THE CONTACT HOMOLOGY OF LEGENDRIAN SUBMANIFOLDS IN R^{2n+1}

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Abstract. We define the contact homology for Legendrian submanifolds in standard contact $(2n+1)$-space using moduli spaces of holomorphic disks with Lagrangian boundary conditions in complex n-space. This homology provides new invariants of Legendrian isotopy which indicate that the theory of Legendrian isotopy is very rich. Indeed, in [4] the homology is used to detect infinite families of pairwise non-isotopic Legendrian submanifolds which are indistinguishable using previously known invariants.

The motivating problem for this paper is the classification of Legendrian submanifolds up to Legendrian isotopy. Here we restrict attention to the standard contact structure on $\mathbb{R}^{2n+1}$. For $n = 1$ the Legendrian isotopy problem has been extensively studied, [2, 6, 9, 10, 11], but there have been few results for $n > 1$. In this paper we give a rigorous definition of contact homology, a potent new invariant originally described in [5]. This new invariant was applied in [4] to construct infinite families of non-Legendrian isotopic, Legendrian n-spheres, n-tori and surfaces of arbitrary genus. These are the first such high-dimensional examples. They also demonstrate that the analogues of rotation number and Thurston-Bennequin invariant (and diffeomorphism type) of a Legendrian submanifold are far from complete invariants of Legendrian isotopy. (See [4] for a definition of the high-dimensional analogues of the classical invariants.)

The goal of this paper is to define contact homology and prove that it is a Legendrian isotopy invariant.

Theorem. *The contact homology of Legendrian submanifolds in $\mathbb{R}^{2n+1}$ with the standard contact form is well defined. (It is invariant under Legendrian isotopy.)*

We define the contact homology using punctured holomorphic disks in $\mathbb{C}^n \approx \mathbb{R}^{2n}$ with boundary on the *Lagrangian projection* $\Pi_{\mathbb{C}} \colon \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n$ of the Legendrian submanifold, and which limit to double points of the projection at the punctures. This is analogous to the approach taken by Chekanov [2] in dimension 3 who was the first to prove that the classical invariants are not enough to distinguish isotopy classes. In dimension 3, however, the entire theory can be reduced to combinatorics. As discussed in [4] our contact homology also fits into the over arching philosophy of Symplectic Field Theory outlined in [8]. There it goes by the name of the "relative contact homology" of the standard contact $(2n+1)$-space.

In Section 1, we define contact homology more concretely and outline its invariance under Legendrian isotopy. If $L \subset \mathbb{R}^{2n+1} \approx \mathbb{C}^n \times \mathbb{R}$ is a Legendrian submanifold we associate to L a differential graded algebra (DGA), denoted $(\mathcal{A}, \partial)$, freely generated by the double points of $\Pi_{\mathbb{C}}(L) \subset \mathbb{C}^n$. Since L is embedded one may distinguish upper and lower branches of L at double points of $\Pi_{\mathbb{C}}(L)$ and using this structure we associate a sign to every puncture of a holomorphic disk with boundary on $\Pi_{\mathbb{C}}(L)$. We define the differential of the DGA by counting punctured rigid holomorphic disks with boundary on $\Pi_{\mathbb{C}}(L)$ and with exactly one positive puncture. The contact homology of L is defined to be $\operatorname{Ker}\partial / \operatorname{Im}\partial$. Thus, contact homology is similar to Floer homology of Lagrangian intersections. The proof of its invariance is similar in spirit to Floer's original approach [13, 14]; we study bifurcations of moduli spaces of rigid holomorphic disks under variations of the Legendrian submanifold in a generic 1-parameter family of Legendrian submanifolds. Similar bifurcation analysis is also done in [19, 21, 30, 31]. Our set-up does not seem well suited to the more popular proof of Floer theory invariance which uses an elegant "homotopy of homotopies" argument (see, for example, [15, 29]).

In Section 5, the (formal) dimension of the moduli space of punctured holomorphic disks with boundary on an exact Lagrangian immersion which is an instant in a generic 1-parameter family is expressed in terms of its boundary data. We compute this dimension by relating the linearization of the $\bar{\partial}$-equation for punctured disks with boundary on the exact Lagrangian to the standard vector Riemann-Hilbert problem on the closed disk (i.e. the disk without punctures).

In Section 6 we show that for Legendrian submanifolds (and their 1-parameter families) in an open dense set in the space of such, the moduli-spaces of holomorphic disks are being transversely cut out. That is, we achieve transversality for the $\bar{\partial}$-equation without perturbing the complex structure on $\mathbb{C}^n$. The fact that we can keep the standard complex structure on $\mathbb{C}^n$ is important for computations of contact homology, see [4]. Similar transversality results were obtained by Oh [25] for closed holomorphic disks with Lagrangian boundary condition, under the additional assumption that the disks have an injective point on the boundary. In general, disks without such points cannot be excluded and we manage to prove transversality for disks involved in contact homology using the fact that they have only one positive puncture, and a technical result, established in Section 2, that all Legendrian submanifolds may be assumed real analytic close to the preimages of double points of $\Pi_{\mathbb{C}}$.

In Section 8, we show that moduli-spaces of holomorphic disks have certain compactness properties. We prove a version of Gromov compactness for punctured holomorphic disks with boundary on an immersed exact Lagrangian submanifold in $\mathbb{C}^n$. In particular, it follows that 0-dimensional moduli-spaces are compact and that 1-dimensional moduli-spaces have natural compactifications.

In Section 7 we establish gluing theorems. These are used to prove that the differential ∂ of the DGA $\mathcal{A}$ satisfies $\partial \circ \partial = 0$, and that the homology of $(\mathcal{A}, \partial)$ is left unchanged by the two basic bifurcations which occur in generic 1-parameter families: appearance of disks of formal dimension -1 and self-tangency instances. The most technically difficult results are the so-called degenerate gluing theorems which are necessary to control the changes of the DGA under self-tangencies. Here holomorphic disks with punctures at the self-tangency double point must be glued. To prove these gluing theorems we use results from Section 3 which give the blow up rate of the constant in the elliptic estimate for the linearized $\bar{\partial}$-equation, as the transverse double point at one puncture approaches a self-tangency double point.

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Contents

1. Contact Homology and Differential Graded Algebras

In this section we describe how to associate to a Legendrian submanifold L in standard contact $(2n+1)$-space a differential graded algebra (DGA) $(\mathcal{A}, \partial)$. Up to a certain equivalence relation this DGA is an invariant of the Legendrian isotopy class of L. In Section 1.1 we recall the notion of Lagrangian projection and define the algebra $\mathcal{A}$. The grading on $\mathcal{A}$ is described in Section 1.2. Sections 1.3 and 1.4 are devoted to the definition of ∂ and Section 1.5 proves the invariance of the homology of $(\mathcal{A}, \partial)$, which we call the contact homology. The main proofs of these three subsections rely on much analysis, which will be completed in the subsequent sections. In a sense, these last three subsections can be viewed as an overview of the remainder of the paper.

1.1. **The algebra $\mathcal{A}$.** Throughout this paper we consider the standard contact structure ξ on $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$ which is the hyperplane field given as the kernel of the contact 1-form

$$\alpha = dz - \sum_{j=1}^{n} y_j dx_j, \tag{1.1}$$

where $x_1, y_1, \dots, x_n, y_n, z$ are Euclidean coordinates on $\mathbb{R}^{2n+1}$. A *Legendrian submanifold* of $\mathbb{R}^{2n+1}$ is an n dimensional submanifold $L \subset \mathbb{R}^{2n+1}$ everywhere tangent to ξ. We also recall that the standard symplectic structure on $\mathbb{C}^n$ is given by

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j,$$

and that an immersion $f\colon L \to \mathbb{C}^n$ of an n-dimensional manifold is *Lagrangian* if $f^*\omega = 0$.

The *Lagrangian projection* projects out the z coordinate:

$$\Pi_{\mathbb{C}}\colon \mathbb{R}^{2n+1} \to \mathbb{C}^n; \quad (x_1, y_1, \dots, x_n, y_n, z) \mapsto (x_1, y_1, \dots, x_n, y_n). \tag{1.2}$$

If $L \subset \mathbb{C}^n \times \mathbb{R}$ is a Legendrian submanifold then $\Pi_{\mathbb{C}}\colon L \to \mathbb{C}^n$ is a Lagrangian immersion. Moreover, for L in an open dense subset of all Legendrian submanifolds (with C^∞ topology), the self intersection of $\Pi_{\mathbb{C}}(L)$ consists of a finite number of transverse double points. We call Legendrian submanifolds with this property *chord generic.*

The Reeb vector field X of a contact form α is uniquely defined by the two equations $\alpha(X) = 1$ and $d\alpha(X, \cdot) = 0$. The *Reeb chords* of a Legendrian submanifold L are segments of flow lines of X starting and ending at points of L. We see from (1.1) that in $\mathbb{R}^{2n+1}$, $X = \frac{\partial}{\partial z}$ and thus $\Pi_{\mathbb{C}}$ defines a bijection between Reeb chords of L and double points of $\Pi_{\mathbb{C}}(L)$. If c is a Reeb chord we write $c^* = \Pi_{\mathbb{C}}(c)$.

Let $\mathcal{C} = \{c_1, \dots, c_m\}$ be the set of Reeb chords of a chord generic Legendrian submanifold $L \subset \mathbb{R}^{2n+1}$. To such an L we associate an algebra $\mathcal{A} = \mathcal{A}(L)$ which is the free associative unital algebra over the group ring $\mathbb{Z}_2[H_1(L)]$ generated by $\mathcal{C}$. We write elements in $\mathcal{A}$ as

$$\sum_i t_1^{n_{1,i}} \dots t_k^{n_{k,i}} \mathbf{c}_i, \tag{1.3}$$

where the t_j's are formal variables corresponding to a basis for $H_1(L)$ thought of multiplicatively and $\mathbf{c}_i = c_{i_1} \dots c_{i_r}$ is a word in the generators. It is also useful to consider the corresponding algebra $\mathcal{A}_{\mathbb{Z}_2}$ over $\mathbb{Z}_2$. The natural map $\mathbb{Z}_2[H_1(L)] \to \mathbb{Z}_2$ induces a reduction of $\mathcal{A}$ to $\mathcal{A}_{\mathbb{Z}_2}$ (set $t_j = 1$, for all j).

1.2. **The grading on $\mathcal{A}$.** Let Λ_n be the Grassman manifold of Lagrangian subspaces in the symplectic vector space $(\mathbb{C}^n, \omega)$ and recall that $H_1(\Lambda_n) = \pi_1(\Lambda_n) \cong \mathbb{Z}$. There is a standard isomorphism

$$\mu : H_1(\Lambda_n) \to \mathbb{Z},$$

given by intersecting a loop in Λ_n with the Maslov cycle Σ. To describe μ more fully we follow [26] and refer the reader to this paper for proofs of the statements below.

Fix a Lagrangian subspace Λ in $\mathbb{C}^n$ and let $\Sigma_k(\Lambda) \subset \Lambda_n$ be the subset of Lagrangian spaces that intersects Λ in a subspace of k dimensions. The *Maslov cycle* is

$$\Sigma = \overline{\Sigma_1(\Lambda)} = \Sigma_1(\Lambda) \cup \Sigma_2(\Lambda) \cup \dots \cup \Sigma_n(\Lambda).$$

This in an algebraic variety of codimension one in Λ_n. If $\Gamma : [0,1] \to \Lambda_n$ is a loop then $\mu(\Gamma)$ is the intersection number of Γ and Σ. The contribution of an intersection point t' with $\Gamma(t') \in \Sigma$ to $\mu(\Gamma)$ is calculated as follows. Fix a Lagrangian complement W of Λ. Then for each $v \in \Gamma(t') \cap \Lambda$ there exists a vector $w(t) \in W$ such that $v + w(t) \in \Gamma(t)$ for t near t'. Define the quadratic form $Q(v) = \frac{d}{dt}|_{t=t'} \omega(v, w(t))$ on $\Gamma(t') \cap \Lambda$ and observe that it is independent of the complement W chosen. Without loss of generality, Q can be assumed non-singular and the contribution of the intersection point to $\mu(\Gamma)$ is the signature of Q. Given any loop Γ in Λ_n we say $\mu(\Gamma)$ is the *Maslov index* of the loop.

If $f\colon L \to \mathbb{C}^n$ is a Lagrangian immersion then the tangent planes of $f(L)$ along any loop γ in L gives a loop Γ in Λ_n. We define the Maslov index $\mu(\gamma)$ of γ as $\mu(\gamma) = \mu(\Gamma)$ and note that we may view the Maslov index as a map $\mu : H_1(L) \to \mathbb{Z}$. Let $m(f)$ be the smallest non-negative number that is the Maslov index of some non-trivial loop in L. We call $m(f)$ the *Maslov number* of f. When $L \subset \mathbb{C}^n \times \mathbb{R}$ is a Legendrian submanifold we write $m(L)$ for the Maslov number of $\Pi_{\mathbb{C}}\colon L \to \mathbb{C}^n$.

Let $L \subset \mathbb{R}^{2n+1}$ be a chord generic Legendrian submanifold and let c be one of its Reeb chords with end points $a, b \in L$, $z(a) > z(b)$. Choose a path $\gamma : [0, 1] \to L$ with $\gamma(0) = a$ and $\gamma(1) = b$. (We call such path a *capping path of* c.) Then $\Pi_{\mathbb{C}} \circ \gamma$ is a loop in $\mathbb{C}^n$ and $\Gamma(t) = d\Pi_{\mathbb{C}}(T_{\gamma(t)}L)$, $0 \le t \le 1$ is a path of Lagrangian subspaces of $\mathbb{C}^n$. Since $c^* = \Pi_{\mathbb{C}}(c)$ is a transverse double point of $\Pi_{\mathbb{C}}(L)$, Γ is not a closed loop.

We close Γ in the following way. Let $V_0 = \Gamma(0)$ and $V_1 = \Gamma(1)$. Choose any complex structure I on $\mathbb{C}^n$ which is compatible with ω ($\omega(v, Iv) > 0$ for all v) and with $I(V_1) = V_0$. (Such an I exists since the Lagrangian planes are transverse.) Define the path $\lambda(V_0, V_1)(t) = e^{tI}V_1$, $0 \le t \le \frac{\pi}{2}$. The concatenation, $\Gamma * \lambda(V_0, V_1)$, of Γ and $\lambda(V_0, V_1)$ forms a loop in Λ_n and we define the *Conley–Zehnder index*, $\nu_\gamma(c)$, of c to be the Maslov index $\mu(\Gamma * \lambda(V_0, V_1))$ of this loop. It is easy to check that $\nu_\gamma(c)$ is independent of the choice of I. However, $\nu_\gamma(c)$ might depend on the choice of homotopy class of the path γ. More precisely, if γ_1 and γ_2 are two paths with properties as γ above then

$$\nu_{\gamma_1}(c) - \nu_{\gamma_2}(c) = \mu(\gamma_1 * (-\gamma_2)),$$

where $(-\gamma_2)$ is the path γ_2 traversed in the opposite direction. Thus $\nu_\gamma(c)$ is well defined modulo the Maslov number $m(L)$.

