) = 0$ then all Lagrangian immersions of L are exact. Also note that any Lagrangian regular homotopy $f_t \colon L \to \mathbb{C}^n$ of exact Lagrangian immersions will lift to a Legendrian regular homotopy $\tilde{f}_t \colon L \to \mathbb{C}^n \times \mathbb{R}$.

Example 3.1. Consider $S^n = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : |x|^2 + y^2 = 1\}$ and define $f \colon S^n \to \mathbb{C}^n$ as

$$f(x, y) : S^n \to \mathbb{C}^n : (x, y) \mapsto ((1 + iy)x).$$

Then f is an exact Lagrangian immersion, with one transverse double point, which lifts to a Legendrian embedding into $\mathbb{R}^{2n+1}$. (When $n = 1$ the image of f is a figure eight in the plane with a double point at the origin.)

3.2. **The front projection.** The *front projection* projects out the y_j's:

$$\Pi_F : \mathbb{R}^{2n+1} \to \mathbb{R}^{n+1} : (x_1, y_1, \dots, x_n, y_n, z) \mapsto (x_1, \dots, x_n, z).$$

If $L \subset \mathbb{R}^{2n+1}$ is a Legendrian submanifold then $\Pi_F(L) \subset \mathbb{R}^{n+1}$ is its *front* which is a codimension one subvariety of $\mathbb{R}^{n+1}$. The front has certain singularities. More precisely, for generic L, the set of singular points of Π_F is a hypersurface $\Sigma \subset L$ which is smooth outside a set of codimension 3 in L, and which contains a subset $\Sigma' \subset \Sigma$ of codimension 2 in L with the following property. If p is a smooth point in $\Sigma \setminus \Sigma'$ then there are local coordinates $(x_1, \dots, x_n)$

around p in L, $(\xi_1, \ldots, \xi_n, z)$ around $\Pi_F(p)$ in $\mathbb{R}^{n+1}$, and constants $\delta = \pm 1$, $\beta, \alpha_2, \ldots, \alpha_n$ such that

$$\Pi_F(x_1, \ldots, x_n) = (x_1^2, x_2, \ldots, x_n, \delta x_1^3 + \beta x_1^2 + \alpha_2 x_2 + \cdots + \alpha_n x_n). \tag{3.2}$$

For a reference, see [2] Lemma on page 115. The image under Π_F of the set of smooth points in $\Sigma \setminus \Sigma'$ will be called the *cusp edge* of the front $\Pi_F(L)$. See Figure 3.

Any map $L \to \mathbb{R}^{n+1}$ with singularities of a generic front can be lifted (in a unique way) to a Legendrian immersion. (The singularities of such a map allow us to solve for the y_i-coordinates from the equation $dz = \sum_{i=1}^n y_i dx_i$ and the solutions give an immersion.) In particular, at a smooth point of the front the y_i-coordinate equals the slope of the tangent plane to the front in the $x_i z$-plane.

The Legendrian immersion of a generic front is an embedding and a double point of a Legendrian immersion correspond to a double point of the front with parallel tangent planes. Also note that that $\Pi_F(L)$ cannot have tangent planes containing the z-direction. For a more thorough discussion of singularities occurring in front projections see [2].

FIGURE 3. Front projection of Example 3.1 in dimension 3, on the left, and 5, on the right.

3.3. **The rotation class.** Let (M, ξ) be a contact $(2n+1)$-manifold with a contact form α. That is, α is a 1-form on M with $\xi = \mathrm{Ker}(\alpha)$. The complete non-integrability condition on ξ implies $\alpha \wedge (d\alpha)^n \neq 0$ which in turn implies that for any $p \in M$, $d\alpha_p|\xi_p$ is a symplectic form on $\xi_p \subset T_pM$.

Let $f : L \to (M, \xi)$, be a Legendrian immersion. Then the image of $df_x : T_xL \to T_{f(x)}M$ is a Lagrangian plane in $\xi_{f(x)}$. Pick any complex structure J on ξ which is compatible with its symplectic structure. Then the complexification of df, $df_{\mathbb{C}} : TL \otimes \mathbb{C} \to \xi$ is a fiberwise bundle isomorphism. The homotopy class of $(f, df_{\mathbb{C}})$ in the space of complex fiberwise isomorphisms $TL \otimes \mathbb{C} \to \xi$ is called the *rotation class* of f and is denoted $r(f)$ (or $r(L)$ if $L \subset M$ is a Legendrian submanifold embedded into M by the inclusion). The h-principle for Legendrian immersions [20] implies that $r(f)$ is a complete invariant for f up to regular homotopy through Legendrian immersions.

When the contact manifold under consideration is $\mathbb{R}^{2n+1}$ with the standard contact structure we may further illuminate the definition of $r(f)$. Let $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ be coordinates on $\mathbb{R}^{2n+1}$ as in Section 2.1. If $J \colon \xi_{(x,y,z)} \to \xi_{(x,y,z)}$ is the complex structure defined by $J(\partial_{x_j} + y_j \partial_z) = \partial_{y_j}$, $J(\partial_{y_j}) = -(\partial_{x_j} + y_j \partial_z)$, for $j = 1, \ldots, n$ then the Lagrangian projection $\Pi_{\mathbb{C}} : \mathbb{R}^{2n+1} \to \mathbb{C}^n$ gives a complex isomorphism from (ξ, J) to the trivial bundle with fiber $\mathbb{C}^n$. Thus we may think of $df_{\mathbb{C}}$ as a trivialization $TL \otimes \mathbb{C} \to \mathbb{C}^n$. Moreover, we can choose Hermitian metrics on $TL \otimes \mathbb{C}$ and on $\mathbb{C}^n$ so that $df_{\mathbb{C}}$ is a unitary map. Then f gives rise to an element in $U(TL \otimes \mathbb{C}, \mathbb{C}^n)$. One may check that the group of continuous maps $C(L, U(n))$ acts freely and transitively on $U(TL \otimes \mathbb{C}, \mathbb{C}^n)$ and thus $\pi_0(U(TL \otimes \mathbb{C}, \mathbb{C}^n))$ is in one to one correspondence with $[L, U(n)]$. Thus we may think of $r(f)$ as an element in $[L, U(n)]$.

We note that when $L = S^n$ then

$$r(f) \in \pi_n(U(n)) \approx \begin{cases} \mathbb{Z}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Thus for spheres we will refer to $r(f)$ as the *rotation number*.

3.4. **The Thurston-Bennequin invariant.** Given an orientable connected Legendrian submanifold L in an oriented contact $(2n+1)$-manifold (M,ξ) we define an invariant, called the Thurston–Bennequin invariant of L, describing how the contact structure "twists about L." The invariant was originally conceived by Thurston and Bennequin [4] when $n = 1$ and generalized to higher dimensions by Tabachnikov [32]. Here we only define the Thurston–Bennequin invariant when L is homologically trivial in M (which for $M = \mathbb{R}^{2n+1}$ poses no additional constraints).

Pick an orientation on L. Let X be a Reeb vector field for ξ and push L slightly off of itself along X to get another oriented submanifold L' disjoint from L. The *Thurston–Bennequin invariant* of L is the linking number

$$\mathrm{tb}(L) = \mathrm{lk}(L, L'). \tag{3.3}$$

Note that $\mathrm{tb}(L)$ is independent of the choice of orientation on L since changing it changes also the orientation of L'. The linking number is computed as follows. Pick any $(n+1)$-chain C in M such that $\partial C = L$ then $\mathrm{lk}(L, L')$ equals the algebraic intersection number of C with L'.

For a chord generic Legendrian submanifold $L \subset \mathbb{R}^{2n+1}$, $\mathrm{tb}(L)$ can be computed as follows. Let c be a Reeb chord of L with end points a and b, $z(a) > z(b)$. Let $V_a = d\Pi_{\mathbb{C}}(T_aL)$ and $V_b = d\Pi_{\mathbb{C}}(T_bL)$. Given an orientation on L these are oriented n-dimensional transverse subspaces in $\mathbb{C}^n$. If the orientation of $V_a \oplus V_b$ agrees with that of $\mathbb{C}^n$ then we say the sign, $\mathrm{sign}(c)$, of c is $+1$ otherwise we say it is -1. Then

$$\mathrm{tb}(L) = \sum_c \mathrm{sign}(c), \tag{3.4}$$

where the sum is taken over all Reeb chords c of L. To verify this formula, use the Reeb-vector field ∂_z to shift L off itself and pick the cycle C as the cone over L through some point with a very large negative z-coordinate.

Note that the parity of the number of double points of any generic immersion of an n-manifold into $\mathbb{C}^n$ depends only on its regular homotopy class [33]. Thus the parity of $\mathrm{tb}(L)$ is determined by the rotation class $r(L)$. Some interesting facts [8] concerning the Thurston-Bennequin invariant are summarized in

Proposition 3.2. *Let L be a Legendrian submanifold in standard contact $(2n+1)$-space.*

1. *If $n > 1$ is odd, then for any $k \in \mathbb{Z}$ we can find, C^0 close to L, a Legendrian submanifold L' smoothly isotopic and Legendrian regularly homotopic to L with $\mathrm{tb}(L') = 2k$.*
2. *If n is even, then $\mathrm{tb}(L) = (-1)^{\frac{n}{2}+1}\frac{1}{2}\chi(L)$.*

The ideas associated with (1) are discussed below in Proposition 4.5. For (2), note that if $n = 2k$ is even then the sign of a double point c is independent of the ordering of the subspaces V_a and V_b and in this case $\mathrm{tb}(L)$ equals Whitney's invariant [33] for immersions of orientable $2k$-manifolds into oriented $\mathbb{R}^{4k}$ which in turn equals $-\frac{1}{2}\chi(\nu)$, where ν is the oriented normal bundle of the immersion [23]. Since the immersion is Lagrangian into $\mathbb{C}^n$ its normal bundle is isomorphic to the tangent bundle TL of L (via multiplication with i) and as an oriented bundle it is isomorphic to TL with orientation multiplied by $(-1)^{\frac{n}{2}}$.

If $n = 1$ the situation is much more interesting. In this case there are two types of contact structures: tight and overtwisted. If the contact structure is overtwisted then the above proposition is still true, but if the contact structure is tight (as is standard contact 3-space) then $\mathrm{tb}(L) \leq \chi(L) - |r(L)|$. There are other interesting bounds on $\mathrm{tb}(L)$ in a tight contact structure, see [19, 27].

The Thurston-Bennequin invariant of a chord generic Legendrian submanifold can also be calculated in terms of Conley-Zehnder indices of Reeb chords. Recall $\mathcal{C}$ is the set of Reeb chords of L.

Proposition 3.3. *If $L \subset \mathbb{R}^{2n+1}$ is an orientable chord generic Legendrian submanifold then*

$$\mathrm{tb}(L) = (-1)^{\frac{(n-2)(n-1)}{2}} \sum_{c \in \mathcal{C}} (-1)^{|c|}.$$

Proof. Recall from (3.4) that $\mathrm{tb}(L)$ can be computed by summing $\mathrm{sign}(c)$ over all Reeb chords c, where $\mathrm{sign}(c)$ is the oriented intersection between the upper and lower sheets of $\Pi_{\mathbb{C}}(L)$ at c^*. So to prove the proposition we only need to check that $\mathrm{sign}(c) = (-1)^{\frac{1}{2}(n^2+n+2)}(-1)^{|c|}$. This will be done after the proof of Lemma 3.4. □

3.5. **Index computations in the front projection.** Though it was easier to define contact homology using the complex projection of a Legendrian submanifold it is frequently easier to construct Legendrian submanifolds using the front projection. In preparation for our examples below we discuss Reeb chords and their Conley-Zehnder indices in the front projection.

If $L \subset \mathbb{R}^{2n+1}$ is a Legendrian submanifold then the Reeb chords of L appears in the front projection as vertical line segment (i.e. a line segment in the z-direction) connecting two points of $\Pi_F(L)$ with parallel tangent planes. (See Section 3.2 and note that L may be perturbed so that the Reeb chords as seen in $\Pi_F(L)$ do not have end points lying on singularities of $\Pi_F(L)$.)

A generic arc γ in $\Pi_F(L)$ connecting two such points a, b intersects the cusp edges of $\Pi_F(L)$ transversely and meets no other singularities of $\Pi_F(L)$ (it might also meet double points of the front projection but "singularities" refers to non-immersion parts of $\Pi_F(L)$). Let p be a point on a cusp edge where γ intersects it. Note that $\Pi_F(L)$ has a well defined tangent space at p. Choose a line l orthogonal to this tangent space. Then, since the tangent space does not contain the vertical direction, orthogonal projection to the vertical direction at p gives a linear isomorphism from l to the z-axis through p. Thus the z-axis induces an orientation on l. Let γ_p be a small part of γ around p and let $h_p \colon \gamma_p \to l$ be orthogonal projection. The orientation of γ induces one on γ_p and we say that the intersection point is an *up- (down-) cusp* if h_p is increasing (decreasing) around p.

Let c be a Reeb chord of L with end points a and b, $z(a) > z(b)$. Let q be the intersection point of the vertical line containing $c \subset \mathbb{R}^{n+1}$ and $\{z = 0\} \subset \mathbb{R}^{n+1}$. Small parts of $\Pi_F(L)$ around a and b, respectively, can be viewed as the graphs of functions h_a and h_b from a neighborhood of q in $\mathbb{R}^n$ to $\mathbb{R}$ (the z-axis). Let $h_{ab} = h_a - h_b$. Since the tangent planes of $\Pi_F(L)$ at a and b are parallel, the differential of h_{ab} vanishes at q. If the double point c^* of $\Pi_{\mathbb{C}}(L)$ corresponding to c is transverse then the Hessian $d^2 h_{ab}$ is a non-degenerate quadratic form (see the proof below). Let $\mathrm{Index}(d^2 h_{ab})$ denote its number of negative eigenvalues.

Lemma 3.4. *If γ is a generic path in $\Pi_F(L)$ connecting a to b then*

$$\nu_\gamma(c) = D(\gamma) - U(\gamma) + \mathrm{Index}(d^2 h_{ab}),$$

where $D(\gamma)$ and $U(\gamma)$ is the number of down- and up-cusps of γ, respectively.

Proof. To compute the Maslov index as described in Section 2.3 we use the Lagrangian reference space $x = 0$ in $\mathbb{R}^{2n}$ (that is, the subspace $\mathrm{Span}(\partial_{y_1}, \dots, \partial_{y_n})$) with Lagrangian complement $y = 0$ ($\mathrm{Span}(\partial_{x_1}, \dots, \partial_{x_n})$).

We must compute the Maslov index of the loop $\Gamma * \lambda(V_b, V_a)$ where V_b and V_a are the Lagrangian subspaces $d\Pi_{\mathbb{C}}(T_b L)$ and $d\Pi_{\mathbb{C}}(T_a L)$ and $\Gamma(t)$ is the path of Lagrangian subspaces induced from γ. We first note that $\Gamma(t)$ intersects our reference space transversely (in 0) if

$\gamma(t)$ is a smooth point of $\Pi_F(L)$, since near such points $\Pi_F(L)$ can be thought of as a graph of a function over some open set in x-space (i.e. $\{z = 0\} \subset \mathbb{R}^{n+1}$). Thus, for generic γ, the only contributions to the Maslov index come from cusp-edge intersections and the path $\lambda(V_b, V_a)$.

We first consider the contribution from $\lambda(V_b, V_a)$. There exists orthonormal coordinates $u = (u_1, \dots, u_n)$ in x-space so that in these coordinates

$$d^2 h_{ab} = \mathrm{Diag}(\lambda_1, \dots, \lambda_n).$$

We use coordinates (u, v) on $\mathbb{R}^{2n} = \mathbb{C}^n$ where u is as above, $\partial_j = \partial_{u_j}$, and $\partial_{v_j} = i\partial_j$ ($i = \sqrt{-1}$). In these coordinates our symplectic form is simply $\omega = \sum_{j=1}^n du_j \wedge dv_j$, and our two Lagrangian spaces are given by $V_a = \mathrm{Span}_{j=1}^n(\partial_j + id^2h_a\partial_j)$, $V_b = \mathrm{Span}_{j=1}^n(\partial_j + id^2h_b\partial_j)$. One easily computes

$$\omega(\partial_j + id^2h_b\partial_j, \partial_j + id^2h_a\partial_j) = \omega(\partial_j, id^2h_{ab}\partial_j) = \lambda_j.$$

Moreover, let

$$W_j = \mathrm{Span}(\partial_j + id^2h_a\partial_j, \partial_j + id^2h_b\partial_j) = \mathrm{Span}(\partial_j + id^2h_a\partial_j, i\partial_j) = \mathrm{Span}(\partial_j + id^2h_b\partial_j, i\partial_j),$$

then W_j and W_k are symplectically orthogonal, $\omega(W_j, W_k) = 0$, for $j \neq k$.

Let $v_a(j)$ be a unit vector in direction $\partial_j + id^2h_a\partial_j$ and similarly for $v_b(j)$. Define the almost complex structure I as follows

$$I(v_b(j)) = \mathrm{Sign}(\lambda_j)v_a(j),$$

and note that it is compatible with ω. Then $e^{sI}v_b(j)$, $0 \leq s \leq \frac{\pi}{2}$, intersects the line in direction $i\partial_j$ if and only if $\lambda_j < 0$ and does so in the positive direction.

It follows that the contribution of $e^{sI}V_b$, $0 \leq s \leq \frac{\pi}{2}$ (i.e. $\lambda(V_b, V_a)$) to the Conley-Zehnder index is $\mathrm{Index}(d^2h_{ab})$.

Second we consider cusp-edge intersections: at a cusp-edge intersection p (which we take to be the origin) there are coordinates $u = (u_1, \dots, u_n)$ such that the front locally around $p = 0$ is given by $u \mapsto (x(u), z(u))$, where

$$x(u) = (u_1^2, u_2, \dots, u_n), \quad z(u) = \delta u_1^3 + \beta u_1^2 + \alpha_2 u_2 + \dots + \alpha_n u_n,$$

where δ is ± 1, and β and α_j are real constants. We can assume the oriented curve γ is given by $u(t) = (\epsilon t, 0, \dots, 0)$, where $\epsilon = \pm 1$. If we take the coorienting line l to be in the direction of the vector

$$v(p) = (-\beta, -\alpha_2, \dots, -\alpha_n, 1).$$

then the function h_p is

$$h_p(t) = \delta\epsilon^3 t^3,$$

and we have an up-cusp if $\delta\epsilon > 0$ and a down cusp if $\delta\epsilon < 0$.

The curve $\Gamma(t)$ of Lagrangian tangent planes of $\Pi_{\mathbb{C}}(L^n)$ along γ is given by

$$\Gamma(t) = \mathrm{Span}(2\epsilon t\partial_1 + i\frac{3\delta}{2}\partial_1, \partial_2, \dots, \partial_n).$$

The plane $\Gamma(0)$ intersects our reference plane at $t = 0$ along the line in direction $i\partial_1$. As described in Section 2.2 the sign of the intersection is given by the sign of

$$\frac{d}{dt}\omega(i\tfrac{3\delta}{2\sigma}\partial_1, 2\sigma\epsilon\partial_1) = -3\delta\epsilon$$

Thus, we get negative signs at up-cusps and positive at down-cusps. The lemma follows. □

Completion of Proof of Proposition 3.3. We say the orientations on two hyperplanes transverse the the z-axis in $\mathbb{R}^{n+1}$ *agree* if their projection to $\{z=0\}\subset\mathbb{R}^{n+1}$ induce the same orientation on this n-dimensional subspace. Let c be a Reeb chord of L and let a and b denote its end points on $\Pi_F(L)$. If the orientations on $T_a\Pi_F(L)$ and $T_b\Pi_F(L)$ agree then the above proof shows that the bases

$$\begin{aligned}&\Big(\partial_1+id^2h_a\partial_1,\dots,\partial_n+id^2h_a\partial_n,\partial_1+id^2h_b\partial_1,\dots,\partial_n+id^2h_b\partial_n\Big)\simeq\\ &\Big(id^2h_{ab}\partial_1,\dots,id^2h_{ab}\partial_n,\partial_1,\dots,\partial_n\Big),\end{aligned}\tag{3.5}$$

provide oriented bases for $d\Pi_{\mathbb{C}}(T_aL)\oplus d\Pi_{\mathbb{C}}(T_bL)$. Note that the standard orientation of $\mathbb{C}^n$ is given by the positive basis $(\partial_1,i\partial_1,\dots,\partial_n,i\partial_n)$ which after multiplication with $(-1)^{\frac{n(n+1)}{2}}$ agrees with the orientation given by the basis $(i\partial_1,\dots,i\partial_n,\partial_1,\dots,\partial_n)$. Thus

$$\operatorname{sign} c=(-1)^{\frac{n(n+1)}{2}}(-1)^{\operatorname{Index}(d^2(h_{ab}))}.$$

However, the orientations of $T_a\Pi_F(L)$ and $T_b\Pi_F(L)$ do not always agree. Let γ be the path in L connecting a to b. The orientations on $T_{\gamma(t)}(\Pi_F(L))$ do not change as long as γ does not pass a cusp edge. It follows from the local model for a cusp edge that each time γ transversely crosses a cusp edge the orientation on $T_{\gamma(t)}(\Pi_F(L))$ changes. Thus $\operatorname{sign} c=(-1)^{\frac{n(n+1)}{2}}(-1)^{D(\gamma)+U(\gamma)}(-1)^{\operatorname{Index}(d^2h_{ab})}=(-1)^{\frac{1}{2}(n^2+n+2)}(-1)^{|c|}$ as we needed to show. □

4. Examples and constructions

Before describing our examples, we discuss the linearized contact homology in Section 4.1. This is an invariant of Legendrian submanifolds derived from the DGA. Its main advantage over contact homology it that it is easier to compute. In Section 4.2, we do several simple computations of contact homology. In Sections 4.3 and 4.4, we describe two constructions: *stabilization* and *front spinning.* In these subsections, we construct infinite families of pairwise non-isotopic Legendrian n-spheres, n-tori and surfaces which are indistinguishable by the classical invariants. In Section 4.6, we summarize the information gleaned from our examples and discuss what this says about Legendrian submanifolds in general.