Let $\mathcal{C} = \{c_1, \dots, c_m\}$ be the set of Reeb chords of L. Choose a capping path γ_j for each c_j and define the *grading* of c_j to be

$$|c_j| = \nu_{\gamma_j}(c_j) - 1,$$

and for any $t \in H_1(L)$ define its grading to be $|t| = -\mu(t)$. This makes $\mathcal{A}(L)$ into a graded ring. Note that the grading depends on the choice of capping paths but, as we will see below, this choice will be irrelevant.

The above grading on Reeb chords c_j taken modulo $m(L)$ makes $\mathcal{A}_{\mathbb{Z}_2}$ a graded algebra with grading in $\mathbb{Z}_{m(L)}$. (Note that this grading does not depend on the choice of capping paths.) In addition the map from $\mathcal{A}$ to $\mathcal{A}_{\mathbb{Z}_2}$ preserves gradings modulo $m(L)$.

1.3. **The moduli spaces.** As mentioned in the introduction, the differential of the algebra associated to a Legendrian submanifold is defined using spaces of holomorphic disks. To describe these spaces we need a few preliminary definitions.

Let D_{m+1} be the unit disk in $\mathbb{C}$ with $m + 1$ punctures at the points $p_0, \dots p_m$ on the boundary. The orientation of the boundary of the unit disk induces a cyclic ordering of the punctures. Let $\partial\hat{D}_{m+1} = \partial D_{m+1} \setminus \{p_0, \dots, p_m\}$.

Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a Legendrian submanifold with isolated Reeb chords. If c is a Reeb chord of L with end points $a, b \in L$, $z(a) > z(b)$ then there are small neighborhoods $S_a \subset L$ of a and $S_b \subset L$ of b that are mapped injectively to $\mathbb{C}^n$ by $\Pi_{\mathbb{C}}$. We call $\Pi_{\mathbb{C}}(S_a)$ the *upper sheet* of $\Pi_{\mathbb{C}}(L)$ at c^* and $\Pi_{\mathbb{C}}(S_b)$ the *lower sheet*. If $u\colon (D_{m+1}, \partial D_{m+1}) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(L))$ is a continuous map with $u(p_j) = c^*$ then we say p_j is *positive* (respectively *negative*) if u maps points clockwise of p_j on ∂D_{m+1} to the lower (upper) sheet of $\Pi_{\mathbb{C}}(L)$ and points anti-clockwise of p_i on ∂D_{m+1} to the upper (lower) sheet of $\Pi_{\mathbb{C}}(L)$ (see Figure 1).

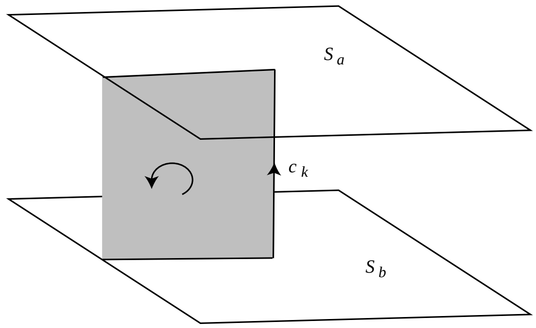

FIGURE 1. Positive puncture lifted to $\mathbb{R}^{2n+1}$. The gray region is the holomorphic disk and the arrows indicate the orientation on the disk and the Reeb chord.

If a is a Reeb chord of L and if $\mathbf{b} = b_1 \dots b_m$ is an ordered collection (a word) of Reeb chords then let $\mathcal{M}_A(a;\mathbf{b})$ be the space, modulo conformal reparameterization, of maps $u : (D_{m+1}, \partial D_{m+1}) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(L))$ which are continuous on D_{m+1}, holomorphic in the interior of D_{m+1}, and which have the following properties

- p_0 is a positive puncture, $u(p_0) = a^*$,
- p_j are negative punctures for $j > 0$, $u(p_j) = b_j^*$,
- the restriction $u|\partial \hat{D}_{m+1}$ has a continuous lift $\tilde{u}\colon \partial \hat{D}_{m+1} \to L \subset \mathbb{C}^n \times \mathbb{R}$, and
- the homology class of $\tilde{u}(\partial D^*_{m+1}) \cup (\cup_j \gamma_j)$ equals $A \in H_1(L)$,

where γ_j is the capping path chosen for c_j, $j = 1, \dots, m$. Elements in $\mathcal{M}_A(a;\mathbf{b})$ will be called *holomorphic disks with boundary on* L or sometimes simply holomorphic disks.

There is a useful fact relating heights of Reeb chords and the area of a holomorphic disk with punctures mapping to the corresponding double points. The *action* (or height) $\mathcal{Z}(c)$ of a Reeb chord c is simply its length and the action of a word of Reeb chords is the sum of the actions of the chords making up the word.

Lemma 1.1. *If* $u \in \mathcal{M}_A(a;\mathbf{b})$ *then*

$$\mathcal{Z}(a) - \mathcal{Z}(\mathbf{b}) = \int_{D_m} u^*\omega = \text{Area}(u) \geq 0. \tag{1.4}$$

Proof. By Stokes theorem, $\int_{D_m} u^*\omega = \int_{\partial D_m} u^*(-\sum_j y_j dx_j) = \int \tilde{u}^*(-dz) = \mathcal{Z}(a) - \mathcal{Z}(\mathbf{b})$. The second equality follows since u is holomorphic and $\omega = \sum_{j=1}^n dx_j \wedge dy_j$. □

Note that the proof of Lemma 1.1 implies that any holomorphic disk with boundary on L must have at least one positive puncture. (In contact homology, only disks with exactly one positive puncture are considered.)

We now proceed to describe the properties of moduli spaces $\mathcal{M}_A(a;\mathbf{b})$ that are needed to define the differential. We prove later that the moduli spaces of holomorphic disks with boundary on a Legendrian submanifold L have these properties provided L is generic among (belongs to a Baire subset of the space of) *admissible* Legendrian submanifolds (L is admissible if it is chord generic and it is real analytic in a neighborhood of all Reeb chord end points). For more precise definitions of these concepts, see Section 2, where it is shown that admissible Legendrian submanifolds are dense in the space of all Legendrian submanifolds. In Section 4, we express moduli spaces $\mathcal{M}_A(a;\mathbf{b})$ as 0-sets of certain C^1-maps between infinite-dimensional Banach manifolds. We say a moduli space is *transversely cut out* if 0 is a regular value of the corresponding map.

Proposition 1.2. *For a generic admissible Legendrian submanifold $L \subset \mathbb{C}^n \times \mathbb{R}$ the moduli space $\mathcal{M}_A(a; \mathbf{b})$ is a transversely cut out manifold of dimension*

$$d = \mu(A) + |a| - |\mathbf{b}| - 1, \tag{1.5}$$

provided $d \leq 1$. (In particular, if $d < 0$ then the moduli space is empty.)

Proposition 1.2 is proved in Section 6.8. If $u \in \mathcal{M}_A(a; \mathbf{b})$ we say that $d = \mu(A) + |a| - |\mathbf{b}|$ is the *formal dimension of* u, and if v is a transversely cut out disk of formal dimension 0 we say that v is a *rigid disk*.

The moduli spaces we consider might not be compact, but their lack of compactness can be understood. It is analogous to "convergence to broken trajectories" in Morse/Floer homology and gives rise to natural compactifications of the moduli spaces. This is also called Gromov compactness, which we cover in more detail in Section 8.

A *broken holomorphic curve*, $u = (u^1, \dots, u^N)$, is a union of holomorphic disks, $u^j : (D_{m_j}, \partial D_{m_j}) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(L))$, where each u^j has exactly one positive puncture p^j, with the following property. To each p^j with $j \geq 2$ is associated a negative puncture $q_j^k \in D_{m_k}$ for some $k \neq j$ such that $u^j(p^j) = u^k(q_j^k)$ and $q_{j'}^{k'} \neq q_j^k$ if $j \neq j'$, and such that the quotient space obtained from $D_{m_1} \cup \dots \cup D_{m_N}$ by identifying p^j and q_j^k for each $j \geq 2$ is contractible. The broken curve can be parameterized by a single smooth $v : (D_m, \partial D) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(L))$. A sequence u_α of holomorphic disks *converges* to a broken curve $u = (u^1, \dots, u^N)$ if the following holds:

- For every $j \leq N$, there exists a sequence $\phi_\alpha^j : D_m \to D_m$ of linear fractional transformations and a finite set $X^j \subset D_m$ such that $u_\alpha \circ \phi_\alpha^j$ converges to u^j uniformly with all derivatives on compact subsets of $D_m \setminus X^j$
- There exists a sequence of orientation-preserving diffeomorphisms $f_\alpha : D_m \to D_m$ such that $u_\alpha \circ f_\alpha$ converges in the C^0-topology to a parameterization of u.

Proposition 1.3. *Any sequence u_α in $\mathcal{M}_A(a; \mathbf{b})$ has a subsequence converging to a broken holomorphic curve $u = (u^1, \dots, u^N)$. Moreover, $u^j \in \mathcal{M}_{A_j}(a^j; \mathbf{b}^j)$ with $A = \sum_{j=1}^N A_j$ and*

$$\mu(A) + |a| - |\mathbf{b}| = \sum_{j=1}^{N} \left(\mu(A_j) + |a^j| - |\mathbf{b}^j| \right). \tag{1.6}$$

Heuristically this is the only type of non-compactness we expect to see in $\mathcal{M}_A(a; \mathbf{b})$: since $\pi_2(\mathbb{C}^n) = 0$, no holomorphic spheres can "bubble off" at an interior point of the sequence u_α, and since $\Pi_{\mathbb{C}}(L)$ is exact no disks without positive puncture can form either. Moreover, since $\Pi_{\mathbb{C}}(L)$ is compact, and since $\mathbb{C}^n$ has "finite geometry at infinity" (see Section 8), all holomorphic curves with a uniform bound on area must map to a compact set.

Proof. The main step is to prove convergence to some broken curve, which we defer to Section 8. The statement about the homology classes follows easily from the definition of convergence. Equation (1.6) follows from the definition of broken curves. □

We next show that a broken curve can be glued to form a family of non-broken curves. For this we need a little notation. Let $\mathbf{c}^1, \dots, \mathbf{c}^r$ be an ordered collection of words of Reeb chords. Let the length of (number of letters in) $\mathbf{c}^j$ be $l(j)$ and let $\mathbf{a} = a_1 \dots a_k$ be a word of Reeb-chords of length $k > 0$. Let $S = \{s_1, \dots, s_r\}$ be r distinct integers in $\{1, \dots, k\}$. Define the word $\mathbf{a}_S(\mathbf{c}^1, \dots, \mathbf{c}^r)$ of Reeb-chords of length $k - r + \sum_{j=1}^r l(j)$ as follows. For each index $s_j \in S$ remove a_{s_j} from the word $\mathbf{a}$ and insert at its place the word $\mathbf{c}^j$.

Proposition 1.4. *Let L be a generic admissible Legendrian submanifold. Let $\mathcal{M}_A(a;\mathbf{b})$ and $\mathcal{M}_B(c;\mathbf{d})$ be 0-dimensional transversely cut out moduli spaces and assume that the j-th Reeb chord in $\mathbf{b}$ is c. Then there exist a $\rho > 0$ and an embedding*

$$G\colon \mathcal{M}_A(a;\mathbf{b}) \times \mathcal{M}_B(c;\mathbf{d}) \times (\rho,\infty) \to \mathcal{M}_{A+B}(a;\mathbf{b}_{\{j\}}(\mathbf{d})).$$

Moreover, if $u \in \mathcal{M}_A(a;\mathbf{b})$ and $u' \in \mathcal{M}_B(c;\mathbf{d})$ then $G(u,u',\rho)$ converges to the broken curve (u,u') as $\rho \to \infty$, and any disk in $\mathcal{M}_A(a;\mathbf{b}_{\{j\}}(\mathbf{d}))$ with image sufficiently close to the image of (u,u') is in the image of G.

This follows from Proposition 7.1 and the definition of convergence to a broken curve.

1.4. **The differential and contact homology.** Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a generic admissible Legendrian submanifold, let $\mathcal{C}$ be its set of Reeb chords, and let $\mathcal{A}$ denote its algebra. For any generator $a \in \mathcal{C}$ of $\mathcal{A}$ we set

$$\partial a = \sum_{\dim\, \mathcal{M}_A(a;\mathbf{b})=0} (\#\mathcal{M}_A(a;\mathbf{b}))A\mathbf{b}, \tag{1.7}$$

where $\#\mathcal{M}$ is the number of points in $\mathcal{M}$ modulo 2, and where the sum ranges over all words $\mathbf{b}$ in the alphabet $\mathcal{C}$ and $A \in H_1(L)$ for which the above moduli space has dimension 0. We then extend ∂ to a map $\partial : \mathcal{A} \to \mathcal{A}$ by linearity and the Leibniz rule.

Since L is generic admissible, it follows from Propositions 1.3 and 1.4 that the moduli spaces considered in the definition of ∂ are compact 0-manifolds and hence consist of a finite number of points. Thus ∂ is well defined. Moreover,

Lemma 1.5. *The map $\partial : \mathcal{A} \to \mathcal{A}$ is a differential of degree -1. That is, $\partial \circ \partial = 0$ and $|\partial(a)| = |a| - 1$ for any generator a of $\mathcal{A}$.*

Proof. After Propositions 1.3 and 1.4 the standard proof in Morse (or Floer) homology [28] applies. It follows from (1.5) that ∂ lowers degree by 1. □

The *contact homology of L* is

$$HC_*(\mathbb{R}^{2n+1}, L) = \mathrm{Ker}\ \partial / \mathrm{Im}\ \partial.$$

It is essential to notice that since ∂ respects the grading on $\mathcal{A}$ the contact homology is a graded algebra.

We note that ∂ also defines a differential of degree -1 on $\mathcal{A}_{\mathbb{Z}_2}(L)$.

1.5. **The invariance of contact homology under Legendrian isotopy.** In this section we show

Proposition 1.6. *If $L_t \subset \mathbb{R}^{2n+1}$, $0 \le t \le 1$ is a Legendrian isotopy between generic admissible Legendrian submanifolds then the contact homologies $HC_*(\mathbb{R}^{2n+1}, L_0)$, and $HC_*(\mathbb{R}^{2n+1}, L_1)$ are isomorphic.*

In fact we show something, that at least appears to be, stronger. Given a graded algebra $\mathcal{A} = \mathbb{Z}_2[G]\langle a_1, \dots, a_n\rangle$, where G is a finitely generated abelian group, a graded automorphism $\phi : \mathcal{A} \to \mathcal{A}$ is called *elementary* if there is some $1 \le j \le n$ such that

$$\phi(a_i) = \begin{cases} A_i a_i, & i \neq j \\ \pm A_j a_j + u, & u \in \mathcal{A}(a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n),\ i = j, \end{cases}$$

where the A_i are units in $\mathbb{Z}_2[G]$. The composition of elementary automorphisms is called a *tame* automorphism. An isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ is tame if it is the composition of a tame automorphism with an isomorphism sending the generators of $\mathcal{A}$ to the generators of $\mathcal{A}'$. An isomorphism of DGA's is called *tame* if the isomorphism of the underlying algebras is tame.

Let $(\mathcal{E}_i, \partial_i)$ be a DGA with generators $\{e^i_1, e^i_2\}$, where $|e^i_1| = i, |e^i_2| = i-1$ and $\partial_i e^i_1 = e^i_2, \partial_i e^i_2 = 0$. Define the *degree i stabilization* $S_i(\mathcal{A}, \partial)$ of $(\mathcal{A}, \partial)$ to be the graded algebra generated by $\{a_1, \dots, a_n, e^i_1, e^i_2\}$ with grading and differential induced from $\mathcal{A}$ and $\mathcal{E}_i$. Two differential graded algebras are called *stable tame isomorphic* if they become tame isomorphic after each is stabilized a suitable number of times.

Proposition 1.7. *If $L_t \subset \mathbb{R}^{2n+1}$, $0 \leq t \leq 1$ is a Legendrian isotopy between generic admissible Legendrian submanifolds then the DGA's $(\mathcal{A}(L_0), \partial)$ and $(\mathcal{A}(L_1), \partial)$ are stable tame isomorphic.*

Note that Proposition 1.7 allows us to associate the stable tame isomorphism class of a DGA to a Legendrian isotopy class of Legendrian submanifolds: any Legendrian isotopy class has a generic admissible representative and by Proposition 1.7 the DGA's of any two generic admissible representatives agree.

It is straightforward to show that two stable tame isomorphic DGA's have the same homology, see [2, 11]. Thus Proposition 1.6 follows from Proposition 1.7. The proof of the later given below is, in outline, the same as the proof of invariance of the stable tame isomorphism class of the DGA of a Legendrian 1-knot in [2]. However, the details in our case require considerably more work. In particular we must substitute analytic arguments for the purely combinatorial ones that suffice in dimension three.

In Section 2 we show that any two admissible Legendrian submanifolds of dimension $n > 2$ which are Legendrian isotopic are isotopic through a special kind of Legendrian isotopy: a Legendrian isotopy $\phi_t \colon L \to \mathbb{C}^n \times \mathbb{R}$, $0 \leq t \leq 1$, is *admissible* if $\phi_0(L)$ and $\phi_1(L)$ are admissible Legendrian submanifolds and if there exist a finite number of instants $0 < t_1 < t_2 < \dots < t_m < 1$ and a $\delta > 0$ such that the intervals $[t_j - \delta, t_j + \delta]$ are disjoint subsets of $(0, 1)$ with the following properties.

(A) For $t \in [0, t_1 - \delta] \cup \left(\bigcup_{j=1}^m [t_j + \delta, t_{j+1} - \delta]\right) \cup [t_m + \delta, 1]$, $\phi_t(L)$ is an isotopy through admissible Legendrian submanifolds.

(B) For $t \in [t_j - \delta, t_j + \delta]$, $j = 1, \dots, m$, $\phi_t(L)$ undergoes a *standard self-tangency move.* That is, there exists a point $q \in \mathbb{C}^n$ and neighborhoods $N \subset N'$ of q with the following properties. The intersection $N \cap \Pi_{\mathbb{C}}(\phi_t(L))$ equals $P_1 \cup P_2(t)$ which, up to biholomorphism looks like $P_1 = \gamma_1 \times P_1'$ and $P_2 = \gamma_2(t) \times P_2'$. Here γ_1 and $\gamma_2(t)$ are subarcs around 0 of the curves $y_1 = 0$ and $x_1^2 + (y_1 - 1 \pm t)^2 = 1$ in the z_1-plane, respectively, and P_1' and P_2' are real analytic Lagrangian $(n-1)$-disks in $\mathbb{C}^{n-1} = \{z_1 = 0\}$ intersecting transversely at 0. Outside $N' \times \mathbb{R}$ the isotopy is constant. See Figure 2. (The full definition of a standard self tangency move appears in Section 2. For simplicity, one technical condition there has been omitted at this point.)

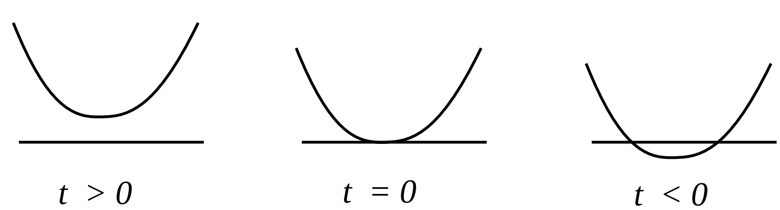

FIGURE 2. Type B double point move.

Note that two Legendrian isotopic admissible Legendrian submanifolds of dimension 1 are in general not isotopic through an admissible Legendrian isotopy. In this case one must allow also a "triple point move" see [2, 11].

To prove Proposition 1.7 we need to check that the differential graded algebra changes only by stable tame isomorphisms under Legendrian isotopies of type (A) and (B). We start with type (A) isotopies.

Lemma 1.8. *Let $L_t, t \in [0,1]$ be a type (A) isotopy between generic admissible Legendrian submanifolds. Then the DGA's associated to L_0 and L_1 are tame isomorphic.*

To prove this we use a parameterized version of Proposition 1.2. If $L_t, t \in I = [0,1]$ is a type (A) isotopy then the double points of $\Pi_{\mathbb{C}}(L_t)$ trace out continuous curves. Thus, when we refer to a Reeb chord c of $L_{t'}$ for some $t' \in [0,1]$ this unambiguously specifies a Reeb chord for all L_t. For any t we let $\mathcal{M}^t_A(a;\mathbf{b})$ denote the moduli space $\mathcal{M}_A(a;\mathbf{b})$ for L_t and define

$$\mathcal{M}^I_A(a;\mathbf{b}) = \{(u,t) | u \in \mathcal{M}^t_A(a;\mathbf{b})\}. \tag{1.8}$$

As above "generic" refers to a member of a Baire subset, see Section 6.2 for a more precise formulation of this term for 1-parameter families.

Proposition 1.9. *For a generic type (A) isotopy $L_t, t \in I = [0,1]$ the following holds. If $a, \mathbf{b}, A$ are such that $\mu(A) + |a| - |\mathbf{b}| = d \leq 1$ then the moduli space $\mathcal{M}^I_A(a;\mathbf{b})$ is a transversely cut out d-manifold. If X is the union of all these transversely cut out manifolds which are 0-dimensional then the components of X are of the form $\mathcal{M}^{t_j}_{A_j}(a_j, \mathbf{b}_j)$, where $\mu(A_j) + |a_j| - |\mathbf{b}_j| = 0$, for a finite number of distinct instances $t_1, \dots, t_r \in [0,1]$. Furthermore, $t_1, \dots, t_r$ are such that $\mathcal{M}^{t_j}_B(c;\mathbf{d})$ is a transversely cut out 0-manifold for every $c, \mathbf{d}, B$ with $\mu(B) + |c| - |\mathbf{d}| = 1$.*

Proposition 1.9 is proved in Section 6.9. At an instant $t = t_j$ in the above proposition we say a *handle slide* occurs, and an element in $\mathcal{M}^{t_j}_{A_j}(a_j, \mathbf{b}_j)$ will be called a *handle slide disk.* (The term handle slide comes form the analogous situation in Morse theory.)

The proof of Lemma 1.8 depends, just as the proof of Lemma 1.5, on one compactness- and one gluing result; the following.

Proposition 1.10. *Any sequence u_α in $\mathcal{M}^I_A(a;\mathbf{b})$ has a subsequence that converges to a broken holomorphic curve with the same properties as in Proposition 1.3.*

The proof of this proposition is identical to that of Proposition 1.3, see Section 8.

Proposition 1.11. *Let $\delta > 0$ and let $L_t, t \in I = [-\delta, \delta]$ be a small neighborhood of a handle slide at $t = 0$ in a generic type (A) isotopy. Then for δ sufficiently small, $L_{\pm\delta}$ are generic admissible and, with $u \in \mathcal{M}^0_A(a;\mathbf{b})$ denoting the handle slide disk, the following holds.*

(1) *Assume that c is the j-th letter in $\mathbf{b}$. Let $\mathcal{M}^0_B(c;\mathbf{d})$ be a moduli space of rigid holomorphic disks. Then there exist $\rho_0 > 0$ and an embedding*

$$G\colon \mathcal{M}^0_B(c;\mathbf{d}) \times [\rho_0, \infty) \to \mathcal{M}^I_{A+B}(a;\mathbf{b}_{\{j\}}(\mathbf{d})).$$

Given $v \in \mathcal{M}^0_B(c;\mathbf{d})$, $G(v,\rho)$ converges to the broken curve (v,u) as $\rho \to \infty$. Moreover, any curve in $\mathcal{M}^I_{A+B}(a;\mathbf{b}_{\{j\}}(\mathbf{d}))$ with image sufficiently close to the image of (v,u) is in the image of G.

(2) *Let $\mathcal{M}^0_B(c;\mathbf{d})$ be a moduli space of rigid holomorphic disks, where $S = \{s_1, \dots, s_r\}$, and $\mathbf{d}$ has a at every position of an element in S. Then there exist $\rho_0 > 0$, and an embedding*

$$G'\colon \mathcal{M}^0_B(c;\mathbf{d}) \times [\rho_0, \infty) \to \mathcal{M}^I_{B+r\cdot A}(c;\mathbf{d}_S(\mathbf{b}, \dots, \mathbf{b})).$$

Given $v \in \mathcal{M}^0_B(c;\mathbf{d})$, $G'(v,\rho)$ converges to the broken curve $(v, u, \dots, u)$. Moreover, any curve in $\mathcal{M}^I_{B+r\cdot A}(c;\mathbf{d}_S(\mathbf{b}, \dots, \mathbf{b}))$ with image sufficiently close to the image of $(v, u, \dots, u)$ is in the image of G'.

Proof. This proposition follows from Theorem 7.2 and Theorem 7.3. We show here why the above are the only kind of broken curves to consider gluing. If the broken curve lives in the compactification of the one-dimensional $\mathcal{M}^I_B(c_0;\mathbf{c})$, then by (1.6) at least one of its pieces must have negative formal dimension. Since the handle slide disk u is the only disk with negative formal dimension, all but one of the pieces of the broken disk must be u. The requirement that our disks have just one positive puncture and Lemma 1.1 reduce all possible configurations of the broken curve to the ones considered above. □

We now prove Lemma 1.8 in two steps. First consider type (A) isotopies without handle slides.

Lemma 1.12. *Let $L_t, t \in [0,1]$ be a generic type (A) isotopy of Legendrian submanifolds for which no handle slides occur. Then the boundary maps ∂_0 and ∂_1 on $\mathcal{A} = \mathcal{A}(L_0) = \mathcal{A}(L_1)$ satisfies $\partial_0 = \partial_1$.*

Proof. Propositions 1.10 and 1.11 imply that $\mathcal{M}^I_A(a;\mathbf{B})$ is compact when its dimension is one. Since if a sequence in this space converged to a broken curve $(u^1,\dots,u^N)$ then at least one u^j would have negative formal dimension. This contradicts the assumptions that no handle slide occurs and that the type (A) isotopy is generic. Thus the corresponding 0 dimensional moduli spaces $\mathcal{M}^0_A$ and $\mathcal{M}^1_A$ used in the definitions of ∂_0 and ∂_1, respectively, form the boundary of a compact 1-manifold. Hence their modulo 2 counts are equal. □

We consider what happens around a handle slide instant. Let L_t, $t \in [-\delta,\delta]$ and $\mathcal{M}^0_A(a;\mathbf{b})$ be as in Lemma 1.11. Let ∂_- denote the differential on $\mathcal{A} = \mathcal{A}(L_{-\delta})$, and ∂_+ the one on $\mathcal{A} = \mathcal{A}(L_\delta)$. For generators c in $\mathcal{A}$ define

$$\phi_a(c) = \begin{cases} c & \text{if } c \neq a, \\ a + A\mathbf{b} & \text{if } c = a. \end{cases}$$

and extend ϕ_a to a tame algebra automorphism of $\mathcal{A}$.

Lemma 1.13. *Let c be a generator of $\mathcal{A}$ then*

$$\partial_+ c = \begin{cases} \phi_a(\partial_- c) & \text{if } c \neq a, \\ \partial_-(\phi_a(c)) & \text{if } c = a. \end{cases}$$

Proof. Any $\alpha \in \mathcal{A}$ can be expressed in a unique way as a $\mathbb{Z}_2$-linear combination of elements $C\mathbf{w}$, where $C \in H_1(L)$ and $\mathbf{w}$ is a word in the generators of $\mathcal{A}$, see (1.3). Let $\langle \alpha, C\mathbf{w}\rangle$ denote the coefficient (0 or 1) in such an expansion. It follows from Proposition 1.11 that for any generator $c \neq a$

$$\langle (\partial_+ - \partial_-)c, B\mathbf{w}_1\mathbf{b}\mathbf{w}_2\rangle = \langle \partial_- c, (BA^{-1})\mathbf{w}_1 a \mathbf{w}_2\rangle.$$

From this, the formula for $\partial_+ c$ follows when $c \neq a$. The formula when $c = a$ follows similarly. □

Lemma 1.14. *The map $\phi_a : \mathcal{A} \to \mathcal{A}$ is a tame isomorphism from $(\mathcal{A}, \partial_-)$ to $(\mathcal{A}, \partial_+)$.*

Proof. As ϕ_a is clearly a tame isomorphism of algebras we only need to check that it is also a chain map. If $c \neq a$ is a generator then $\phi_a\partial_- c = \partial_+ c = \partial_+\phi_a c$. It follows from Lemma 1.1 that $\partial_+ a$ contains no terms which contain an a and that the word $\mathbf{b}$ does not contain the letter a. Thus $\partial_+ A\mathbf{b} = \partial_- A\mathbf{b}$ and hence

$$\phi_a\partial_- a = \phi_a\partial_-(\phi_a(a + A\mathbf{b})) = \phi_a(\partial_+ a + \partial_+ A\mathbf{b}) = \partial_+(a + A\mathbf{b}) = \partial_+\phi_a a.$$

□

Proof of Lemma 1.8. The lemma follows from Lemmas 1.12, 1.13, and 1.14. □

We consider elementary isotopies of type (B). Let $L_t, t \in I = [-\delta, \delta]$ be an isotopy of type (B) where two Reeb chords $\{a, b\}$ are born as t passes through 0. Let o be the degenerate Reeb chord (double point) at $t = 0$ and let $\mathcal{C}' = \{a_1, \dots, a_l, b_1, \dots, b_m\}$ be the other Reeb chords. Again we note that $c_i \in \mathcal{C}'$ unambiguously defines a Reeb chord for all L_t and a and b unambiguously define two Reeb chords for all L_t when $t > 0$. It is easy to see that (with the appropriate choice of capping paths) the grading on a and b differ by 1 so let $|a| = j$ and $|b| = j - 1$. Let $(\mathcal{A}_-, \partial_-)$ and $(\mathcal{A}_+, \partial_+)$ be the DGA's associated to $L_{-\delta}$ and L_δ, respectively.