4.1. **Linearized homology.** To distinguish Legendrian submanifolds using contact homology one must find computable invariants of stable tame isomorphism classes of DGA's. We use an idea of Chekanov [5] to "linearize" the homology of such algebras. To keep the discussion simple we will only consider algebras generated over $\mathbb{Z}_2$ and not $\mathbb{Z}_2[H_1(L)]$.

Let $\mathcal{A}$ be an algebra generated by $\{c_1,\dots,c_m\}$. For $j=0,1,2,\dots$ let $\mathcal{A}_j$ denote the ideal of $\mathcal{A}$ generated by all words $\mathbf{c}$ in the generators with $l(\mathbf{c})\ge j$. A differential $\partial\colon\mathcal{A}\to\mathcal{A}$ is called *augmented* if $\partial(A_1)\subset A_1$ (in other words if ∂c_j does not contain 1 for any j). If $(\mathcal{A},\partial)$ is augmented then $\partial(A_j)\subset A_j$ for all j. A DGA $(\mathcal{A},\partial)$ is called *good* if its differential is augmented.

Let $(\mathcal{A},\partial)$ be a DGA with generators $\{c_1,\dots,c_m\}$ and consider the vector space $\mathcal{V}=\mathcal{A}_1/\mathcal{A}_2$ over $\mathbb{Z}_2$. If $(\mathcal{A},\partial)$ is good then $\partial\colon\mathcal{A}\to\mathcal{A}$ induces a differential $\partial_1\colon\mathcal{V}\to\mathcal{V}$. Note that $\{c_1,\dots,c_m\}$ gives a basis in $\mathcal{V}$ and that in this basis ∂_1c_j equals the part of ∂c_j which is linear in the generators. We define the *linearized homology* of a $(\mathcal{A},\partial)$ as

$$\operatorname{Ker}(\partial_1)/\operatorname{Im}(\partial_1),$$

which is a graded vector space over $\mathbb{Z}_2$.

We want to apply this construction to DGA's associated to Legendrian isotopy classes. Let $L\subset\mathbb{R}^{2n+1}$ be an admissible Legendrian submanifold with algebra $(\mathcal{A}(L),\partial)$ generated by $\{c_1,\dots,c_m\}$. Let G be the set of tame isomorphisms of $\mathcal{A}(L)$ and for $g\in G$ let $\partial^g\colon\mathcal{A}(L)\to\mathcal{A}(L)$ be $\partial^g=g\partial g^{-1}$. We define the *linearized contact homology of* L, $HLC_*(\mathbb{R}^{2n+1},L)$ to be the set of isomorphism classes of linearized homologies of $(\mathcal{A}, \partial^g)$, where $g \in G$ is such that $(\mathcal{A}, \partial^g)$ is good. (Note that this set may be empty.) Define $G_0 \subset G$ to be the subgroup of tame isomorphisms g_0 such that $g_0(c_j) = c_j + a_j$ for all j, where $a_j = 0$ or $a_j = 1$. Note that $a_j = 0$ if $|c_j| \neq 0$ since g_0 is graded and that $G_0 \approx \mathbb{Z}_2^k$, where k is the number of generators of $\mathcal{A}$ of degree 0.

Lemma 4.1. *If $L_t \subset \mathbb{R}^{2n+1}$ is a Legendrian isotopy between admissible Legendrian submanifolds then $HLC_*(\mathbb{R}^{2n+1}, L_0)$ is isomorphic to $HLC_*(\mathbb{R}^{2n+1}, L_1)$. Moreover, if $L \subset \mathbb{R}^{2n+1}$ is an admissible Legendrian submanifold then $HLC_*(\mathbb{R}^{2n+1}, L)$ is equal to the set of isomorphism classes of linearized homologies of $(\mathcal{A}, d^{g_0})$, where $g_0 \in G_0$ is such that $(\mathcal{A}, \partial^{g_0})$ is good.*

Proof. The first statement follows from the observation that the stabilization $(S_j(\mathcal{A}), \partial)$ of a good DGA $(\mathcal{A}, \partial)$ is good and that the linearized homologies of $(S_j(\mathcal{A}), \partial)$ and $(\mathcal{A}, \partial)$ are isomorphic. The second statement is proved in [5]. □

Let $L \subset \mathbb{R}^{2n+1}$ be an admissible Legendrian submanifold. Note that if $\mathcal{A}(L)$ has no generator of degree 1 then $(\mathcal{A}, \partial)$ is automatically good and if $\mathcal{A}$ has no generator of degree 0 then G_0 contains only the identity element. If the set $HLC_*(\mathbb{R}^{2n+1}, L)$ contains only one element we will sometimes below identify this set with its only element.

4.2. **Examples.** In this subsection we describe several relatively simple examples in which the contact homology is easy to compute and defer more complicated computations to the following subsections.

Example 4.2. The simplest example in all dimensions is L_0 described in Example 3.1, with a single Reeb chord c. Using Lemma 3.4 and the fact that the difference of the z-coordinates at the end points of c is a local maximum we find $|c| = n$. So $\mathcal{A}(L_0) = \langle c \rangle$ and the differential is $\partial c = 0$, showing (if $n > 1$) the contact homology is

$$HC_k(\mathbb{R}^{2n+1}, L_0) = \begin{cases} 0, & k \not\equiv 0 \mod n, \text{ or } k < 0, \\ \mathbb{Z}_2, & \text{otherwise.} \end{cases}$$

If $n = 1$ this is still true but $\mathcal{M}(c; \emptyset)$ is not empty (it contains two elements [5]).

Example 4.3. Generalizing Example 4.2 above we can consider the Legendrian sphere L' in $\mathbb{R}^{2n+1}$ with 3 cusp edges in its front projection. See Figure 4. If one draws the pictures with

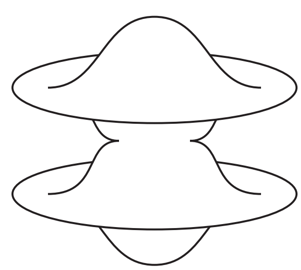

FIGURE 4. The sphere L' with 3 cusps.

an $SO(n)$ symmetry about the z-axis then there will be one Reeb chord running from the top of the sphere to the bottom, call it c and a $(n-1)$-spheres worth of Reeb chords. Perturbing the symmetric picture slightly yields two Reeb chords a, b in place of the spheres worth in the symmetric picture. The gradings are

$$|c| = n + 2, \quad |a| = 1, \quad |b| = n.$$

It is clear that whatever the contact homology of L' is, it is different from that in the example above. When $n > 2$ (respectively $n = 2$) there are two (respectively three) possibilities for the boundary map. Thus for n even we have examples of non-isotopic Legendrian spheres with the same classical invariants.

Given two Legendrian submanifolds K and K' we can form their "(cusp) connected sum" as follows: isotope K and K' so that their fronts are separated by a hyperplane in $\mathbb{R}^{n+1}$ containing the z-direction and let c be an arc beginning at a cusp edge of K and ending at a cusp edge of K' and parameterized by $s \in [-1, 1]$. Take a neighborhood N of c whose vertical cross sections consist of round balls whose radii vary with s and have exactly one minimum at $s = 0$ and no other critical points. Introducing cusps along N as indicated in Figure 5. Define

FIGURE 5. The neighborhood of c and its deformation into the front projection of a Legendrian tube.

the "connected sum" $K\#K'$ to be the Legendrian submanifold obtained from the joining of $K \setminus (K \cap N), K' \setminus (K' \cap N)$ and ∂N. Note this operation might depend on the cusp edges one chooses on K and K' but we will make this choice explicit in our examples. In dimension 3 it can be shown that the connected sum of two knots is well defined [5, 14]. It would be interesting to understand this operation better in higher dimensions. See Remark 4.28 below.

Lemma 4.4. *Let $\mathcal{C}$ and $\mathcal{C}'$ be the sets of Reeb chords of K and K', respectively, and let $|\cdot|_K$, $|\cdot|_{K'}$, and $|\cdot|$ denote grading in $\mathcal{A}(K)$, $\mathcal{A}(K')$, and $\mathcal{A}(K\#K')$, respectively. It is possible to perform the connected sum so that the set of Reeb chords of $K\#K'$ is $\mathcal{C} \cup \mathcal{C}' \cup \{h\}$ and so that the following holds.*

1. *If $c \in \mathcal{C}$ then $|c|_K = |c|$, if $c' \in \mathcal{C}'$ then $|c|_{K'} = |c|$, and $|h| = n - 1$.*
2. *$\partial h = 0$.*
3. *If $\mathcal{A}_K$ and $\mathcal{A}_{K'}$ denote the subalgebras of $\mathcal{A}(K\#K')$ generated by $\mathcal{C} \cup \{h\}$ and $\mathcal{C}' \cup \{h\}$, respectively, then $\partial(\mathcal{A}_K) \subset \mathcal{A}_K$ and $\partial(\mathcal{A}_{K'}) \subset \mathcal{A}_{K'}$.*
4. *If $c \in \mathcal{C}$ then $\partial c \in \mathcal{A}(K\#K')_1$ if and only if $\partial_K c \in \mathcal{A}(K)_1$ and similarly for $c' \in \mathcal{C}'$. (In other words, the constant part of ∂c ($\partial c'$) does not change after the connected summation.)*

Proof. We may assume that K and K' are on opposite sides of the hyperplane $\{x_1 = 0\}$ and there is a unique point p, respectively p', on a cusp edge of K, respectively K', that is closest to K', respectively K. We may further assume that all the coordinates but the x_1 coordinate of p and p' agree. Define $K\#K'$ using c, the obvious horizontal arc connecting p and p. It is now clear that all the Reeb chords in K and K' are in $K\#K'$ and there is exactly one extra chord h, coming from the minimum in the neighborhood N of c. It is also clear that the gradings of the inherited chords are unchanged and that $|h| = n - 1$.