Lemma 1.15. *The stabilized algebra $S_j(\mathcal{A}_-, \partial_-)$ is tame isomorphic to $(\mathcal{A}_+, \partial_+)$.*

Proof of Proposition 1.7 and 1.6 . The first proposition follows from Lemmas 1.8 and 1.15 and implies in its turn the second. □

We prove Lemma 1.15 in several steps below. Label the Reeb chords of L_t so that

$$\mathcal{Z}(b_m) \leq \dots \leq \mathcal{Z}(b_1) \leq \mathcal{Z}(b) < \mathcal{Z}(a) \leq \mathcal{Z}(a_1) \leq \dots \leq \mathcal{Z}(a_l),$$

let $\mathcal{B} = \mathbb{Z}_2[H_1(L)]\langle b_1, \dots, b_m\rangle$ and note that $\mathcal{B}$ is a subalgebra of both $\mathcal{A}_-$ and $\mathcal{A}_+$. Then

Lemma 1.16. *For $\delta > 0$ small enough*

$$\partial_+ a = b + v,$$

where $v \in \mathcal{B}$.

Proof. Let $\mathbf{0} \in H_1(L)$ denote the zero element. In the model for the type (B) isotopy there is an obvious disk in $\mathcal{M}^t_{\mathbf{0}}(a; b)$ for $t > 0$ small which is contained in the z_1-plane. We argue that this is the only point in the moduli space. We restrict attention to the neighborhood N of o^* that is biholomorphic to the origin in $\mathbb{C}^n$ as in the description of a type (B) move. Let $\pi_i : \mathbb{C}^n \to \mathbb{C}$ be the projection onto the i^{th} coordinate. If $u : D \to \mathbb{C}^n$ is a holomorphic map in $\mathcal{M}^t_{\mathbf{0}}(a; b)$ then $\pi_i \circ u$ will either be constant or not. If $\pi_i \circ u$ is non-constant for $i > 1$ then the image of $\pi_1 \circ u$ intersected with N has boundary on two transverse Lagrangian submanifolds. As such it will have a certain area A_i. Since $\mathcal{Z}(a) - \mathcal{Z}(b) \to 0$ as $t \to 0+$ we can choose t small enough so that $\mathcal{Z}(a) - \mathcal{Z}(b) < A_i$, for all $i > 1$. Then $\pi_i \circ u$ must be a point for all $i > 1$ and for $i = 1$, it can only be the obvious disk. Lemma 6.24 shows that $\mathcal{M}^t_{\mathbf{0}}(a; b)$ is transversely cut out and thus contributes to $\partial_+ a$. If $u \in \mathcal{M}^t_A(a; b)$, where $A \neq \mathbf{0}$ then the image of u must leave N. Thus, the above argument shows that $\mathcal{M}^t_A(a; b) = \emptyset$ for t small enough. Also, for $t > 0$ sufficiently small $\mathcal{Z}(a) - \mathcal{Z}(b) < \mathcal{Z}(b_m)$. Hence by Lemma 1.1, $v \in \mathcal{B}$. □

Define the elementary isomorphism $\Phi_0 : \mathcal{A}_+ \to S_j(\mathcal{A}_-)$ (on generators) by

$$\Phi_0(c) = \begin{cases} e^j_1 & \text{if } c = a, \\ e^j_2 + v & \text{if } c = b \\ c & \text{otherwise.} \end{cases}$$

The map Φ_0 fails to be a tame isomorphism since it is not a chain map. However, we use it as the first step in an inductive construction of a tame isomorphism $\Phi_l \colon \mathcal{A}_+ \to S_j(\mathcal{A}_-)$. To this end, for $0 \leq i \leq l$, let $\mathcal{A}_i$ be the subalgebra of $\mathcal{A}_+$ generated by $\{a_1, \dots, a_i, a, b, b_1, \dots, b_m\}$ (note that $\mathcal{A}_l = \mathcal{A}_+$). Then, with $\tau : S_j(\mathcal{A}_-) \to \mathcal{A}_-$ denoting the natural projection and with ∂^s_- denoting the differential induced on $S_j(\mathcal{A}_-)$, we have

Lemma 1.17.

$$\Phi_0 \circ \partial_+ w = \partial^s_- \circ \Phi_0 w \tag{1.9}$$

for $w \in \mathcal{A}_0$ and

$$\tau \circ \Phi_0 \circ \partial_+ = \tau \circ \partial^s_- \circ \Phi_0. \tag{1.10}$$

Before proving this lemma, we show how to use it in the inductive construction which completes the proof of Lemma 1.15.

Proof of Lemma 1.15. The proof is similar to the proof of Lemmas 6.3 and 6.4 in [11] (*cf* [2]). Define the map $H : S_j(\mathcal{A}_-) \to S_j(\mathcal{A}_-)$ on words $\mathbf{w}$ in the generators by

$$H(\mathbf{w}) = \begin{cases} 0 & \text{if } \mathbf{w} \in \mathcal{A}_-, \\ 0 & \text{if } \mathbf{w} = \alpha e_1^j \beta \text{ and } \alpha \in \mathcal{A}_- \\ \alpha e_1^j \beta & \text{if } \mathbf{w} = \alpha e_2^j \beta \text{ and } \alpha \in \mathcal{A}_-, \end{cases}$$

and extend it linearly. Assume inductively that we have defined a graded isomorphism $\Phi_{i-1} : \mathcal{A}_+ \to S_j(\mathcal{A}_-)$ so that it is a chain map when restricted to $\mathcal{A}_{i-1}$ and so that $\Phi_{i-1}(a_k) = a_k$, for $k > i-1$. (Note that Φ_0 has these properties by Lemma 1.17.)

Define the elementary isomorphism $g_i : S_j(\mathcal{A}_-) \to S_j(\mathcal{A}_-)$ on generators by

$$g_i(c) = \begin{cases} c & \text{if } c \neq a_i, \\ a_i + H \circ \Phi_{i-1} \circ \partial_+(a_i) & \text{if } c = a_i \end{cases}$$

and set $\Phi_i = g_i \circ \Phi_{i-1}$. Then Φ_i is a graded isomorphism. To see that Φ_i is a chain map when restricted to $\mathcal{A}_i$ observe the following facts: $\tau \circ H = 0, \tau \circ g_i = \tau$, and $\tau \circ \Phi_i = \tau \circ \Phi_0$ for all i. Moreover, $\partial_+ a_i \in \mathcal{A}_{i-1}$ and $\tau - \mathrm{id}_{S_j(\mathcal{A}_-)} = \partial_-^s \circ H + H \circ \partial_-^s$, where in the last equation we think of $\tau\colon S_j(\mathcal{A}_-) \to S_j(\mathcal{A}_-)$ as $\tau\colon S_j(\mathcal{A}_-) \to \mathcal{A}_-$ composed with the natural inclusion.

Using these facts we compute

$$\begin{aligned} \partial_-^s g_i(a_i) =& \partial_-^s(a_i) + (\partial_-^s H)\Phi_{i-1}\partial_+(a_i) = \partial_-^s(a_i) + (H\partial_-^s + \tau + \mathrm{id})\Phi_{i-1}\partial_+(a_i) \\ =& \partial_-^s(a_i) + \tau\Phi_0\partial_+(a_i) + \Phi_{i-1}\partial_+(a_i) = \Phi_{i-1}\partial_+(a_i). \end{aligned}$$

Thus $\Phi_i \circ \partial_+(a_i) = \partial_-^s \circ g_i(a_i) = \partial_-^s \circ \Phi_i(a_i)$. Since Φ_i and Φ_{i-1} agree on $\mathcal{A}_{i-1}$ it follows that Φ_i is a chain map on $\mathcal{A}_i$. Continuing we eventually get a tame chain isomorphism $\Phi_l : \mathcal{A}_+ \to S_j(\mathcal{A}_-)$. □

The proof of Lemma 1.17 depends on the following two propositions.

Proposition 1.18. *Let $L_t, t \in I = [-\delta, \delta]$ be a generic Legendrian isotopy of type (B) with notation as above (that is, o is the degenerate Reeb chord of L_0 and the Reeb chords a and b are born as t increases past 0).*

(1) *Let $\mathcal{M}_A^0(o, \mathbf{c})$ be a moduli space of rigid holomorphic disks. Then there exist $\rho > 0$ and a local homeomorphism*

$$S\colon \mathcal{M}_A^0(o; \mathbf{c}) \times [\rho, \infty) \to \mathcal{M}_A^{(0,\delta]}(a; \mathbf{c}),$$

with the following property. If $u \in \mathcal{M}_A^0(o; \mathbf{c})$ then any disk in $\mathcal{M}_A^{(0,\delta]}(a; \mathbf{c})$ sufficiently close to the image of u is in the image of S.

(2) *Let $\mathcal{M}_A^0(c, \mathbf{d})$ be a moduli space of rigid holomorphic disks. Let $S \subset \{1, \dots, m\}$ be the subset of positions of $\mathbf{d}$ where the Reeb chord o appears (to avoid trivialities, assume $S \neq \emptyset$). Then there exists $\rho > 0$ and a local homeomorphism*

$$S'\colon \mathcal{M}_A^0(c, \mathbf{d}) \times [\rho, \infty) \to \mathcal{M}_A^{(0,\delta]}(c, \mathbf{d}_S(b)),$$

with the following property. If $u \in \mathcal{M}_0(c; \mathbf{d})$ then any disk in $\mathcal{M}_A^{(0,\delta]}(c; \mathbf{d}_S(b))$ sufficiently close to the image of u is in the image of S'.

This is a rephrasing of Theorem 7.4 and the following proposition is a restatement of Theorem 7.5

Proposition 1.19. *Let $L_t, t \in I = [-\delta, \delta]$ be a generic isotopy of type (B). Let $\mathcal{M}^0_{A_1}(o; \mathbf{c}^1)$, $\dots$, $\mathcal{M}^0_{A_r}(o; \mathbf{c}^r)$, and $\mathcal{M}^0_B(c; \mathbf{d})$ be moduli spaces of rigid holomorphic disks. Let $S \subset \{1, \dots, m\}$ be the subset of positions in $\mathbf{d}$ where the Reeb chord o appears and assume that S contains r elements. Then there exists $\rho > 0$ and an embedding*

$$G\colon \mathcal{M}^0_B(c; \mathbf{d}) \times \Pi_{j=1}^r \mathcal{M}^0_{A_j}(o; \mathbf{d}^j) \times [\rho, \infty) \to \mathcal{M}^{[-\delta,0)}_{B+\sum A_j}(c; \mathbf{d}_S(\mathbf{c}^1, \dots, \mathbf{c}^r)),$$

with the following property. If $v \in \mathcal{M}_0(c; \mathbf{d})$ and $u_j \in \mathcal{M}_0(o; \mathbf{c}^j)$, $j = 1, \dots, r$ then any disk in $\mathcal{M}^{[-\delta,0)}_{B+\sum A_j}(c; \mathbf{d}_S(\mathbf{c}^1, \dots, \mathbf{c}^r))$ sufficiently close to the image of $(v, u_1, \dots, u_r)$ is in the image of G.

Proof of Lemma 1.17. Equation (1.9) follows from arguments similar to those in Lemma 1.8. Specifically, one can use these arguments to show that $\partial_+ b_i = \partial_- b_i$. Then since $\partial_+ b_i \in \mathcal{B}$ and since Φ_0 is the identity on $\mathcal{B}$,

$$\Phi_0 \partial_+ b_i = \partial_+ b_i = \partial_- b_i = \partial^s_- \Phi_0 b_i.$$

We also compute

$$\Phi_0 \partial_+ a = \Phi_0(b + v) = e^j_2 + v + v = e^j_2 = \partial^s_- \Phi_0 a,$$

and, since $\partial_+ b$ and $\partial_+ v$ both lie in $\mathcal{B}$,

$$\Phi_0 \partial_+ b = \partial_+ b, \quad \partial^s_- \Phi_0 b = \partial^s_-(e^j_1 + v) = \partial_- v = \partial_+ v.$$

Since $0 = \partial_+ \partial_+ a = \partial_+ b + \partial_+ v$, we conclude that (1.9) holds.

To check (1.10), we write $\partial_+ a_i = W_1 + W_2 + W_3$, where W_1 lies in the subalgebra generated by $\{a_1, \dots, a_l, b_1, \dots, b_m\}$, where W_2 lies in the ideal generated by a and where W_3 lies in the ideal generated by b in the subalgebra generated by $\{a_1, \dots, a_l, b, b_1, \dots, b_m\}$.

Let u_t, be a family of holomorphic disks with boundary on L_t. As $t \to 0$, u_t converges to a broken disk $(u^1, \dots, u^N)$ with boundary on L_0. This together with the genericity of the type (B) isotopy implies that for $t \neq 0$ small enough there are no disks of negative formal dimension with boundary on L_t since a broken curve which is a limit of a sequence of such disks would have at least one component u^j with negative formal dimension.

Let $u_s\colon D \to \mathbb{C}^n$, $s \neq 0$ be rigid disks with boundary on L_s. If, the image $u_{-t}(\partial D)$ stays a positive distance away from o^* as $t \to 0+$ then the argument above implies that u_{-t} converges to a non-broken curve. Hence $\partial_- a_i = W_1 + W_4$ where for each rigid disk $u_{-t}\colon D \to \mathbb{C}^n$ contributing to a word in W_4 there exists points $q_{-t} \in \partial D$ such that $u_{-t}(q_{-t}) \to o^*$ as $t \to 0+$. The genericity assumption on the type (B) isotopy implies that no rigid disk with boundary on L_0 maps any boundary point to o^*, see Corollary 6.22. Hence u_{-t} must converge to a broken curve $(u^1, \dots, u^N)$ which brakes at o^*. Moreover, by genericity and (1.6), every component u^j of the broken curve must be a rigid disk with boundary on L_0. Proposition 1.19 shows that any such broken curve may be glued and Proposition 1.18 determines the pieces which we may glue. It follows that $W_4 = \hat{W}_2$ where $\hat{W}_2$ is obtained from W_2 by replacing each occurrence of b with v. Therefore,

$$\tau \Phi_0 \partial_+(a_i) = \tau \Phi_0(W_1 + W_2 + W_3) = W_1 + \hat{W}_2 = \partial_-(a_i) = \tau \partial^s_- \Phi_0(a_i).$$

□

2. Admissible Legendrian submanifolds and isotopies

2.1. **Chord genericity.** Recall that a Legendrian submanifold $L \subset \mathbb{R}^{2n+1}$ is chord generic if all its Reeb chords correspond to transverse double points of the Lagrangian projection $\Pi_{\mathbb{C}}$. For a dense open set in the space of paths of Legendrian embeddings, the corresponding 1-parameter families L_t, $0 \le t \le 1$, are chord generic except for a finite number of parameter values $t_1, \dots, t_k$ where $\Pi_{\mathbb{C}}(L_{t_j})$ has one double point with self-tangency, and where for some

$\delta > 0$ $\Pi_{\mathbb{C}}(L_t)$, $(t_j - \delta, t_j + \delta)$, is a versal deformation of $\Pi_{\mathbb{C}}(L_{t_j})$, for $j = 1, \ldots, k$. We call 1-parameter families L_t with this property *chord generic 1-parameter families.*

2.2. **Local real analyticity.** For technical reasons, we require our Legendrian submanifolds to be real analytic in a neighborhood of the endpoints of their Reeb chords and that self-tangency instants in 1-parameter families have a very special form.

Definition 2.1. A chord generic Legendrian submanifold $L \subset \mathbb{C}^n \times \mathbb{R}$ is *admissible* if for any Reeb chord c of L with endpoints q_1 and q_2 there are neighborhoods $U_1 \subset L$ and $U_2 \subset L$ of q_1 and q_2, respectively, such that $\Pi_{\mathbb{C}}(U_1)$ and $\Pi_{\mathbb{C}}(U_2)$ are real analytic submanifolds of $\mathbb{C}^n$.

We will require that self-tangency instants in 1-parameter families have the following special form. Consider $0 \in \mathbb{C}^n$ and coordinates $(z_1, \ldots, z_n)$ on $\mathbb{C}^n$. Let P_1 and P_2 be Lagrangian submanifolds of $\mathbb{C}^n$ passing through 0. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ be coordinates on P_1 and P_2, respectively. Let $R_1 \subset P_1$ and $R_2 \subset P_2$ be the boxes $|x_j| \leq 1$ and $|y_j| \leq 1$, $j = 1, \ldots, n$. Let $B_j(2)$ and $B_j(2+\epsilon)$ for some small $\epsilon > 0$ be the balls of radii 2 and $2+\epsilon$ around $0 \in P_j$, $j = 1, 2$. We require that in R_1, P_1 has the form

$$\gamma_1 \times \hat{P}_1 \tag{2.1}$$

where γ_1 is an arc around 0 in the real line in the z_1-plane and where $\hat{P}_1$ is a Lagrangian submanifold of $\mathbb{C}^{n-1} \approx \{z_1 = 0\}$. We require that in R_2, P_2 has the form

$$\gamma_2(t) \times \hat{P}_2, \tag{2.2}$$

where γ_2 is an arc around 0 in the unit-radius circle centered at i in z_1-plane and where $\hat{P}_2$ is a Lagrangian submanifold of $\mathbb{C}^{n-1} \approx \{z_1 = 0\}$, which meets $\hat{P}_1$ transversally at 0.

If $q \in \mathbb{C}^n$ let λ_q denote the complex line in $T_q\mathbb{C}^n$ parallel to the z_1-line. We also require that for every point $p \in B_j(2+\epsilon) \setminus B_j(2)$ the tangent plane T_pP_j satisfies

$$T_pP_j \cap \lambda_p = 0, \quad j = 1, 2. \tag{2.3}$$

Definition 2.2. Let L_t be a chord generic 1-parameter family of Legendrian submanifolds such that L_0 has a self-tangency. We say that the self-tangency instant L_0 is *standard* if there is some neighborhood U of the self-tangency point and a biholomorphism $\phi\colon U \to V \subset \mathbb{C}^n$ such that

$$\phi(L_t \cap U) = P_1 \cup P_2(t) \cap N, \tag{2.4}$$

where N is some neighborhood of $0 \in \mathbb{C}^n$, and where $P_2(t)$ is P_2 transalted t units in the y_1-direction.

Definition 2.3. Let L_t, $0 \leq t \leq 1$ be a chord generic 1-parameter family of Legendrian submanifolds. Let $t_1, \ldots, t_k$ be its self tangency instants. We say that L_t is an *admissible* 1-parameter family if L_t is admissible for all $t \neq t_k$, if there exists small disjoint intervals $(t_j - \delta, t_j + \delta)$ where the 1-parameter family is constant outside some small neighborhood W of the self-tangency point, and if all self-tangency instants are standard.

Definition 2.4. A Legendrian submanifold $L \subset \mathbb{R} \times \mathbb{C}^n$ which is a self-tangency instant of an admissible 1-parameter family will be called *semi-admissible.*

2.3. **Reducing the Legendrian isotopy problem.** We prove a sequence of lemmas which reduce the classification of Legendrian submanifolds up to Legendrian isotopy to the classification of admissible Legendrian submanifolds up to admissible Legendrian isotopy.

We start with a general remark concerning lifts of Hamiltonian isotopies in $\mathbb{C}^n$. If h is a smooth function with compact support in $\mathbb{C}^n$ then the Hamiltonian vector field

$$X_h = -\frac{\partial h}{\partial y_i}\partial_{x_i} + \frac{\partial h}{\partial x_i}\partial_{y_i}$$

associated to h generates a 1-parameter family of diffeomorphisms Φ_h^t of $\mathbb{C}^n$. Moreover, the vector field X_h lifts to a contact vector field

$$\tilde{X}_h = -\frac{\partial h}{\partial y_i}\partial_{x_i} + \frac{\partial h}{\partial x_i}\partial_{y_i} + \left(h - y_i\frac{\partial h}{\partial y_i}\right)\partial_z$$

on $\mathbb{C}^n \times \mathbb{R}$, which generates a 1-parameter family $\tilde{\Phi}_h^t$ of contact diffeomorphisms of $\mathbb{C}^n \times \mathbb{R}$ which is a lift of Φ_h^t. We write $\Phi_h = \Phi_h^1$ and similarly $\tilde{\Phi}_h = \tilde{\Phi}_h^1$.

We note for future reference that in case the preimage of the support of h in L has more than one connected component we may define a Legendrian isotopy of L by moving only one of these components (for a short time) using $\tilde{X}_h$ and keeping the rest of them fixed.

An *ϵ-isotopy* is an isotopy during which no point moves a distance larger than $\epsilon > 0$.

Lemma 2.5. *Let L be a Legendrian submanifold. Then, for any $\epsilon > 0$, there is an admissible Legendrian submanifold L_ϵ which is Legendrian ϵ-isotopic to L.*

Proof. As mentioned, we may after arbitrarily small Legendrian isotopy assume that L is chord generic. Thus, it is enough to consider one transverse double point. We may assume that one of the sheets of L close to this double point is given by $x \mapsto (x, df(x), f(x))$ for some smooth function f. Let g be a real analytic function approximating f (e.g. its Taylor polynomial of some degree). Consider a Hamiltonian h which is $h(x,y) = g(x) - f(x)$ in this neighborhood and 0 outside some slightly larger neighborhood. It is clear that the corresponding Hamiltonian vector field can be made arbitrarily small. Its flow map at time 1 is given by $\Phi_h^1(x,y) = (x, y + dg(x) - df(x))$. Using this and suitable cut-off functions for the lifted Legendrian isotopies the lemma follows. □

Lemma 2.6. *Let L_t be any chord generic Legendrian isotopy from an admissible Legendrian submanifold L_0 to another one L_1. Then for any $\epsilon > 0$, there is an admissible Legendrian isotopy ϵ-close to L_t connecting L_0 to L_1.*

Proof. Let $t_1, \dots, t_k$ be the self tangency instants of the isotopy. First change the isotopy so that there exists small disjoint intervals $(t_j - \delta, t_j + \delta)$ where the 1-parameter family is constant outside some small neighborhood W of the self-tangency point. Consider the restriction of the isotopy to the self-tangency free regions. The 1-parametric version of the proof of Lemma 2.5 clearly applies to transform this part of the isotopy into one consisting of admissible Legendrian submanifolds. Then change the isotopy in the neighborhoods of the self tangency instants, using essentially the same argument as above, to a self-tangency of the from given in (2.1) and (2.2).

It remains to show how to fulfill the condition (2.3). To this end, consider a Lagrangian submanifold of the form (2.1). Locally it is given by $(x, df(\hat{x}))$, where $\hat{x} = (x_2, \dots, x_n)$. Let $\phi(x)$ be a function which equals 0 in $B(2 - \frac{1}{2}\epsilon)$ and outside $B(2 + 2\epsilon)$ and has $\frac{\partial^2 \phi}{\partial x_1 x_j} \neq 0$ for some j at all points in $B(2+\epsilon) \setminus B(2)$. (For example if K is a small constant a suitable cut-off of the function $Kx_1(x_2 + \dots x_n)$ has this property). We see as above that our original Legendrian is Legendrian isotopic to $(x, df(x) + d\phi(x))$. The tangent space of the latter submanifold is spanned by the vectors

$$\partial_{x_1} + \sum_j \frac{\partial^2 \phi}{\partial x_j \partial x_1}\partial_{y_j}, \tag{2.5}$$

$$\partial_{x_r} + \sum_j \frac{\partial^2 \phi}{\partial x_j \partial x_r}\partial_{y_j} + \frac{\partial^2 f}{\partial x_j \partial x_r}\partial_{y_j}, \quad 2 \le r \le n. \tag{2.6}$$

Any non-trivial linear combination of the last $n-1$ vectors projects non-trivially to the subspace $dx_1 = dy_1 = dy_2 = \dots = dy_n = 0$. The first vector lies in the subspace $dx_2 = \dots =$

$dx_n = 0$; thus, since the first vector does not lie in the z_1-line because $\frac{\partial^2\phi}{\partial x_1 x_j} \neq 0$ for some $j \neq 1$, no linear combination of the vectors does either. P_2 can be deformed in a similar manner.

After the self-tangency moment is passed it is easy to Legendrian isotope back to the original family through admissible Legendrian submanifolds. □

3. Holomorphic disks

In this section we establish notation and ideas that will be used throughout the rest of the paper.

3.1. **Reeb chord notation.** Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a Legendrian submanifold and let c be a Reeb chord of L. The z-coordinate of the upper and lower end points of c will be denoted by c^+ and c^-, respectively. See Figure 3. So as a point set $c = c^* \times [c^-, c^+]$ and the action of c is simply $\mathcal{Z}(c) = c^+ - c^-$.

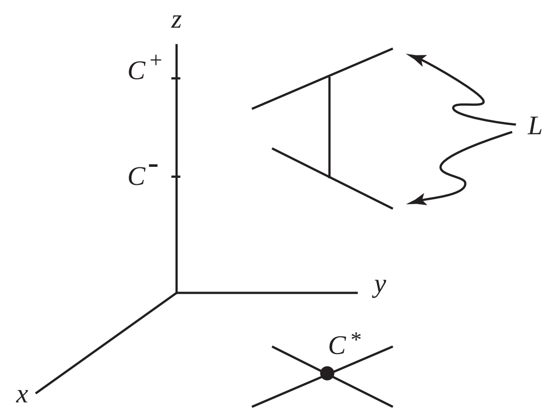

Figure 3. A Reeb chord in $\mathbb{R}^3$.

If $r > 0$ is small enough so that $\Pi_{\mathbb{C}}^{-1}(B(c^*, r))$ intersects L is exactly two disk about the upper and lower end points of c, then we define $U(c^\pm, r)$ to be the component of $\Pi_{\mathbb{C}}^{-1}(B(c^*, r)) \cap L$ containing $c^* \times c^\pm$.

3.2. **Definition of holomorphic disks.** If M is a smooth manifold then let $\mathcal{H}_k^{\mathrm{loc}}(M, \mathbb{C}^n)$ denote the Frechet space of all functions which agree locally with a function with k derivatives in L^2. Let $\Delta_m \subset \mathbb{C}$ denote the unit disk with m punctures on the boundary, let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a (semi-)admissible Legendrian submanifold.

Definition 3.1. A *holomorphic disk with boundary on* L consists of two functions $u \in \mathcal{H}_2^{\mathrm{loc}}(\Delta_m, \mathbb{C}^n)$ and $h \in \mathcal{H}_{\frac{3}{2}}(\partial\Delta_m, \mathbb{R})$ such that

$$\bar\partial u(\zeta) = 0, \text{ for } \zeta \in \operatorname{int}(\Delta_m), \tag{3.1}$$

$$(u(\zeta), h(\zeta)) \in L, \text{ for } \zeta \in \partial\Delta_m, \tag{3.2}$$

and such that for every puncture p on $\partial\Delta_m$ there exists a Reeb chord c of L such that

$$\lim_{\zeta \to p} u(\zeta) = c^*. \tag{3.3}$$

When (3.3) holds we say that (u, h) *maps the puncture* p *to the Reeb chord* c.

Remark 3.2. Since $u \in \mathcal{H}_2^{\mathrm{loc}}(\Delta_m, \mathbb{C}^n)$, the restriction of u to the boundary lies in $\mathcal{H}_{\frac{3}{2}}^{\mathrm{loc}}(\partial\Delta_m, \mathbb{C}^n)$. Therefore both u and its restriction to the boundary are continuous. Hence (3.2) and (3.3) make sense.

Remark 3.3. If $u \in \mathcal{H}_2^{\mathrm{loc}}(\Delta_m, \mathbb{C}^n)$ then $\bar\partial u \in \mathcal{H}_1^{\mathrm{loc}}(\Delta_m, T^{*0,1}D_m \otimes \mathbb{C}^n)$ and hence the trace of $\bar\partial u$ (its restriction to the boundary $\partial\Delta_m$) lies in $\mathcal{H}_{\frac{1}{2}}^{\mathrm{loc}}(\partial\Delta_m, T^{*0,1}D_m \otimes \mathbb{C}^n)$. If u is a holomorphic disk then $\bar\partial u = 0$ and hence its trace $\bar\partial u|\partial\Delta_m$ is also 0.