Denote $z_j = x_j + iy_j$. The image of a holomorphic disk $u : D \to \mathbb{C}^n$ with positive puncture at h^* must lie in the complex hyperplane $\{z_1 = 0\}$. To see this notice that the projection of $K\#K'$ onto the z_1-plane is as shown in Figure 6. Let u_1 be the composition of u with this projection. If u_1 is not constant then $u(\partial D)$ must lie in the shaded region in Figure 6. Thus the corner at h^* (note h^* projects to 0 in this figure) must be a negative puncture. Since any holomorphic disk with positive puncture at h^* must lie entirely in the hyperplane $\{z_1 = 0\}$ it cannot have any negative punctures. Thus ∂h has only a constant part. For $n > 2$ this implies $\partial h = 0$ immediately. If $n = 2$ then $\partial h = 0$ since in this case, there are exactly two

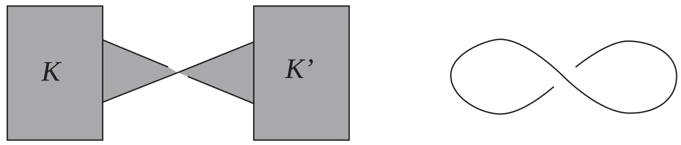

FIGURE 6. The projection onto the z_1-plane (left). The intersection with the z_2-plane (right).

holomorphic disks in the z_2-plane, see Figure 6, and Proposition 2.4 implies both these disks contribute to the boundary map.

To see (3) consider the projection of $K\#K'$ onto the z_1-plane, see Figure 6. If a holomorphic disk D intersected the projection of K and K' then it would intersect the y_1-axis in a closed interval, with non-trivial interior, containing the origin. This contradicts the maximum principle since the intersection of the boundary of D with the y_1-axis can contain only the origin.

For the last statement, consider Reeb chords in K, those in K' can be handled in exactly the same way. We use Proposisiton 2.3 which implies that we may choose the points p so that no rigid holomorphic disk $u\colon D \to \mathbb{C}^n$ with boundary on K maps any point in ∂D to p. Since the space of rigid disks is a compact 0 manifold there are only finitely many rigid disks, $u_1, \dots, u_r$, say. Since each u_k is continuous on the boundary ∂D we find that $u_1(\partial D) \cup \dots \cup u_r(\partial D)$ stays a positive distance d away from p. Consider the ball $B(p, \frac{1}{2}d)$ and use a tube attached inside $B(p, \frac{1}{4}d)$ for the connected sum. If $c \in \mathcal{C}$ and v is a rigid holomorphic disk with boundary on $K\#K'$, with positive puncture at c, no other punctures, and such that the image $v(\partial D)$ is disjoint from $\partial B(p, \frac{1}{2}d)$ then v is also a disk with boundary on K and hence $v = u_j$ for some j.

Since no holomorphic disk with boundary on $K\#K'$ which touches a point in K can pass the hyperplane $\{x_1 = 0\} \subset \mathbb{C}^n$ it will also represent a disk on the connected sum $K\#L_0$, where L_0 is a small standard sphere. Pick a generic Legendrian isotopy K_t, $0 \le t \le 1$ of $K\#L_0$ to K which is supported in $(K \cap B(p, \frac{1}{4}d))\#L_0$. Then either there exists $t < 1$ such that all rigid disks v on K_t for $t > 0$ satisfies $v(\partial D) \cap (B(p, \frac{1}{2}d) \setminus B(p, \frac{1}{4}d)) = \emptyset$ or there exists a sequence of rigid disks v_j with boundary on L_{t_j}, $t_j \to 1$ as $j \to \infty$ such that $v_j(\partial D) \cap (B(p, \frac{1}{2}d) \setminus B(p, \frac{1}{4}d)) \neq \emptyset$. In the first case the lemma follows from the observation above. We show the second case cannot appear: by Gromov compactness, the sequence v_j has a subsequence which converges to a broken disk $(v^1, \dots, v^N)$ with boundary on K. Since K is generic there are no disks with negative formal dimension and all components of $(v^1, \dots, v^N)$ must be rigid. But since v_j is rigid the broken disk must in fact be unbroken by (2.6). Thus we find a rigid disk v^1 with boundary on K such that $v^1(\partial D) \cap B(p, d) \neq \emptyset$ contradicting our choice of p. The lemma follows. □

4.3. **Stabilization and the proof of Theorem 1.1.** In this subsection we describe a general construction that can be applied to Legendrian submanifolds called *stabilization*. Using the stabilization technique, we prove Theorem 1.1. We begin with a model situation.

In $\mathbb{R}^{n+1}$ consider two unit balls F and E in the hyperplanes $\{z = 0\}$ and $\{z = 1\}$, respectively. Let M be a k-manifold embedded in F. Let N be a regular ϵ-neighborhood of M in F for some positive $\epsilon \ll 1$. Deform F to F' by pushing M up to $z = \epsilon$ and deform N so that the z-coordinate of $p \in N$ is $\epsilon - \operatorname{dist}(p, M)$. Note that are many Reeb chords, one for each point in M and $F \setminus N$. To deform this into a generic picture choose a Morse function $f : M \to [0, 1]$ and $g : \overline{(F \setminus N)} \to [0, 1]$ such that $g^{-1}(1) = \partial F$ and $g^{-1}(0) = \partial N$. (It is important to notice that we may, if we wish, modify the boundary conditions on $g|_{\partial F}$ depending on our circumstances.) Take a positive $\delta \ll \epsilon$ and further deform F' by adding

$\delta f(p)$ to the z coordinate of points in M and subtracting $\delta g(p)$ from the z coordinate of points in $F \setminus N$. The result is a generic pair of Lagrangian disks F' and E with one Reeb chord for each critical point of f and g. Define F'' as we defined F' but begin by dragging M up to $z = 1 + \epsilon$ (instead of $z = \epsilon$ as we did for F').

Now if $\Pi_F(L)$ is the front projection of a Legendrian submanifold L and there are two horizontal disks in $\Pi_F(L)$ we can identify them with F and E above. (Note we can always assume there are horizontal disks by either looking near a cusp and flattening out a region, or letting F and E be the disks obtained by flattening out the regions around the top and bottom of a Reeb chord.) Legendrian isotope L so that F becomes F'. Replacing F' in $\Pi_F(L)$ by F'' will result in the front of a Legendrian submanifolds L' which is called the *stabilization of L along M.*

Proposition 4.5. *If L' is the stabilization of L with notation as above, then*

(1) *The rotation class of L' is the same as that of L.*
(2) *The invariant* tb *is given by*

$$\mathrm{tb}(L') = \begin{cases} \mathrm{tb}(L), & n \text{ even}, \\ \mathrm{tb}(L) + (-1)^{(D-U)} 2\chi(M), & n \text{ odd}, \end{cases}$$

where D, U, is the number of down, up, cusps along a generic path from E to F in $\Pi_F(L)$.

(3) *The Reeb chords of L and L' are naturally identified. The grading of any chord not associated with M, F and E is the same for both L and L'. Let c be a chord associated to M, F and E and let $|c|_L$ be its grading in L and $|c|_{L'}$ its grading in L'. Then*

$$|c|_{L'} = n - 2 - |c|_L$$

This theorem may seem a little strange if one is used to Legendrian knots in $\mathbb{R}^3$. In particular, it is well known that in 3 dimensions there are two different stabilizations and both change the rotation number. What is called a stabilization in dimension 3 is really a "half stabilization," as defined here. (Recall such a "half stabilization" corresponds to adding zig-zags to the front projection and looks like a "Reidemeister Type 1" move in the Lagrangian projection [13].) In particular if one does the above described stabilization near a cusp in dimension 3 it will be equivalent to doing both types of half stabilizations. See Figure 7.

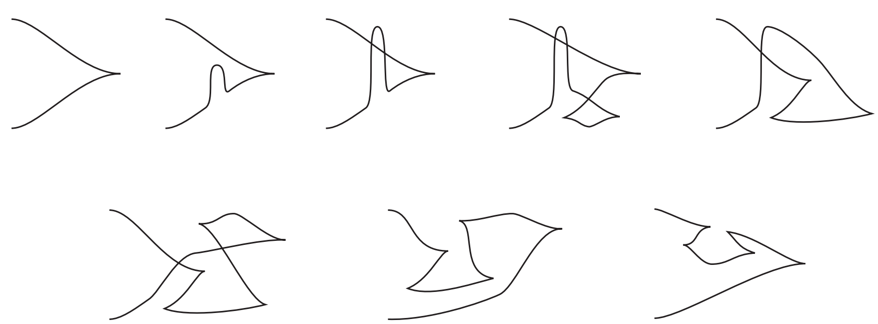

FIGURE 7. Our stabilization in dimension 3 is equivalent to two normal 3 dimensional stabilizations [13].

Remark 4.6. The stabilization procedure will typically produce non-topologically isotopic knots when done in dimension 3.

Proof. Recall the rotation class is defined as the Legendrian regular homotopy class. Now (1) is easy to see since the straight line homotopy from $\Pi_F(L)$ to $\Pi_F(L')$ will give a regular Legendrian homotopy between L and L'. Statement (2) follows from (3). As for (3) let c be a chord corresponding to a critical point of the Morse function f then Lemma 3.4 implies that $|c|_L = (k - \text{Morse Index}_c(f)) + D - U - 1$ and $|c|_{L'} = \text{Morse Index}_c(f) + (n-k) - D + U - 1$ where k is the dimension of M. □

We now consider some examples to see what effect stabilization has on contact homology.

Example 4.7. Let L be a Legendrian submanifold in $\mathbb{R}^{2n+1}$ and p a point on a cusp of $\Pi_F(L)$. Consider a small ball B around p in $\mathbb{R}^{n+1}$. We can isotope the front projection so as to create two new Reeb chords c_1 and c_2 in B, see Figure 7, such that $|c_1| = 0$ and $|c_2| = 1$. Let F' be the front obtained by pushing the lower end point of c_1 past the upper sheet of $\Pi_F(L)$ in B and let L' be the corresponding Legendrian submanifold.

Proposition 4.8. *The contact homology of L' is*

$$HC_k(\mathbb{R}^{2n+1}, L') = 0.$$

Proof. We can assume that p is at the origin in $\mathbb{R}^{n+1}$. For any ϵ define B_ϵ to be the product of the ball of radius ϵ about p in the x_1z-plane times $[-\epsilon,\epsilon]^{n-1}$ (in $x_2 \dots x_n$-space). We may now assume that $\Pi_F(L) \cap B_\epsilon$ is the cusp shown in Figure 7 times $[-\epsilon,\epsilon]^{n-1}$ and that the stabilization is done in $B_{\frac{\epsilon}{2}}$. A "monotonicity" argument shows that any disk with a positive puncture at c_2 (or c_1) and leaving B_ϵ has area bounded below. (See for example, [7, Lemma 11.3].) However, the action of c_2 can be made arbitrarily small. Therefore any disk with a positive corner at c_2 must stay in the ball B_ϵ. The projection of $\Pi_{\mathbb{C}}(L') \cap B_\epsilon$ to a z_j-plane $j \neq 1$ is shown on the left hand side of Figure 8. (The reason for the appearance of this picture is that we can choose the front so that $\frac{\partial z}{\partial x_j} \cdot x_j \leq 0$, for all $j > 1$.) The boundary of a projection

FIGURE 8. $\Pi_{\mathbb{C}}(L')$ projected onto a z_j-line, $j \neq 1$ (left) and intersected with the z_1-plane (right).

of a holomorphic curve must lie in the shaded region of the figure; moreover, the corner at c_2 of such a disk is negative. Thus any holomorphic curve with positive puncture at c_2 must lie entirely in the z_1-plane. The right hand side of Figure 8 shows $\Pi_{\mathbb{C}}(L') \cap B_\epsilon \cap \{z_1 - \text{plane}\}$. We see there one disk which by Proposition 2.4 contributes to the boundary of c_2. Thus $\partial c_2 = 1$, and one may easily check this implies $HC_k(\mathbb{R}^{2n+1}, L') = 0$. □

This last example is not particularly surprising given the analogous theorem, long known in dimension 3 [5], that stabilizations (or actually "half stabilizations" even) kill the contact homology. With this in mind the following examples might be a little surprising. It shows that in higher dimensions stabilization does not always kill the contact homology. The main difference with dimension 3 is the stabilizations we do below would, in dimension 3, change the knot type.