Remark 3.4. It turns out, see Section 8.5, that if (u, f) is a holomorphic disk with boundary on a smooth L, then the function u is in fact smooth up to and including the boundary and thus f is also smooth. Hence, it is possible to rephrase Definition 3.1 in terms of smooth functions. (Also, it follows that the definition above agrees with that given in Section 1.3.) The advantage of the present definition is that it allows for Legendrian submanifolds of lower regularity. (The Legendrian condition applies to submanifolds L which are merely C^1-smooth.)

3.3. **Conformal structures.** We describe the space of conformal structures on Δ_m as follows. If $m \le 3$, then the conformal structure is unique. Let $m > 4$ and let the punctures of Δ_m be $p_1, \dots, p_m$. Then fixing the positions of the punctures p_1, p_2, p_3 the conformal structure on Δ_m is determined by the position of the remaining $m-3$ punctures. In this way we identify the space of conformal structures $\mathcal{C}_m$ on Δ_m with an open simplex of dimension $m-3$.

3.4. **A family of metrics.** Let Δ denote the unit disk in the complex plane. Consider Δ_m with m punctures $p_1, \dots, p_m$ on the boundary and conformal structure κ. Let d be the smallest distance along $\partial\Delta$ between two punctures and take

$$\delta = \min\left\{\frac{d}{100}, \frac{\pi}{100}\right\}.$$

Define $D(p,\delta)$ to be a disk such that $\partial\Delta(p,\delta)$ intersects $\partial\Delta$ orthogonally at two points a_+ and a_- of distance δ (in $\partial\Delta$) from p.

Let L_p be the oriented tangent-line of $\partial\Delta$ at p and let g_p be the unique Möbius transformation which fixes p, maps a_+ to the point of distance δ from p along L_p, maps a_- to the point of distance $-\delta$ from p along L_p, and such that the image of $g_p(\Delta)$ intersects the component of $\mathbb{C} - L_p$ which intersects Δ.

The function $h_p\colon D(p,\delta) \cap \Delta_m \to [0,\infty) \times [0,1]$ defined by

$$h_p(\zeta) = -\frac{1}{\pi}\Big(\log\left(-i\bar p(g_p(\zeta) - p)\right) - \log(\delta)\Big),$$

is a conformal equivalence. Let g_0 denote the Euclidean metric on $\mathbb{C}$. Then there exists a function $s\colon [0,\frac{1}{2}] \times [0,1] \to \mathbb{R}$ such that $h_p^{-1*}g_0 = s(\zeta)g_0$ on $[0,\frac{1}{2}]\times[0,1]$. Let $\phi\colon [0,\infty) \to [0,1]$ be a smooth function which is 0 in a neighborhood of 0 and 1 in a neighborhood of $\frac{1}{2}$ for $\tau > \frac{1}{2}$. Let g_p be the metric

$$g_p(\tau + it) = \Big(\phi(\tau) + (1-\phi(\tau))s(t+it)\Big)g_0,$$

on $[0,\infty)\times[0,1]$.

Now consider Δ_m with the metric $g(\kappa)$ which agrees with $h_{p_j}^* g_{p_j}$ on $h_{p_j}^{-1}([\frac{1}{2\pi},\infty)\times[0,1])$ for each puncture p_j, and with g_0 on $\Delta_m - (D(p_1,\delta) \cup \dots, D(p_m,\delta))$. Then $(\Delta_m, g(\kappa))$ is conformally equivalent to (Δ_m, g_0).

We denote by $D_m(\kappa)$ the disk Δ_m with the metric $g(\kappa)$. If the specific κ is unimportant or clear from context we will simply write D_m. Also $E_{p_j} \subset D_m$ will denote the Euclidean neighborhood $[1,\infty)\times[0,1]$ of the j^{th} puncture p_j of D_m. We use coordinates $\zeta=\tau+it$ on E_{p_j} and let $E_{p_j}[M]$ denote the subset of $\tau+it\in E_{p_j}$ with $|\tau|\geq M$.

3.5. **Sobolev spaces.** Consider D_m with metric $g(\kappa)$ for some $\kappa\in\mathcal{C}_m$. Let $\hat{D}_m$ denote the open Riemannian manifold which is obtained by adding an open collar to ∂D_m and extending the metric in a smooth and bounded way to $\hat{g}(c)$.

The Sobolev spaces $\mathcal{H}_k^{\mathrm{loc}}(\hat{D}_m,\mathbb{C}^n)$ are now defined in the standard way as the space of $\mathbb{C}^n$-valued functions (distributions) the restrictions of which to any open ball B in any relatively compact coordinate chart $\approx\mathbb{R}^2$ lies in the usual Sobolev space $\mathcal{H}_k(B,\mathbb{C}^n)$.

Using the metric $\hat{g}(\kappa)$ and the finite cover

$$\bigcup_j \operatorname{int}(\hat{E}_{p_j}[1])\cup(\hat{D}_m-\cup_j\hat{E}_{p_j}[2]),$$

where $\hat{E}_{p_j}$ is the union of E_{p_j} and the corresponding part of the collar of $\hat{D}_m$, we define, for each integer k, the space $\mathcal{H}_k(\hat{D}_m,\mathbb{C}^n)$ as the subspace of all $f\in\mathcal{H}_k^{\mathrm{loc}}(\hat{D}_m,\mathbb{C}^n)$ with $\|f\|_k<\infty$.

We consider $\mathcal{H}_k(\hat{D}_m,\mathbb{C}^n)$ as a space of distributions acting on $C_0^\infty(\hat{D}_m,\mathbb{C}^n)$. We write

- $\mathcal{H}_k(D_m,\mathbb{C}^n)$ for the space of restrictions to $\operatorname{int}(D_m)\subset\hat{D}_m$ of elements in $\mathcal{H}_k(\hat{D}_m,\mathbb{C}^n)$, and
- $\dot{\mathcal{H}}_k(A,\mathbb{C}^n)$ for the set of distributions in $\mathcal{H}_k(\hat{D}_m,\mathbb{C}^n)$ supported in $A\subset\hat{D}_m$.

Then $\dot{\mathcal{H}}_k(D_m,\mathbb{C}^n)$ is a closed subspace of $\mathcal{H}_k(\hat{D}_m,\mathbb{C}^n)$ and if $K_m=\hat{D}_m-\operatorname{int}(D_m)$ then

$$\mathcal{H}_k(D_m,\mathbb{C}^n)=\mathcal{H}_k(\hat{D}_m,\mathbb{C}^n)/\dot{\mathcal{H}}_k(K_m,\mathbb{C}^n).$$

We endow $\mathcal{H}_k(D_m,\mathbb{C}^n)$ and $\dot{\mathcal{H}}_k(D_m,\mathbb{C}^n)$ with the quotient- and induced topology, respectively. Let $C_0^\infty(D_m,\mathbb{C}^n)$ denote the space of restrictions of elements in $C_0^\infty(\hat{D}_m,\mathbb{C}^n)$ to D_m.

Lemma 3.5. *$C_0^\infty(D_m,\mathbb{C}^n)$ is dense in $\mathcal{H}_k(D_m,\mathbb{C}^n)$, $C_0^\infty(\operatorname{int}(D_m),\mathbb{C}^n)$ is dense in $\dot{\mathcal{H}}_k(D_m,\mathbb{C}^n)$, and the spaces $\mathcal{H}_k(D_m,\mathbb{C}^n)$ and $\dot{\mathcal{H}}_{-k}(D_m,\mathbb{C}^n)$ are dual with respect to the extension of the bilinear form*

$$\int_{D_m}\langle u,v\rangle\,dA$$

where $u\in C_0^\infty(D_m,\mathbb{C}^n), v\in C_0^\infty(\operatorname{int}(D_m),\mathbb{C}^n)$ and $\langle\,,\,\rangle$ denotes the standard Riemannian inner product on $\mathbb{C}^n\approx\mathbb{R}^{2n}$.

This is essentially Theorem B.2.1 p. 479 in [20].

We will also use weighted Sobolev spaces: for $a\in\mathbb{R}$, let $e_a^j\colon D_m\to\mathbb{R}$ be a smooth function such that $e_a^j(\tau+it)=e^{a\tau}$ for $\tau+it\in E_{p_j}[3]$ and $e_a(\zeta)=1$ for $\zeta\in D_m-E_{p_j}[2]$. For $\mu=(\mu_1,\dots,\mu_m)\in\mathbb{R}^m$, let $\mathbf{e}_\mu\colon D_m\to\mathbb{R}\otimes\mathrm{id}\subset\mathbf{GL}(\mathbb{C}^n)$ be

$$\mathbf{e}_\mu(\zeta)=\Pi_{j=1}^m e_{\mu_j}^j(\zeta)\,\mathrm{id},$$

Note that $\mathbf{e}_\mu(\zeta)$ preserves Lagrangian subspaces. We can now define $\mathcal{H}_{k,\mu}(D_m,\mathbb{C}^n)=\{u\in\mathcal{H}_k^{\mathrm{loc}}(D_m,\mathbb{C}^n):\mathbf{e}_\mu u\in\mathcal{H}_k(D_m,\mathbb{C}^n)\}$, with norm $\|u\|_{k,\mu}=\|\mathbf{e}_\mu u\|_k$.

3.6. **Asymptotics.** Let Λ_0 and Λ_1 be (ordered) Lagrangian subspaces of $\mathbb{C}^n$. Define the *complex angle* $\theta(\Lambda_0,\Lambda_1)\in[0,\pi)^n$ inductively as follows:

If $\dim(\Lambda_0\cap\Lambda_1)=r\geq 0$ let $\theta_1=\dots=\theta_r=0$ and let $\mathbb{C}^{n-r}$ denote the Hermitian complement of $\mathbb{C}\otimes\Lambda_0\cap\Lambda_1$ and let $\Lambda_i'=\Lambda_i\cap\mathbb{C}^{n-r}$ for $i=0,1$. If $\dim(\Lambda_0\cap\Lambda_1)=0$ then let $\Lambda_i'=\Lambda_i$, $i=0,1$ and let $r=0$. Then Λ_0' and Λ_1' are Lagrangian subspaces. Let α be smallest angle such that $\dim(e^{i\alpha}\Lambda_0\cap\Lambda_1)=r'>0$. Let $\theta_{r+1}=\dots=\theta_{r+r'}=\alpha$. Now repeat the construction until θ_n has been defined. Note that $\theta(A\Lambda_0,A\Lambda_1)=\theta(\Lambda_0,\Lambda_1)$ for every $A\in\mathbf{U}(n)$ since multiplication with $e^{i\alpha}$ commutes with everything in $\mathbf{U}(n)$.

Proposition 3.6. *Let (u, h) be a holomorphic disk with boundary on a (semi-)admissible Legendrian submanifold L. Let p be a puncture on D_m such that p maps to the Reeb chord c. For $M > 0$ sufficiently large the following is true:*

If $\Pi_{\mathbb{C}}(L)$ self-intersects transversely at c^* then

$$|u(\tau + it)| = \mathcal{O}(e^{-\theta\tau}), \quad \tau + it \in E_p[M], \tag{3.4}$$

where $\theta > 0$ is the smallest complex angle of c.

If $\Pi_{\mathbb{C}}(L)$ has a self-tangency at c^* then either there exists a real number c_0 such that

$$u(\tau + it) = \Big(\frac{\pm 2}{c_0 + \tau + it}, 0, \dots, 0\Big) + \mathcal{O}(e^{-\theta\tau}) \quad \tau + it \in E_p[M], \tag{3.5}$$

or

$$|u(\tau + it)| = \mathcal{O}(e^{-\theta\tau}), \quad \tau + it \in E_p[M], \tag{3.6}$$

where θ is the smallest non-zero complex angle of L at p.

In particular, if the punctures $p_1, \dots, p_m$ on D_m map to Reeb chords $c_1, \dots, c_m$ and if $f \colon D_m \to \mathbb{C}^n$ is any smooth function which is constantly equal to $c_1^*, \dots, c_m^*$ in neighborhoods of $p_1, \dots, p_m$, then $u - f \in \mathcal{H}_2(D_m, \mathbb{C}^n)$.

Proof. Equation (3.4) is a consequence of Theorem B in [27]. To prove the corresponding statement for a self-tangency double point we may assume that the self-tangency point is $0 \in \mathbb{C}^n$ and that around 0, $\Pi_{\mathbb{C}}(L)$ agrees with the local model in Definition 2.3. Elementary complex analysis (see Lemma 5.2 below) shows that for a standard self tangency the first component u_1 of a holomorphic disk is given by

$$u_1(\zeta) = \frac{\pm 2}{\zeta - c_0 + \sum_{n \in \mathbb{Z}} c_n \exp(n\pi\zeta)}, \tag{3.7}$$

where c_j are real constants, in $E_p[M]$. Applying [27] again to the remaining components u' of u gives the claim. The last statement follows immediately from the asymptotics at punctures. □

4. Functional analytic setup

As explained in Section 1, contact homology is built using moduli-spaces of holomorphic disks. In this section we construct Banach manifolds of maps of punctured disks into $\mathbb{C}^n$ which satisfy certain boundary conditions. In this setting, moduli-spaces will appear as the zero-sets of bundle maps.

In Section 4.1 we define our Banach manifolds as submanifolds in a natural bundle of Banach spaces. To find atlases for our Banach manifolds we proceed in the standard way: construct an "exponential map" from the proposed tangent space and show it is a diffeomorphism near the origin. To do this, in Section 4.2, we turn our attention to a special metric on the tangent bundle of the Legendrian submanifold. From this we construct a family of metrics on $\mathbb{C}^n$ in Section 4.3 and use it to define a preliminary version of the "exponential map" for the Banach manifold. Section 4.4 contains some technical results needed to deal with families of Legendrian submanifolds. In Section 4.5 we show how to construct the atlas. Section 4.6 discusses how to invoke variations of the conformal structure of the source space into the present setup. In Section 4.7 we linearize the bundle map, the zero set of which is the moduli-space. Section 4.8 discusses some issues involving the semi-admissible case.

4.1. **Bundles of affine Banach spaces.** Let $L_\lambda \subset \mathbb{C}^n \times \mathbb{R}$, $\lambda \in \Lambda$, where Λ is an open subset of a Banach space, denote a smooth family of chord generic admissible Legendrian submanifolds. That is, Λ is smoothly mapped into the space of admissible Legendrian embeddings of L endowed with the C^∞-topology.

We also study the semi-admissible case. To this end we also let L_λ, $\lambda \in \Lambda$, be a smooth family of semi-admissible Legendrian submanifolds. For simplicity, and since it will suffice for our applications, we assume that in this case, the self tangency point of $\Pi_{\mathbb{C}}(L_\lambda)$ remain fixed as λ varies and that in a neighborhood of this point the product structure of $\Pi_{\mathbb{C}}(L_\lambda)$ is preserved and the first components γ_1 and γ_2, shown in Figure 2 remain fixed as λ varies.

Let $\mathbf{a}(\lambda) = (a_1(\lambda), \dots, a_m(\lambda))$, $\lambda \in \Lambda$ be an ordered collection of Reeb chords of L_λ depending continuously on λ. Consider D_m with punctures $p_1, \dots, p_m$, and a conformal structure $\kappa \in \mathcal{C}_m$.

Fix families, smoothly depending on $(\lambda, \kappa) \in \Lambda \times \mathcal{C}_m$, of smooth reference functions

$$u_{\mathrm{ref}}[\mathbf{a}(\lambda), \kappa] \colon D_m \to \mathbb{C}^n$$

such that $u_{\mathrm{ref}}[\mathbf{a}(\lambda), \kappa]$ is constantly equal to a_k^* in E_{p_k}, and

$$h_{\mathrm{ref}}[\mathbf{a}(\lambda), \kappa] \colon \partial D_m \to \mathbb{R}$$

such that $h_{\mathrm{ref}}[\mathbf{a}(\lambda), \kappa]$ is constantly equal to $a_1^-(\lambda)$ and $a_1^+(\lambda)$ on $[1,\infty) \subset E_{p_1}$, and $[1,\infty)+i \subset E_{p_1}$, respectively, and, for $k \geq 2$, constantly equal to $a_k^+(\lambda)$ and $a_k^-(\lambda)$ on $[1,\infty) \subset E_{p_k}$, and $[1,\infty)+i \subset E_{p_k}$, respectively.

Let $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in [0,\infty)^m$. For $u \colon D_m \to \mathbb{C}^n$ and $h \colon \partial D_m \to \mathbb{R}$ consider the conditions

$$u - u_{\mathrm{ref}}[\mathbf{a}(\lambda), \kappa] \in \mathcal{H}_{2,\epsilon}(D_m, \mathbb{C}^n), \tag{4.1}$$

$$h - h_{\mathrm{ref}}[\mathbf{a}(\lambda), \kappa] \in \mathcal{H}_{\frac{3}{2},\epsilon}(\partial D_{m+1}, \mathbb{R}). \tag{4.2}$$

(Note that the κ-dependence of the right hand sides in (4.1) and (4.2) has been dropped from the notation.) Define the affine Banach space

$$\mathcal{F}_{2,\epsilon}(\mathbf{a}(\lambda), \kappa) = \Big\{ (u,h) \colon D_m \to \mathbb{C}^n \times \mathbb{R} \,:\, u \text{ satisfies (4.1)}, \, h \text{ satisfies (4.2)} \Big\},$$

endowed with the norm which is the sum of the norms of the components. Let

$$\mathcal{F}_{2,\epsilon,\Lambda}(\mathbf{a}, \kappa) = \bigcup_{\lambda \in \Lambda} \mathcal{F}_{2,\epsilon}(\mathbf{a}(\lambda), \kappa)$$

be the metric space with distance function

$$\begin{aligned} d((v,f,\lambda),(w.g,\mu)) =& \|(v - u_{\mathrm{ref}}[\mathbf{a}(\lambda),\kappa]) - (w - u_{\mathrm{ref}}[\mathbf{a}(\mu),\kappa])\|_{2,\epsilon} \\ &+ \|(f - h_{\mathrm{ref}}[\mathbf{a}(\lambda),\kappa]) - (g - h_{\mathrm{ref}}[\mathbf{a}(\mu),\kappa])\|_{\frac{3}{2},\epsilon} \\ &+ |\lambda - \mu|. \end{aligned} \tag{4.3}$$

We give $\mathcal{F}_{2,\epsilon,\Lambda}(\mathbf{a}, \kappa)$ the structure of a Banach manifold by producing an atlas as follows. Let $(w, f, \lambda) \in \mathcal{F}_{2,\epsilon,\Lambda}(\mathbf{a}, \kappa)$. Let (w_μ, f_μ, μ) be any family such that $(w_\lambda, f_\lambda, \lambda) = (w, f, \lambda)$ and such that

$$\mu \mapsto (w_\mu - u_{\mathrm{ref}}[\mathbf{a}(\mu), \kappa], f_\mu - h_{\mathrm{ref}}[\mathbf{a}(\mu), \kappa])$$

is a smooth map into $\mathcal{H}_{2,\epsilon}(D_m, \mathbb{C}^n) \times \mathcal{H}_{\frac{3}{2},\epsilon}(\partial D_m, \mathbb{R})$. Then a chart is given by

$$\begin{aligned} &\mathcal{H}_{2,\epsilon}(D_m, \mathbb{C}^n) \times \mathcal{H}_{\frac{3}{2},\epsilon}(\partial D_m, \mathbb{R}) \times \Lambda \to \mathcal{F}_{2,\epsilon,\Lambda}(\mathbf{a}, \kappa); \\ &(g, r, \mu) \mapsto (w_\mu + g, f_\mu + r, \mu). \end{aligned} \tag{4.4}$$

If $(u, h, \lambda) \in \mathcal{F}_{2,\epsilon,\Lambda}(\mathbf{a}(\lambda), \kappa)$ then $\bar\partial u \in \mathcal{H}_{1,\epsilon}(D_m, T^{*0,1}D^m \otimes \mathbb{C}^n)$ and its trace $\bar\partial u | \partial D_m$ lies in $\mathcal{H}_{\frac{1}{2}}(D_m, T^{*0,1}D_m \otimes \mathbb{C}^n))$.

Definition 4.1. Let $\mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa) \subset \mathcal{F}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa)$ denote the subset of elements (u,h,λ) which satisfy

$$(u,h)(\zeta) \in L_\lambda \text{ for all } \zeta \in \partial D_m, \tag{4.5}$$

$$\int_{\partial D_m} \langle \bar{\partial} u, v\rangle \, ds = 0, \text{ for every } v \in C_0^\infty(\partial D_m, T^{*0,1} D_m \otimes \mathbb{C}^n), \tag{4.6}$$

where $\langle\,,\rangle$ denotes the inner product on $T^{*0,1}\otimes \mathbb{C}^n$ induced from the standard (Riemannian) inner product on $\mathbb{C}^n$.

Lemma 4.2. $\mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa)$ *is a closed subset.*

Proof. If (u_k,h_k,λ_k) is a sequence in $\mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa)$ which converges in $\mathcal{F}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa)$ then $\lambda_k \to \lambda$ and the sequence $(u_k|\partial D_m, h_k)$ converges in sup-norm. Hence (4.5) is a closed condition. Also, $\bar{\partial}$ is continuous as is the trace map. It follows that (4.6) is a closed condition as well. □

4.2. **The normal bundle of a Lagrangian immersion with a special metric.** Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a an instant of a chord generic 1-parameter family of Legendrian submanifolds. Then $\Pi_{\mathbb{C}} \colon L \to \mathbb{C}^n$ is a Lagrangian immersion and the normal bundle of $\Pi_{\mathbb{C}}$ is isomorphic to the tangent bundle TL of L. On the restriction $T_L(TL)$ of the tangent bundle $T(TL)$ of TL to the zero-section L there is a natural endomorphism $J \colon T_L(TL) \to T_L(TL)$ such that $J^2 = -1$. It is defined as follows. If $p \in L$ then $T_{(p,0)}(TL)$ is a direct sum of the space of horizontal vectors tangent to L at p and the space of vertical vectors tangent to the fiber of $\pi \colon TL \to L$ at p. If $v \in T_L(TL)$ is tangent to L at $p \in L$ then Jv is the vector v viewed as a tangent vector to the fiber T_pL of $\pi \colon TL \to L$ at $(p,0)$, and if w is a vector tangent to the fiber of π at $(p,0)$ then $Jw = -w$, where $-w$ is viewed as a tangent vector in T_pL. This defines J on the two direct summands. Extend it linearly.

The immersion $\Pi_{\mathbb{C}} \colon L \to \mathbb{C}^n$ extends to an immersion P of a neighborhood of the zero-section in TL and P can be chosen so that along L, $i \circ dP = dP \circ J$.

From a Riemannian metric g on L, we construct a metric $\hat{g}$ on a neighborhood of the zero section in TL in the following way. Let $v \in TL$ with $\pi(v) = p$. Let X be a tangent vector of TL at v. The Levi-Civita connection of g gives the decomposition $X = X^H + X^V$, where X^V is a vertical vector, tangent to the fiber, and X^H lies in the horizontal subspace at v determined by the connection. Thus X^V is a vector in T_pL with its endpoint at v. It can be translated linearly to the origin $0 \in T_pL$. We use the same symbol X^V to denote this vector translated to $0 \in T_pL$. Write $\pi X \in T_pL$ for the image of X under the differential of the projection π and let R denote the curvature tensor of g.

Define the field of quadratic forms $\hat{g}$ on TL as

$$\hat{g}(v)(X,Y) = g(p)(\pi X, \pi Y) + g(p)(X^V, Y^V) + g(p)(R(\pi X, v)\pi Y, v), \tag{4.7}$$

where $v \in TL$, $\pi(v) = p$, and $X, Y \in T_v(TL)$.

Proposition 4.3. *There exists $\rho > 0$ such that $\hat{g}$ is a Riemannian metric on*

$$\{v \in TL \colon g(v,v) < \rho\}.$$

In this metric, the zero section L is totally geodesic and the geodesics in L are exactly those in the metric g. Moreover, if γ is a geodesic in L and X is a vector field in $T(TL)$ along γ then X satisfies the Jacobi equation if and only if JX does.

Proof. Since $g(R(\pi X, v)\pi Y, v) = g(R(\pi Y, v)\pi X, v)$, $\hat{g}$ is symmetric. When restricted to the 0-section $\hat{g}$ is non-degenerate. The first statement follows from the compactness of L.

In Lemmas 4.5 and 4.6 below we show L is totally geodesic and the statement about Jacobi-fields, respectively. □

Let $x = (x_1, \dots, x_n)$ be local coordinates around $p \in L$ and let $(x, \xi) \in \mathbb{R}^{2n}$ be the corresponding coordinates on TM, where $\xi = \xi_s \partial_s$ (here, and in the rest of this section, we use the Einstein summation convention, repeated indices are summed over) where ∂_j is the tangent vector of TL in the x_j-direction. We write ∂_{j^*} for the tangent vector of TL in the ξ_j-direction. Let ∇, $\hat{\nabla}$ denote the Levi-Civita connections of g and $\hat{g}$, respectively. Let Roman and Greek indices run over the sets $\{1, \dots, n\}$ and $\{1, 1^*, 2, 2^*, \dots, n, n^*\}$, respectively and recall the following standard notation:

$$\begin{aligned} &g_{ij} = g(\partial_i, \partial_j), \quad \hat{g}_{\alpha\beta} = \hat{g}(\partial_\alpha, \partial_\beta), \\ &\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k, \quad \hat{\nabla}_{\partial_\alpha} \partial_\beta = \hat{\Gamma}_{\alpha\beta}^\gamma \partial_\gamma \\ &R(\partial_i, \partial_j)\partial_k = R_{ijk}^l \partial_l, \quad g(R(\partial_i, \partial_j))\partial_k, \partial_r) = R_{ijkr}. \end{aligned}$$

Lemma 4.4. *The components of the metric $\hat{g}$ satisfies*

$$\hat{g}_{ij}(x, \xi) = g_{ij}(x) + \xi_s \xi_t \Big(g_{kr}(x) \Gamma_{is}^k(x) \Gamma_{jt}^r(x) + R_{isjt}(x) \Big), \tag{4.8}$$

$$\hat{g}_{i^* j^*}(x, \xi) = g_{ij}(x), \tag{4.9}$$

$$\hat{g}_{ij^*}(x, \xi) = \xi_s g_{jk}(x) \Gamma_{is}^k(x). \tag{4.10}$$

Proof. Since ∂_{j^*} is vertical, (4.9) holds. Note that the horizontal space at (x, ξ) is spanned by the velocity vectors of the curves obtained by parallel translating ξ along the coordinate directions through x. Let $V(t)$ be a parallel vector field through x in the ∂_j-direction with $V(0) = \xi$ and $\dot{V}(0) = a_k \partial_k$. Then

$$V(t) = (\xi_k + t a_k + \mathcal{O}(t^2))\partial_k$$

and applying ∇_{∂_j} to $V(t)$ we get

$$0 = \nabla_{\partial_j} V(t) = \xi_s \nabla_{\partial_j} \partial_s + a_k \partial_k + \mathcal{O}(t).$$

Taking the limit as $t \to 0$ we find $a_k \partial_k = -\xi_s \Gamma_{js}^k(x) \partial_k$. Hence the horizontal space at (x, ξ) is spanned by the vectors $\partial_j - \xi_s \Gamma_{js}^k \partial_{k^*}$, $j = 1, \dots, n$ and therefore,

$$\partial_j^V = \xi_s \Gamma_{js}^k(x) \partial_{k^*}.$$

Straightforward calculation gives (4.8) and (4.10). □

Lemma 4.5. *The Christofel symbols of the metric $\hat{g}$ at $(x, 0)$ satisfies*

$$\hat{\Gamma}_{ij}^k(x, 0) = \hat{\Gamma}_{ij^*}^{k^*}(x, 0) = \Gamma_{ij}^k(x), \tag{4.11}$$

$$\hat{\Gamma}_{ij}^{k^*}(x, 0) = \hat{\Gamma}_{ij^*}^k(x, 0) = \Gamma_{i^* j^*}^{k^*}(x, 0) = 0. \tag{4.12}$$

Hence if γ is a geodesic in (L, g) then it is also a geodesic in $(TL, \hat{g})$.

Proof. The equations

$$\hat{\Gamma}_{\alpha\beta}^\gamma = \frac{1}{2} \hat{g}^{\gamma\delta} (\hat{g}_{\alpha\delta,\,\beta} + \hat{g}_{\beta\delta,\,\alpha} - \hat{g}_{\alpha\beta,\,\delta}),$$

where $\hat{g}^{\alpha\beta}$ denotes the components of the inverse matrix of $\hat{g}$ and Lemma 4.4 together imply (4.11) and (4.12).