Example 4.9. When $n = 2$ we define the sphere L_1 via its front projection, which is described in Figure 9. For $n > 2$ there is an analogous front projection: take two copies, L_0, L_0', of the

Legendrian sphere L_0 from Example 4.2 and arrange them as shown in the figure. Deform L_0 as shown in Figure 9. Take a curve c, parameterized by $s \in [-1, 1]$, from the cusp edge on L_0 to the cusp edge on L_0'. By taking this curve to be very large we can assume the rate of change in its z-coordinate is very small. Moreover we will assume that by the time it passes under L_0 its z-coordinate is less than the z-coordinates of L_0' and thus has to "slope up" to connect with L_0'. (These choices will minimize the number of Reeb chords.) Take a neighborhood N of c whose vertical cross sections consist of round balls whose radii vary with s and have exactly one minimum at $s = 0$ and no other critical points. Introducing cusps along N as indicated in Figure 5 we can join L_0, L_0' and ∂N together to form a front

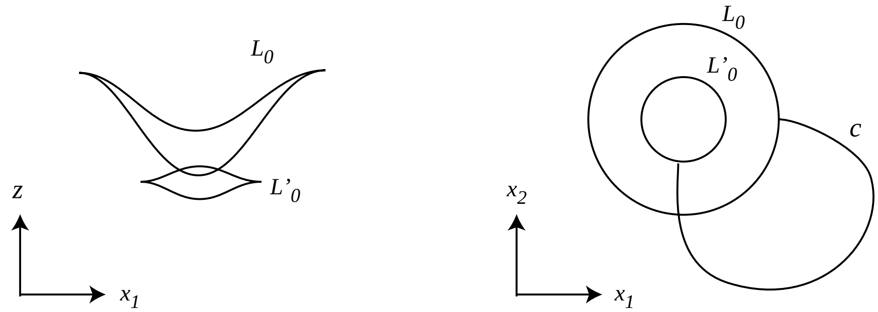

FIGURE 9. On the left hand side the x_1z-slice of part of L_1 is show. To see this portion in $\mathbb{R}^3$ rotate the figure about its center axis. On the right hand side we indicate the arc c connecting the two copies of L_0.

projection for a Legendrian sphere in $\mathbb{R}^{2n+1}$. There are exactly six Reeb chords involving only L_0 and L_0' which we label $a_1, \ldots, a_6$. There is also a Reeb chord b that occurs in N where the radii of the cross sectional balls have a minimum. Using Lemma 3.4 we compute:

$$\begin{aligned} |a_1| &= |a_2| = |a_5| = n, \\ |b| &= n-1, \\ |a_4| &= 0, \\ |a_3| &= |a_6| = -1. \end{aligned}$$

Proposition 4.10. *The following are true*

(1) *L_1 is a stabilization of L_0.*
(2) *For all n the rotation classes of L_1 and L_0 agree.*
(3) *When n is even* $\text{tb}(L_1) = \text{tb}(L_0)$ *and when n is odd* $\text{tb}(L_1) = \text{tb}(L_0) - 2$.
(4) *The linearized contact homology of L_1 in homology grading -1 is*

$$HLC_{-1}(\mathbb{R}^{2n+1}, L_1) = \mathbb{Z}_2.$$

(5) *L_0 and L_1 are not Legendrian isotopic.*

Proof. Let L_1' be the Legendrian sphere whose front is the same as the front of L_1 except that L_0' has been moved down so as to make L_0 and L_0' disjoint. Then L_1' is clearly Legendrian isotopic to L_0 and stabilizing L_1' (using M a point) results in L_1. Thus Statement (1) holds. Statements (2) and (3) follow from (1) and Proposition 4.5. Statement (5) follows from (4).

The Reeb chords for L_1' and L_1 are easily identified and their gradings are the same except for $|a_5|_{L_1'} = -2$. At this point it is clear that $HLC_{-1}(\mathbb{R}^{2n+1}, L_1) = \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. (This is good enough to distinguish L_0 and L_1.) Since L_1' and L_0 are Legendrian isotopic their linearized contact homologies must agree. Furthermore, the linearized contact homology of L_0 is a one element set,

$$HLC_n = HLC_n(\mathbb{R}^{2n+1}, L_0) = \mathbb{Z}_2, \quad HLC_j = HLC_j(\mathbb{R}^{2n+1}, L_0) = 0,\ j \neq n.$$

Thus, if ∂_1' denotes the (linearized) differential on $\mathcal{A}(L_1')_1/\mathcal{A}(L_1')_2$ we conclude the following.

(a) $\partial_1' a_5 = 0$ since a_5 is the generator of lowest grading.
(b) $\operatorname{Im}(\partial_1' | \operatorname{Span}(a_3, a_6)) = \operatorname{Span}(a_5)$ since $HCL_{-2} = 0$ and thus $\operatorname{Ker}(\partial_1' | \operatorname{Span}(a_3, a_6))$ is 1-dimensional.
(c) If $n > 2$ then $\partial_1' a_4$ spans $\operatorname{Ker}(\partial_1' | \operatorname{Span}(a_3, a_4))$ since $HLC_{-1} = 0$, if $n = 2$ then $\operatorname{Im}(\partial_1' | \operatorname{Span}(a_4, b)) = \operatorname{Ker}(\partial_1' | \operatorname{Span}(a_3, a_4))$ and therefore $\operatorname{Ker}(\partial_1' | \operatorname{Span}(a_4, b))$ is 1-dimensional.
(d) If $n > 2$ then $\partial_1' b = 0$. Also, $\operatorname{Im}(\partial_1' | \operatorname{Span}(a_1, a_2)) = \operatorname{Ker}(\partial_1' | \operatorname{Span}(a_4, b))$.

Let $\hat{L}_1$ be the Legendrian immersion "between" L_1' and L_1 with one double point which arises as the length of the Reeb chord a_5 shrinks to 0. Take $\hat{L}_1$ to be generic admissible. Moreover, by Proposition 2.3 we may assume that no rigid holomorphic disk with boundary on $\hat{L}_1$ and without puncture at a_5^* maps any boundary point to a_5^*. As in the proof of Lemma 4.4, we find a ball $B(a_5^*, d)$ such that no rigid disk without puncture at a_5^* maps a boundary point into $B(a_5^*, d)$.

Let $K_t, t \in [-\delta, \delta]$ be a small Legendrian regular homotopy such that $K_0 = \hat{L}_1$, K_δ is Legendrian isotopic to L_1' and $K_{-\delta}$ is Legendrian isotopic to L_1. Moreover, we take K_t supported inside a small neighborhood of a_5 which map into $B(a_5^*, \frac{1}{4}d)$ by $\Pi_{\mathbb{C}}$. Now if $u\colon D \to K_0$ is a disk on K_0 which maps no boundary point into $B(a_5^*, \frac{1}{2}d)$ then u can be viewed as a disk with boundary on K_t and vice versa.

We show that there exists $\epsilon > 0$ such that for $|t| < \epsilon$ there exist no rigid disk with boundary on K_t and without puncture at a_5 which map a boundary point to $B(a_5^*, \frac{1}{2}d)$. If this is not the case we extract a subsequence v_j of such maps which, by Gromov compactness, converges to a broken disk $(v^1, \dots, v^N)$ with boundary on K_0. If $N > 1$ then, by (2.6) at least one of the disks v^j must have negative formal dimension but since K_0 is generic admissible no such disks exists and the limiting disk v^1 is unbroken. Now v^1 is a rigid disk with boundary on K_0 and without puncture at a_5^* which maps boundary points to $B(a_5^*, d)$. This contradicts the choice of K_0 and hence proves the existence of $\epsilon > 0$ with properties as claimed. Thus, (c) above implies, with ∂_1 the differential on $\mathcal{A}(L_1)_1/\mathcal{A}(L_1)_2$, $\partial_1(\operatorname{Span}(a_4))$ ($\partial_1(\operatorname{Span}(a_4, b))$ if $n = 2$) is 1-dimensional and hence (4) holds. □

Let L_2 be the Legendrian sphere obtained by connect summing two copies of L_1. Note L_1 only has one cusp edge so there is no ambiguity in the construction; thus, we choose any arc which is disjoint from the fronts of the two spheres that are being connect summed. We similarly define L_k by connect summing L_{k-1} with L_1.

Theorem 4.11. *The Legendrian spheres L_k are all non Legendrian isotopic and, for n even, have the same classical invariants.*

Proof. This follows since $HLC_{-1}(\mathbb{R}^{2n+1}, L_k)$ equals $\mathbb{Z}_2^k$. □

In order to construct examples in dimensions $2n + 1$ where n is odd we consider a variant of this example.

Example 4.12. Let L_1' be constructed as L_1 is in Example 4.9 except start with L_0 and L_0' as shown in Figure 10. Like L_1, L_1' will have seven Reeb chords which we label in a similar

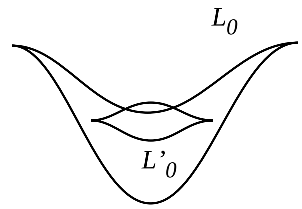

FIGURE 10. The position of L_0 and L_0' to construct L_1'.

manner. Here the grading on the chords is

$$\begin{aligned}|a_1| &= |a_2| = |a_5| = n,\\ |a_3| &= |a_6| = |b| = n-1,\\ |a_4| &= 0.\end{aligned}$$

Proposition 4.13. *The following are true*

1. (1) *L_1' is a stabilization of L_0.*
2. (2) *For all n the rotation class of L_1' and L_0 agree.*
3. (3) *When n is even* $\mathrm{tb}(L_1') = \mathrm{tb}(L_0)$ *and when n is odd* $\mathrm{tb}(L_1') = \mathrm{tb}(L_0) + 2$.
4. (4) *The linearized contact homology of L_1' has only one element and in homology grading 0 is*
$$HLC_0(\mathbb{R}^{2n+1}, L_1') = \mathbb{Z}_2.$$
5. (5) *L_0 and L_1' are not Legendrian isotopic.*

The proof of this proposition is identical to the proof of Proposition 4.10. To obtain interesting examples when n is odd we let K_1 be the connected sum of L_1 and L_1' and let K_k be the connected sum of K_{k-1} with K_1.

Theorem 4.14. *The classical invariants of K_k agree with those of L_0 but K_k and K_j are not Legendrian isotopic if $k \neq j$.*

This follows from Propositions 3.3, 4.10, 4.13, Lemma 4.4, and the computations of the linearized contact homology for L_1 and L_1'.

Example 4.15. Let F_g be the Legendrian surface of genus g with front obtained by "connect summing" several standard 2-spheres as shown in Figure 11 . Then $\mathcal{A}(F_g)$ is generated by

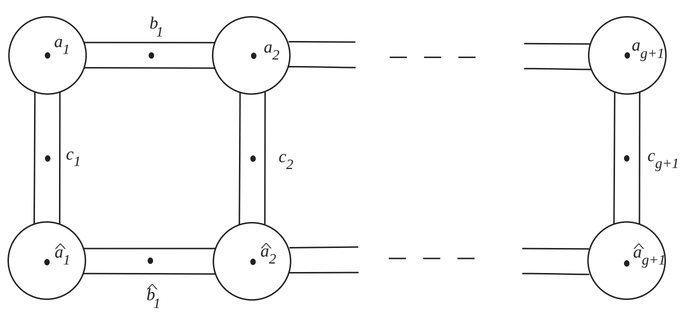

FIGURE 11. Top view of F_g.

$$\{a_j, \hat{a}_j, b_k, \hat{b}_k, c_j\}_{1\le j\le 1+g,\ 1\le k\le g},$$

where $|a_j| = |\hat{a}_j| = 2$ and $|b_k| = |\hat{b}_k| = |c_j| = 1$. Using projection to and slicing with the z_1- and z_2-planes as above we find

$$\begin{aligned}
\partial a_1 &= b_1 + c_1,\\
\partial \hat{a}_1 &= \hat{b}_1 + c_1,\\
\partial a_j &= b_{j-1} + b_j + c_j, \text{ for } j \neq 1, 1+g,\\
\partial \hat{a}_j &= \hat{b}_{j-1} + \hat{b}_j + c_j, \text{ for } 1 < j < 1+g,\\
\partial a_{1+g} &= b_g + c_{1+g},\\
\partial \hat{a}_{1+g} &= \hat{b}_g + c_{1+g},\\
\partial b_k &= \partial \hat{b}_k = \partial c_j = 0, \text{ for all } j, k.
\end{aligned}$$

We find $HC_*(\mathbb{R}^5, F_g) = \mathbb{Z}_2\langle a, b_1, \dots, b_g\rangle$ where $|a| = 2$ and $|b_i| = 1$. Let L_1 be as in Example 4.9 and define $F_g^0 = F_g$ and $F_g^k = F_g^{k-1} \# L_1$. Then the subspace of elements of grading -1 in $HLC_*(\mathbb{R}^5, F_g^k)$ is k-dimensional. Thus, F_g^k and F_g^j are not Legendrian isotopic if $j \neq k$. Clearly $tb(F_g^k) = tb(F_g^j)$. To see that $r(F_g^k) = r(F_g^j)$, it suffices to check, via front projections, that the Maslov classes are the same on the generators of $H_1(F_g^j) = H_1(F_g^j)$.

4.4. **Front spinning.** Given a Legendrian manifold $L \subset \mathbb{R}^{2n+1}$ we construct the *suspension* of L, denoted ΣL as follows: let $f\colon M \to \mathbb{R}^{2n+1}$ be a parameterization of L, and write

$$f(p) = (x_1(p), y_1(p), \dots, x_n(p), y_n(p), z(p)),$$

The front projection $\Pi_F(L)$ of L is the subvariety of $\mathbb{R}^{n+1}$ parameterized by $\Pi_F \circ f(p) = (x_1(p), \dots, x_n(p), z(p))$. We may assume that L has been translated so that $\Pi_F(L) \subset \{x_1 > 0\}$. If we embed $\mathbb{R}^{n+1}$ into $\mathbb{R}^{n+2}$ via $(x_1, \dots, x_n, z) \mapsto (x_0, x_1, \dots x_n, z)$ then $\Pi_F(\Sigma L)$ is obtained from $\Pi_F(L) \subset \mathbb{R}^{n+1}$ by rotating it around the subspace $\{x_0 = x_1 = 0\}$. See Figure 12. We can parameterize $\Pi_F(\Sigma L)$ by $(\sin\theta x_1(p), \cos\theta x_1(p), x_2(p), \dots, x_n(p))$, $\theta \in S^1$.

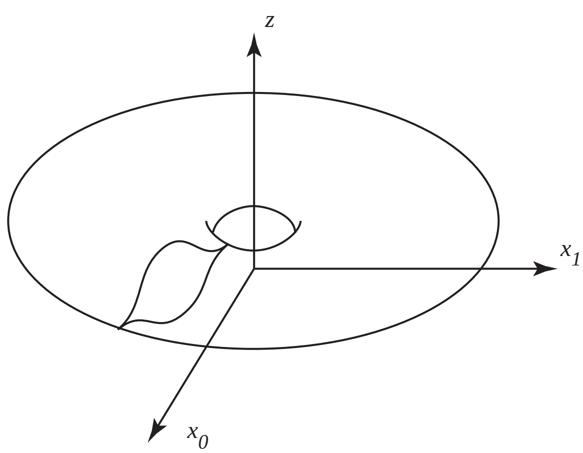

FIGURE 12. The front of ΣL.

Thus, $\Pi_F(\Sigma L)$ is the front for a Legendrian embedding $L \times S^1 \to \mathbb{R}^{2n+3}$ we denote the corresponding Legendrian submanifold ΣL. We have the following simple lemma.

Lemma 4.16. *The Legendrian submanifold $\Sigma L \subset \mathbb{R}^{2n+3}$ has*

(1) *the topological type or $L \times S^1$,*
(2) *the Thurston–Bennequin invariant* $\mathrm{tb}(\Sigma L) = 0$,

(3) *Maslov class determined by*

$$\mu_{\Sigma L}(g) = \begin{cases} \mu_L(h), & \text{if } g = \iota h \text{ where } \iota : \pi_1(L) \to \pi_1(\Sigma L) \text{ is the natural inclusion} \\ 0, & \text{if } g = [\text{point} \times S^1], \end{cases}$$

(4) *the same Maslov number as* L, $m(\Sigma L) = m(L)$, *and*

(5) *the rotation class of* ΣL *is determined by the rotation class of* L.

Though it seems difficult to compute the full contact homology of ΣL we can extract useful information about its linear part. To this end we introduce the following notation. Let $L \subset \mathbb{R}^{2n+1}$ be a Legendrian submanifold and let $\mathcal{A} = \mathcal{A}(L) = \mathbb{Z}_2[H_1(L)]\langle c_1, \dots, c_m\rangle$ be the graded algebra generated by its Reeb chords. We associate auxiliary algebras to L which are free unital algebras over $\mathbb{Z}_2[H_1(\Sigma L)]$. For any integer N, let $\mathbb{Z}^0_{2N} \subset \mathbb{Z}_{2N}$ denote the subgroup of even elements and let $\mathbb{Z}^1_{2N} = \mathbb{Z}_{2N} \setminus \mathbb{Z}^0_{2N}$.

- Let
$$\mathcal{A}^N_\Sigma(L) = \mathbb{Z}_2[H_1(\Sigma L)]\Big\langle c_j[\alpha], \hat{c}_j[\beta]\Big\rangle_{1\le j\le m,\ \alpha\in\mathbb{Z}^0_{2N},\ \beta\in\mathbb{Z}^1_{2N}},$$
where $|c_j[\alpha]| = |c_j|$, $\alpha \in \mathbb{Z}^0_{2N}$ and $|\hat{c}_j[\beta]| = |c_j| + 1$, $\beta \in \mathbb{Z}^1_{2N}$.
- For $\beta \in \mathbb{Z}_{2N}$, define the subalgebra $\mathcal{A}^N_\Sigma[\beta] \subset \mathcal{A}^N_\Sigma = \mathcal{A}^N_\Sigma(L)$ as
$$\mathcal{A}^N_\Sigma[\beta] = \mathbb{Z}_2[H_1(\Sigma L)]\langle c_j[\beta-1], c_j[\beta+1], \hat{c}_j[\beta]\rangle_{1\le j\le m}.$$
- Define the algebra
$$\mathcal{A}_\sigma(L) = \mathbb{Z}_2[H_1(\Sigma L)]\langle c_j, \hat{c}_j\rangle_{1\le j\le m},$$
where $|\hat{c}_j| = |c_j| + 1$.

We note that there is a natural homomorphism $\pi\colon \mathcal{A}^N_\Sigma \to \mathcal{A}_\sigma$ defined on generators by $\pi(c_j[\alpha]) = c_j$, and $\pi(\hat{c}_j[\beta]) = \hat{c}_j$. Also note that for each $\alpha \in \mathbb{Z}^0_{2N}$ there is a natural inclusion $\Delta[\alpha]\colon \mathcal{A} \to \mathcal{A}^N_\Sigma$ defined on generators by $\Delta[\alpha](c_i) = c_i[\alpha]$, and using the natural inclusion $H_1(L) \to H_1(\Sigma L)$ on coefficients.

Viewing $\mathcal{A}(L)$ and $\mathcal{A}_\sigma(L)$ as a vector space over $\mathbb{Z}_2$, see (2.3), and again using $H_1(L) \to H_1(\Sigma L)$ we define the linear map $\Gamma\colon \mathcal{A}(L) \to \mathcal{A}_\sigma(L)$ by

$$\Gamma(1) = 0, \quad \Gamma(t_1^{n_1}\dots t_s^{n_s} c_{i_1}\dots c_{i_r}) = t_1^{n_1}\dots t_r^{n_r}\Big(\sum_{j=1}^{r} c_{i_1}\dots c_{i_{j-1}}\hat{c}_{i_j}c_{i_{j+1}}\dots c_{i_r}\Big).$$

Proposition 4.17. *Let* $c_1, \dots, c_m$ *be the Reeb chords of* L *and let* $(\mathcal{A}, \partial)$ *denote its DGA. Then there exits an even integer* N *and a representative* X *of the Legendrian isotopy class of* ΣL *with associated DGA* $(\mathcal{A}(X), \partial_\Sigma)$ *satisfying*

$$\mathcal{A}(X) = \mathcal{A}^N_\Sigma, \tag{4.1}$$

$$\partial_\Sigma c_i[\alpha] = \Delta[\alpha](\partial c_i), \text{ for all } \alpha \in \mathbb{Z}^0_{2N}, \tag{4.2}$$

$$\partial_\Sigma \hat{c}_i[\beta] = c_i[\beta-1] + c_i[\beta+1] + \gamma^1_i[\beta] + \gamma^2_i[\beta], \text{ for all } \beta \in \mathbb{Z}^1_{2N} \tag{4.3}$$

where, $\gamma^2_i[\beta]$ *lies in the ideal of* $\mathcal{A}^N_\Sigma[\beta]$ *generated by all monomials which are quadratic in the variables* $\hat{c}_1[\beta], \dots, \hat{c}_m[\beta]$,*, where* $\gamma^1_i[\beta] \in \mathcal{A}^N_\Sigma[\beta]$ *is linear in the generators* $\hat{c}_i[\beta]$ *and satisfies*

$$\pi(\gamma^1_i[\beta]) = \Gamma(\partial c_i). \tag{4.4}$$

Moreover, $(\mathcal{A}(X), \partial_\Sigma)$ *is stable tame isomorphic to* $(\mathcal{A}^2_\Sigma, \partial_\Sigma)$.

We will prove this proposition in the next subsection but first we consider its consequences. To simplify notation, we consider the algebra generated over $\mathbb{Z}_2$ instead of $\mathbb{Z}_2[H_1(\Sigma L)]$.

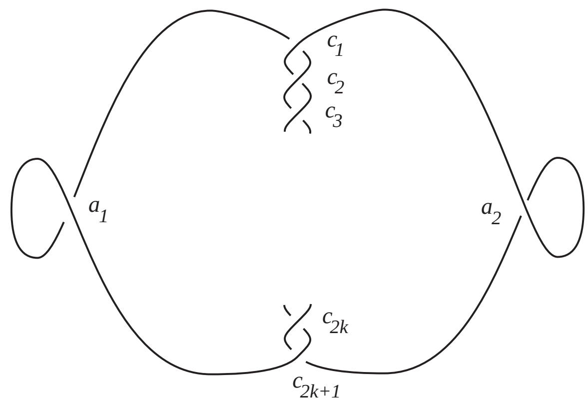

FIGURE 13. The knots T_k.

Example 4.18. Let T_k be the Legendrian torus knot in Figure 13 with rotation number $r(T_k) = 0$. The algebra for T_k is $\mathcal{A}(T_k) = \mathbb{Z}_2\langle a_1, a_2, c_1, \ldots, c_{2k+1}\rangle$ with $|a_1| = |a_2| = 1$ and $|c_j| = 0$ for all j. With Greek letters running over the integers in $[1, 2k+1]$ we have

$$\begin{aligned}
\partial a_1 &= 1 + \sum_{\alpha} c_\alpha + \sum_{\alpha>\beta>\gamma} c_\alpha c_\beta c_\gamma + \cdots + c_{2k+1}c_{2k}\ldots c_1,\\
\partial a_2 &= 1 + \sum_{\alpha} c_\alpha + \sum_{\alpha<\beta<\gamma} c_\alpha c_\beta c_\gamma + \cdots + c_1c_2\ldots c_{2k+1},\\
\partial c_j &= 0, \quad \text{all j.}
\end{aligned}$$

We note that ∂^g, where g is the elementary automorphism with $g(c_1) = c_1 + 1$ and which fixes all other generators, is augmented and that the linearized homology of $(\mathcal{A}, \partial^g)$ is (as a vector space without grading) $\mathbb{Z}_2^{2k+1}$. Applying the suspension operation n times we get a Legendrian n-tori $\Sigma^n T_k$ with $\text{tb}(\Sigma^n T_k) = 0$ for all $n > 0$, with rotation classes independent of k (see Lemma 4.16), and with Maslov number equal to 0. The algebras of $\Sigma^n T_k$ admit an elementary isomorphism (add 1 to each $c_1[\alpha_1][\alpha_2]\ldots[\alpha_n]$ with $|c_1[\alpha_1][\alpha_2]\ldots[\alpha_n]| = 0$) making them good and such that the corresponding linearized homology is isomorphic to $\mathbb{Z}_2^{2^n(2k+1)}$. This implies that every chord generic Legendrian representative of $\Sigma^n T_k$ has at least $2^n(2k+1)$ Reeb chords. Moreover, since $\Sigma^n T_j$ has a representative with $2^n(2j+3)$ Reeb chords it is easy to extract an infinite family of pairwise distinct Legendrian n-tori from the above.

Using Example 4.18 we find

Theorem 4.19. *There are infinitely many Legendrian n-tori in $\mathbb{R}^{2n+1}$ that are pairwise not Legendrian isotopic even though their classical invariants agree.*

Example 4.20. As a final family of examples we consider the Whitehead doubles of the unknot W_s shown in Figure 14. Note that $r(W_s) = 0$. The algebra for W_s is

$$\mathcal{A}(\mathbb{R}^3, W_s) = \mathbb{Z}_2\langle c_0, \ldots, c_s, a_1, \ldots, a_{s+2}\rangle$$

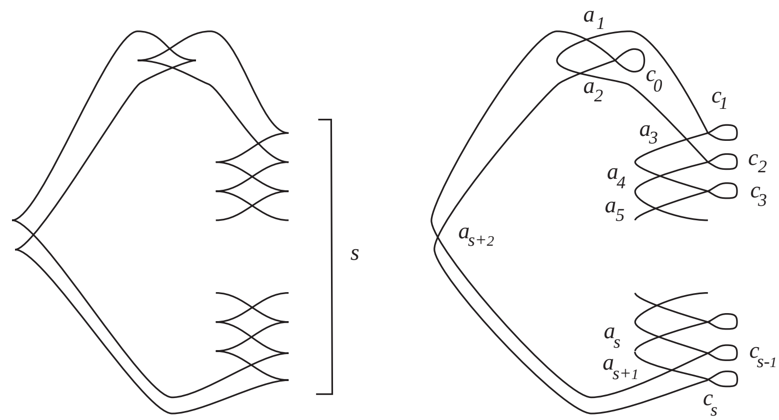

FIGURE 14. The front projection (left) and Lagrangian projection (right) of the knots W_s.

with $|c_i| = 1, |a_1| = -|a_2| = s-2$ and $|a_i| = 0$ for $i > 2$. Moreover,

$$\begin{aligned}\partial c_0 &= 1 + a_1 a_2 + a_{s+2},\\ \partial c_1 &= 1 + a_3,\\ \partial c_i &= 1 + a_{i+1} a_{i+2} \quad \text{for } i > 1,\\ \partial a_i &= 0 \quad \text{for all } i.\end{aligned}$$

The differential is clearly not augmented, but there will be a unique tame graded automorphism making it augmented. The only feature of the linearization we use is that

$$\begin{aligned} HLC_{2-s}(\mathbb{R}^5, \Sigma W_s) &= \mathbb{Z}_2,\\ HLC_i(\mathbb{R}^5, \Sigma W_s) &= 0 \quad \text{for } i < 2-s.\end{aligned}$$

The computation of these groups for $\Sigma^n W_s$ yields the same answer. Thus, they are all distinct and we get another proof of Theorem 4.19.

4.5. **Proof of the front spinning proposition.** To prove Proposition 4.17 we first analyze another Legendrian submanifold. Let $\psi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a smooth small perturbation of the constant function 1 that is a 2-periodic and has non-degenerate local maxima at even integers and local minima at odd integers. Given a Legendrian submanifold $L \subset \mathbb{R}^{2n+1}$ parameterized as above we define the front $\Pi_F(L \times \mathbb{R})$ in $\mathbb{R}^{n+1}$ by

$$G(p,t) = (\psi(t) z(p), t, x_1(p), \dots, x_n(p)). \tag{4.5}$$

Denote the resulting Legendrian submanifold of $\mathbb{R}^{2(n+1)+1}$ by $L \times \mathbb{R}$. Heuristically, $L \times \mathbb{R}$ is a kind of "cover" of ΣL and the boundary map of ΣL shall be determined by studying the boundary map of $L \times \mathbb{R}$. We begin with a simple lemma.

Lemma 4.21. *For each Reeb chord c_j of L there are $\mathbb{Z}$ Reeb chords, $c_j[n]$, for $L \times \mathbb{R}$; moreover, $|c_j[2n]| = |c_j| + 1$ and $|c_j[2n+1]| = |c_j|$.*

Lemma 4.22. *A holomorphic disk in $\mathbb{C}^{n+1}$ with boundary on $\Pi_{\mathbb{C}}(L \times \mathbb{R})$ cannot intersect the hyperplane $z_0 = k, k \in \mathbb{Z}$. In addition any holomorphic disk with a negative corner at $c_j[2k]$ or a positive corner at $c_j[2k+1]$ must lie entirely in the plane $z_0 = 2k, z_0 = 2k+1$, respectively.*

The proof of this lemma is identical to the proof of Lemma 4.4 once one has drawn the projection of $\Pi_{\mathbb{C}}(L \times \mathbb{R})$ onto the z_0-plane. Also, an argument similar to that in the proof of Lemma 4.4 in combination with Proposition 2.4 shows:

Lemma 4.23. *There is a unique holomorphic disk with positive corner at $c_j[2n]$ and negative corner at $c_j[2n \pm 1]$ and this disk is transversely cut out.*

We now discuss perturbations of L necessary to ensure the appropriate moduli spaces are manifolds. To ensure all our moduli spaces are cut out transversely we might have to perturb our Legendrian near positive corners of non-transversely cut out holomorphic disks. Note that due to Lemma 4.22 we see disks with positive corners at a Reeb chord in $x_0 = 2k+1$ lie in $z_0 = 2k+1$. Thus the linear problem splits for these disks and an argument similar to the proof of Proposition 2.3 shows they are all transversely cut out. In perturbing $L \times \mathbb{R}$ to be generic we can assume the perturbation is near the hyperplanes $x_0 = 2k, k \in \mathbb{Z}$ and none of the Reeb chords move.

Now let $\mathcal{B}_{2k} = \mathbb{Z}_2[H_1(L)]\langle c_j[2k-1], c_j[2k], c_j[2k+1]\rangle_{j=1}^m$ and $\mathcal{B}_{2k+1} = \mathbb{Z}_2[H_1(L)]\langle c_j[2k+1]\rangle_{j=1}^m$. These are all sub-algebras of the algebra $\mathcal{B}$ generated by all the Reeb chords for $L\times\mathbb{R}$. Let $\partial_{\mathbb{R}}$ be the boundary map for $L \times \mathbb{R}$. From the above Lemmas we clearly have

$$\partial_{\mathbb{R}}(\mathcal{B}_{2k+1}) \subset \mathcal{B}_{2k+1}$$

and

$$\partial_{\mathbb{R}}(\mathcal{B}_{2k}) \subset \mathcal{B}_{2k}.$$

Moreover Lemma 4.22 and our discussion of the generic perturbation above give

Lemma 4.24. *Let $\Gamma_{\pm 1} : \mathcal{A} \to \mathcal{B}_{\pm 1}$ be given by $\Gamma_{\pm 1}(c_j) = c_j[\pm 1]$. Then*

$$\partial_{\mathbb{R}} c_j[\pm 1] = \Gamma_{\pm 1}(\partial c_j).$$

To understand $\partial_{\mathbb{R}}$ on $\mathcal{B}_0$ we begin with

Lemma 4.25. $\partial_{\mathbb{R}}^2 = 0.$

Proof. Let F be the part of the front of $L \times \mathbb{R}$ between $x_0 = -(2k+\frac{3}{2})$ and $x_0 = 2k + \frac{3}{2}$, say. Let F' be F translated $4k+10$ units in the x_1-direction. For sufficiently large k $F \cap F' = \emptyset$. For such a k let $G \cup G'$ be $F \cup F'$ rotated by $\frac{\pi}{2}$ around the affine subspace $\{x_0 = 0, x_1 = 2k+5\}$. Now make $f \cup F' \cup G \cup G'$ into a closed Legendrian submanifold L' by adding "round corners."

Using the lemmas above and a monotonicity argument as in the proof of Lemma 4.8 it is easy to see that the boundary map for L' agrees with $\partial_{\mathbb{R}}$ on $\mathcal{B}_0$ and B_1. Since we know the square of the boundary map for a closed compact Legendrian is 0 the lemma follows. □

Lemma 4.26. *We can choose $L \times \mathbb{R}$ so that the part of $\partial_{\mathbb{R}}(c_j[0])$ that is constant in the generators $c_0[0], \ldots, c_m[0]$, is*

$$c_j[-1] + c_j[1].$$

Proof. This term is present in $\partial_{\mathbb{R}}(c_j[0])$ by Lemma 4.23.

To see there are no disks D with just one corner (which of course is positive at $c_j[0]$) assume we have such a disk D. Then consider $\psi_s : \mathbb{R} \to \mathbb{R}$ where $\psi_0 = \psi$ and $\psi_1(s) = 1$ and the corresponding Legendrian submanifolds $(L \times \mathbb{R})_s$ whose fronts are defined using ψ_s just as the front of $L \times \mathbb{R}$ used ψ in (4.5). As we isotope $L \times \mathbb{R} = (L \times \mathbb{R})_0$ to $(L \times \mathbb{R})_1$ we see that D will have to converge to a (possibly broken) disk for $(L \times \mathbb{R})_1$. But arguing as in the lemmas above we can see that any such disk will have to have z_0 constant and thus corresponds to a disk for L. However there can be no rigid holomorphic disk for L with corner at c_j since $|c_j| = |c_j[0]| - 1 = 1 - 1 = 0$. Moreover, if we have a broken holomorphic disk one can similarly see that one of the pieces of the broken disk will have negative formal dimension and thus cannot exist since we took L to be generic.

One may similarly argue that there are no holomorphic disks with one positive corner at $c_j[0]$ and all negative corners at Reeb chords $c_k[\pm 1]$, where $k \neq j$ for some j. □

Lemma 4.26 implies

$$\partial_{\mathbb{R}}(c_j[0]) = c_j[-1] + c_j[1] + \eta_j + r_j,$$

where η_j is the part of $\partial_{\mathbb{R}}(c_j[0])$ linear in $c_1[0], \dots, c_m[0]$ and r_j is the remainder (terms which are at least quadratic in the $c_j[0]$'s). Since $\partial_{\mathbb{R}}^2 = 0$ we see that

$$\partial_{\mathbb{R}}(c_j[-1] + c_j[1]) = \sigma(\eta_j), \tag{4.6}$$

where σ is the algebra homomorphism defined by $\sigma c_j[0] = c_j[-1] + c_j[1]$ and $\sigma c_j[\pm 1] = 0$. A straightforward calculation shows that

$$\eta_j = \Gamma_0(\partial c_j)$$

is a solution to (4.6) where $\Gamma_0 : \mathcal{A} \to \mathcal{B}_0$ is the linear map defined on monomials by

$$\begin{aligned}\Gamma_0(c_{j_1} \dots c_{j_r}) =& c_{j_1}[0]c_{j_2}[-1] \dots c_{j_r}[-1] + c_{j_1}[1]c_{j_2}[0]c_{j_3}[-1] \dots c_{j_r}[-1] + \dots \\ &+ c_{j_1}[1] \dots c_{j_{r-1}}[1]c_{j_r}[0].\end{aligned}$$

While this is not the only solution to (4.6), it is unique in the following sense: let $\mathcal{B}' = \mathbb{Z}_2[H_1(L)]\langle c_1, \dots, c_m, c_1[0], \dots, c_m[0]\rangle$ and define $\pi : \mathcal{B}_0 \to \mathcal{B}'$ by $\pi(c_j[\pm 1]) = c_j$ and $\pi(c_j[0]) = c_j[0]$.

Lemma 4.27. *If α is linear in the $c_j[0]$'s and $\sigma(\alpha) = 0$ then $\pi(\alpha) = 0$.*

Proof. Assume α satisfies the hypothesis of the lemma. Let $ac_j[0]b$ be a monomial in α with $l(a) = l_1$ and $l(b) = l_2$ where $l(\cdot) =$ length of monomial. We claim that since $\sigma(\alpha) = 0$, there must be another monomial in α of the form $a'c_j[0]b'$ where $\pi(a) = \pi(a')$ and $\pi(b) = \pi(b')$. The lemma will follow. To see this note that

$$\sigma(ac_j[0]b) = ac_j[-1]b + ac_j[1]b.$$

Since the $(l_1 + 1)$th letter in these monomials is different they can only be canceled by a monomial of the form $\sigma(a'c_j[0]b')$ or by terms coming from two separate monomials w_1 and w_2 in α. So either we are done or two of the four terms in $\sigma(w_1 + w_2)$ are canceled by $\sigma(ac_j[0]b)$ leaving two other terms. Note the two leftover terms still have different $(l_1 + 1)$th letters. So once again either there is a term of the form $a'c_j[0]b'$ to cancel these two terms or there are two more monomials in α. We clearly will eventually find the desired term $a'c_j[0]b'$ (induction on the number of terms in α). One should observe that a' and b' have the properties stated above because in each step in the above cancellation process we are replacing $c_j[\pm 1]$'s with $c_j[\mp 1]$'s. □

We are now ready to prove our main result of this subsection.

Proof of Proposition 4.17. Consider ΣL represented by rotating the front of L around a circle C with radius $\frac{1}{\pi}N$ for some even integer N. Perturb this non-generic front with a function ϕ on C similar to (4.5) so that ϕ approximates the constant function 1, has local minima at angles $2m \cdot \frac{\pi}{N}$ and local maxima at $(2m + 1) \cdot \frac{\pi}{N}$, $m = 1, \dots, N - 1$. Let X_N denote the corresponding Legendrian submanifold. Then there is a natural $1-1$ correspondence between the generators of the algebra $\mathcal{A}(X_N)$ and the generators of $\mathcal{A}_\Sigma^N(L)$.

Considering the projections of X_N to the complex lines in $\mathbb{C}^{n+1}$ which intersect $\mathbb{R}^{n+1}$ in lines through antipodal local minima of ϕ we see that the differential ∂_Σ of $\mathcal{A}(X_N) = \mathcal{A}_\Sigma^N$ preserves the subalgebras $\mathcal{A}_\Sigma^N[\beta]$ for every $\beta \in \mathbb{Z}_{2N}^1$. Moreover, a finite part of the front of X_N over an arc on C between two minima is for N sufficiently large an arbitrarily good approximation of the part of the front of $L \times \mathbb{R}$ between -1 and 1. In fact, since all spaces of rigid disks on $L \times \mathbb{R}$ are transversely cut out there is a neighborhood of $L \times \mathbb{R}$ in the space of (admissible) Legendrian submanifolds such that the moduli spaces of rigid disks on any Legendrian submanifold in this neighborhood is canonically isomorphic to those of $L \times \mathbb{R}$.

This can be seen as follows: by Gromov compactness, there exists a neighborhood of $L \times \mathbb{R}$ in the space of admissible Legendrian submanifolds such that for any Y in this neighborhood there are no holomorphic disks with boundary on Y and with negative formal dimension, pick a generic type (A) isotopy from $L \times \mathbb{R}$ to Y and apply Lemma 2.11. Thus, for sufficiently large N the subalgebras $(\mathcal{A}_\Sigma^N[\beta], \partial_\Sigma)$ are all isomorphic to the algebras $(\mathcal{B}_0, \partial)$. The first part of the proposition now follows from Lemmas 4.26 and 4.27.

For the statement of stable tame isomorphism class, note that the subalgebras of $\mathcal{A}_\Sigma^N$ generated by all Reeb chords corresponding to maxima and minima over the circle in the closed upper (lower) half planes are both isomorphic to the subalgebra of $\mathcal{A}(L\times\mathbb{R})$ generated by Reeb chords between 1 and $N+1$. Change the front of $L \times \mathbb{R}$ by shrinking the minima of ψ over 1 and $N+1$ until the corresponding Reeb chords are shorter than all other Reeb chords. We can then find a Legendrian isotopy which cancels pairs of maxima and minima of ψ between 1 and $N+1$, leaving one maximum. Thus the subalgebra generated by Reeb chords between 1 and $N+1$ is stable tame isomorphic to $\mathcal{B}_0$. Moreover, if the stabilizations and tame isomorphisms corresponding to self tangencies and handle slides in the canceling process are constructed as in the proofs of Lemmas 2.11 and 2.18, respectively, then Reeb chords which are shorter than the positive chord of a handle slide disk and shorter than both chords canceling in a self tangency are left unchanged by the tame isomorphisms. Thus the subalgebra $\mathcal{B}_1$ and $\mathcal{B}_{N+1}$ are left unchanged by this chain of stabilizations and tame isomorphisms and hence it induces a chain of stabilizations and tame isomorphisms connecting $\mathcal{A}_\Sigma^2(L)$ to $\mathcal{A}_\Sigma^N(L)$. □

4.6. **Remarks on the examples.**

Remark 4.28. In dimension 3 the connected sum of Legendrian knots is well defined [14]. However in higher dimensions there are several ways to make a Legendrian version of the connected sum. Lemma 4.4 discusses one such way. However there are other direct generalization of the 3 dimensional connected sum. Thus the correct definition of connected sum is not clear. Even if we just consider the "cusp connected sum" (from Lemma 4.4) it is still not clear if it is well defined. So we ask

Is the connected sum well defined?

And more specifically

Does the cusp connected sum depend on the cusps chosen in the construction?

Remark 4.29. Colin, Giroux and Honda have announced the following result: in dimension 3 if you fix the tb, r and a knot type there are only finitely many Legendrian knots realizing this data. If one considers this question in higher dimensions most of our examples described above provide counterexamples to the corresponding assertion. In particular consider Theorem 4.14.

Remark 4.30. Given a Legendrian submanifold L we can define an invariant $N(L)$ to be the minimal number of Reeb chords associated to a generic representative of L. One can ask

Is there an effective bound on $N(L)$ in terms of the Thurston-Bennequin invariant and rotation class?

The result of Colin, Giroux and Honda mentioned above indicate a positive answer to this question in dimension 3. While our examples above show that these answer is a resounding NO in dimensions above 3.

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