Let $x(t)$ be a geodesic in (L, g). Then $(x, x^*) = (x(t), 0)$ satisfies

$$\begin{aligned} &\ddot{x}_k + \hat{\Gamma}_{ij}^k \dot{x}_i \dot{x}_j + \hat{\Gamma}_{i^* j}^k \dot{x}_{i^*} \dot{x}_j + \hat{\Gamma}_{ij^*}^k \dot{x}_i \dot{x}_{j^*} + \hat{\Gamma}_{i^* j^*}^k \dot{x}_{i^*} \dot{x}_{j^*} = \ddot{x}_k + \Gamma_{ij}^k \dot{x}_i \dot{x}_j = 0, \\ &\ddot{x}_{k^*} + \hat{\Gamma}_{ij}^{k^*} \dot{x}_i \dot{x}_j + \hat{\Gamma}_{i^* j}^{k^*} \dot{x}_{i^*} \dot{x}_j + \hat{\Gamma}_{ij^*}^{k^*} \dot{x}_i \dot{x}_{j^*} + \hat{\Gamma}_{i^* j^*}^{k^*} \dot{x}_{i^*} \dot{x}_{j^*} = 0. \end{aligned}$$

This proves the second statement. □

Lemma 4.6. *If γ is a geodesic in $(TL,\hat{g})$ which lies in L then X is a Jacobi-field along γ if and only if JX is.*

Proof. We establish the following two properties of the metric $\hat{g}$ and the endomorphism J. If γ is a curve in L with tangent vector T and X is any vector field in $T(TL)$ along γ then

$$\hat{\nabla}_T JX = J\hat{\nabla}_T X. \tag{4.13}$$

If X, Y, and Z are tangent vectors to TL at $(p,0)\in L$ such that Y and Z are horizontal (i.e. tangent to L) and if $\hat{R}$ denotes the curvature tensor of $\hat{g}$ at $(p,0)$ then

$$\hat{R}(JX,Y)Z = J\hat{R}(X,Y)Z. \tag{4.14}$$

For (4.13), use local coordinates and write, for $\gamma(t)=x(t)$, $T(x)=a_k(x)\partial_k$, $X(x)=b_j(x)\partial_j+b_{j^*}(x)\partial_{j^*}$. By Lemma 4.5,

$$\begin{aligned}
\hat{\nabla}_T JX &= a_k\hat{\nabla}_{\partial_k}(-b_{j^*}\partial_j+b_j\partial_{j^*})\\
&= a_k\Big[-(\partial_k b_{j^*})\partial_j+(\partial_k b_j)\partial_{j^*}\\
&\qquad - b_{j^*}(\hat{\Gamma}^r_{kj}\partial_r+\hat{\Gamma}^{r^*}_{kj}\partial_{r^*})+b_j(\hat{\Gamma}^r_{kj^*}\partial_r+\hat{\Gamma}^{r^*}_{kj^*}\partial_{r^*})\Big]\\
&= a_k\Big[-(\partial_k b_{j^*})\partial_j+(\partial_k b_j)\partial_{j^*}-b_{j^*}\hat{\Gamma}^r_{kj}\partial_r+b_j\hat{\Gamma}^{r^*}_{kj^*}\partial_{r^*}\Big]\\
&= Ja_k\Big[(\partial_k b_{j^*})\partial_{j^*}+(\partial_k b_j)\partial_j+b_{j^*}\hat{\Gamma}^{r^*}_{kj^*}\partial_{r^*}+b_j\hat{\Gamma}^r_{kj}\partial_r\Big] = J\hat{\nabla}_T X.
\end{aligned}$$

For (4.14), introduce normal coordinates x around p. Then

$$g_{ij}(0)=\delta_{ij},\quad \Gamma^k_{ij}(0)=0 \tag{4.15}$$

for all i,j,k, and hence Lemma 4.4 implies,

$$\hat{g}_{ij}(0,\xi)=\delta_{ij}+\mathcal{O}(\xi^2),\quad \hat{g}_{i^*j}(0,\xi)=0,\quad \hat{g}_{i^*j^*}(0,\xi)=\delta_{ij}.$$

Therefore,

$$\hat{g}^{ij}(0,\xi)=\delta^{ij}+\mathcal{O}(\xi^2),\quad \hat{g}^{i^*j}(0,\xi)=0,\quad \hat{g}^{i^*j^*}(0,\xi)=\delta^{ij}. \tag{4.16}$$

We show that, in these normal coordinates,

$$\hat{R}(\partial_{i^*},\partial_j)\partial_k = J\hat{R}(\partial_i,\partial_j)\partial_k \tag{4.17}$$

at $(0,0)$. Since $\hat{R}$ is a tensor field, (4.17) implies (4.14).

Lemma 4.5 implies that all Christofel symbols of $\hat{g}$ vanishes at $(x,\xi)=(0,0)$ and also that $\partial_i\Gamma^{r^*}_{jk}(x,0)=0$ all i,j,k,r^*. Hence,

$$\begin{aligned}
\hat{R}(\partial_i,\partial_j)\partial_k &= \hat{\nabla}_{\partial_i}\hat{\nabla}_{\partial_j}\partial_k-\hat{\nabla}_{\partial_j}\hat{\nabla}_{\partial_i}\partial_k\\
&= (\partial_i\hat{\Gamma}^r_{jk})\partial_r+(\partial_i\hat{\Gamma}^{r^*}_{jk})\partial_{r^*}-(\partial_j\hat{\Gamma}^r_{ik})\partial_r-(\partial_j\hat{\Gamma}^{r^*}_{ik})\partial_{r^*}\\
&= (\partial_i\Gamma^r_{jk})\partial_r-(\partial_j\Gamma^r_{ik})\partial_r,
\end{aligned}$$

and thus

$$J\hat{R}(\partial_i,\partial_j)\partial_k = (\partial_i\Gamma^r_{jk})\partial_{r^*}-(\partial_j\Gamma^r_{ik})\partial_{r^*}. \tag{4.18}$$

We compute the left hand side of (4.17):

$$\begin{aligned}
\hat{R}(\partial_{i^*},\partial_j)\partial_k &= \hat{\nabla}_{\partial_{i^*}}\hat{\nabla}_{\partial_j}\partial_k-\hat{\nabla}_{\partial_j}\hat{\nabla}_{\partial_{i^*}}\partial_k\\
&= \hat{\nabla}_{\partial_{i^*}}\left(\hat{\Gamma}^r_{jk}\partial_r+\hat{\Gamma}^{r^*}_{jk}\partial_{r^*}\right)-\hat{\nabla}_{\partial_j}\left(\hat{\Gamma}^r_{i^*k}\partial_r+\hat{\Gamma}^{r^*}_{i^*k}\partial_{r^*}\right).
\end{aligned} \tag{4.19}$$

Lemma 4.5 gives $\partial_j \hat\Gamma^r_{i^*k} = 0$, and Lemma 4.4 in combination with (4.16) give $\partial_{i^*}\hat\Gamma^r_{jk} = 0$. Hence,

$$\hat R(\partial_{i^*},\partial_j)\partial_k = (\partial_{i^*}\hat\Gamma^{r^*}_{jk})\partial_{r^*} - (\partial_j\hat\Gamma^{r^*}_{ik^*})\partial_{r^*} = (\partial_{i^*}\hat\Gamma^{r^*}_{jk})\partial_{r^*} - (\partial_j\Gamma^{r}_{ik})\partial_{r^*}. \tag{4.20}$$

It thus remains to compute $\partial_{i^*}\hat\Gamma^{r^*}_{jk}$.

$$\begin{aligned}
\partial_{i^*}\hat\Gamma^{r^*}_{jk} &= \frac12 \partial_{i^*}\Big(\hat g^{r^*l^*}(\hat g_{jl^*,k} + \hat g_{kl^*,j} - \hat g_{jk,l^*}) + \hat g^{r^*l}(\hat g_{jl,k} + \hat g_{kl,j} - \hat g_{jk,l})\Big)\\
&= \frac12 \hat g^{rl}\partial_{i^*}(\hat g_{jl^*,k} + \hat g_{kl^*,j} - \hat g_{jk,l^*}) \quad \text{[by (4.16)]}\\
&= \frac12 g^{rl}\Big((\partial_k\Gamma^m_{ji})g_{ml} + (\partial_j\Gamma^m_{ki})g_{ml} - (R_{jikl} + R_{jlki})\Big) \quad \text{[Lemma 4.4, (4.15)]}\\
&= \frac12 (\partial_k\Gamma^r_{ji} + \partial_j\Gamma^r_{ki} - (R_{jikr} + R_{jrki})) \quad \text{[(4.15)]}.
\end{aligned} \tag{4.21}$$

But

$$R_{jikr} = g(\nabla_{\partial_j}\nabla_{\partial_i}\partial_k - \nabla_{\partial_i}\nabla_{\partial_j}\partial_k, \partial_r) = \partial_j\Gamma^r_{ik} - \partial_i\Gamma^r_{jk} = \partial_j\Gamma^r_{ki} - \partial_i\Gamma^r_{jk},$$

and

$$R_{jrki} = R_{kijr} = \partial_k\Gamma^r_{ij} - \partial_i\Gamma^r_{kj} = \partial_k\Gamma^r_{ji} - \partial_i\Gamma^r_{jk}.$$

Hence

$$\partial_{i^*}\hat\Gamma^{r^*}_{jk} = \partial_i\Gamma^r_{jk},$$

which together with (4.19) and (4.20) imply (4.17).

Consider a geodesic of $(TL,\hat g)$ in L with tangent vector T. By (4.13) and (4.14),

$$\hat\nabla_T\hat\nabla_T JX + \hat R(JX,T)T = J(\hat\nabla_T\hat\nabla_T X + \hat R(X,T)T). \tag{4.22}$$

Thus X is a Jacobi field if and only if JX is. □

4.3. **A family of metrics on $\mathbb{C}^n$.** Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be an instant of a chord generic 1-parameter family of Legendrian submanifolds and fix a Riemannian metric g on L. Using the metric $\hat g$ on TL (see Section 4.2), we construct a 1-parameter family of metrics $g(L,\sigma)$, $0 \le \sigma \le 1$, on $\mathbb{C}^n$ with good properties with respect to $\Pi_{\mathbb{C}}(L)$.

Let $c_1,\dots,c_m$ be the Reeb chords of L. Fix $\delta > 0$ such that all the 6δ-balls $B(c_j^*, 6\delta)$ are disjoint and such that the intersections $B(c_j^*,6\delta)\cap\Pi_{\mathbb{C}}(L)$ are homeomorphic to two n-disks intersecting at a point.

Identify the normal bundle of the immersion $\Pi_{\mathbb{C}}$ with the tangent bundle TL. Consider the metric $\hat g$ on a ρ-neighborhood of the 0-section in TL ($\rho > 0$ as in Proposition 4.3). Let $P\colon W \to \mathbb{C}^n$ be an immersion of a ρ'-neighborhood $N(\rho')$ of the 0-section $\rho' \le \rho$ such that $i \circ dP = dP \circ J$ along the 0-section.

Consider the P-push-forward of the metric $\hat g$ to the image of $N(\rho')$ restricted to $L \setminus \bigcup_j U(c_j^-,\delta)$. Note that if $\rho' > 0$ is small enough this restriction of P is an embedding and the push-forward metric is defined in a neighborhood of $\Pi_{\mathbb{C}}(L \setminus \bigcup_j U(c_j^-, 2\delta))$. Extended it to a metric g^1 on all of $\mathbb{C}^n$, which agrees with the standard metric outside a neighborhood of $\Pi_{\mathbb{C}}(L)$.

Consider the P-push-forward of the metric $\hat g$ to the image of the ρ'-neighborhood of the 0-section restricted to $L\setminus\bigcup_j U(c_j^+,\delta)$. This metric is defined in a neighborhood of $\Pi_{\mathbb{C}}(L \setminus \bigcup_j U(c_j^+, 2\delta))$ and can be extended to a metric g^0 on all of $\mathbb{C}^n$, which agrees with the standard metric outside a neighborhood of $\Pi_{\mathbb{C}}(L)$.

Choose the metrics g^0 and g^1 so that they agree outside $\cup_j B(c_j^*, 3\delta)$ and let g^σ, $0 \le \sigma \le 1$ be a smooth 1-parameter family of metrics on $\mathbb{C}^n$ with the following properties:

- $g^\sigma = g^0$ in a neighborhood of $\sigma = 0$,

- $g^\sigma = g^1$ in a neighborhood of $\sigma = 1$,
- g^σ is constant in σ outside $\cup_j B(c_j^*, 4\delta)$.

We take $g(L, \sigma) = g^\sigma$.

Remark 4.7. If L_λ, $\lambda \in \Lambda$ is a smooth family of chord generic Legendrian submanifolds then, as is easily seen, the above construction can be carried out in such a way that the family of 1-parameter families of metrics $g(L_\lambda, \sigma)$ becomes smooth in λ.

Given a vector field v along a disk $u\colon D_m \to \mathbb{C}^n$ with boundary on L, we would like to be able to exponentiate v to get a variation of u among disks with boundaries on L. We will not be able to use a fixed metric g^σ to do this. To solve this problem let $\sigma\colon \mathbb{C}^n \times \mathbb{R} \to [0,1]$ be a smooth function which equals 0 on

$$\mathbb{C}^n \times \mathbb{R} - \bigcup_j B(c_j^*, 5\delta) \times \left[c_j^+ - \frac{1}{2}\mathcal{Z}(c_j), c_j^+ + 1\right]$$

and equals 1 on

$$\bigcup_j B(c_j^*, 4\delta) \times \left[c_j^+ - \frac{1}{4}\mathcal{Z}(c_j), c_j^+ + \frac{1}{2}\right].$$

Let $\exp_p^g$ denote the exponential map of the metric g at the point p. If $p \in L_\lambda$ and v is tangent to L_λ at p, then write $x(p) = \Pi_{\mathbb{C}}(p)$ and $\xi(v) = \Pi_{\mathbb{C}}(v)$. One may now easily prove the following lemma.

Lemma 4.8. *Let L_λ, $\lambda \in \Lambda$ be a family of (semi-)admissible Legendrian submanifolds. Let $0 \in \Lambda$ and let $\sigma\colon \mathbb{C}^n \times \mathbb{R} \to [0,1]$ be the function constructed from L_0 as above. There exists $\rho > 0$ and a neighborhood $W \subset \Lambda$ of 0 such that if p is any point in L_λ, $\lambda \in W$ and v any vector tangent to L_λ at p with $|\xi(v)| < \rho$ then*

$$\exp_x^{g(L_\lambda, \sigma(p))} t\xi \in \Pi_{\mathbb{C}}(L_\lambda) \text{ for } 0 \le t \le 1.$$

4.4. **Extending families of Legendrian embeddings and their differentials.** In the next subsection we will need to exponentiate vector fields along a disk whose boundary is in L_0 $(0 \in \Lambda)$ to get a disk with boundary in L_λ for λ near 0. To accomplish this we construct diffeomorphisms of $\mathbb{C}^n$.

Consider $L_\lambda \subset \mathbb{C}^n \times \mathbb{R}$, $\lambda \in \Lambda$ and let $0 \in \Lambda$. There exists a smooth family of Legendrian embeddings

$$k_\lambda\colon L_0 \to \mathbb{C}^n \times \mathbb{R},$$

such that k_0 is the inclusion, $k_\lambda(L_0) = L_\lambda$, and $k_\lambda(c_j^\pm(0)) = c_j^\pm(\lambda)$ for each j.

As in Section 4.3, fix $\delta > 0$ such that all the 6δ-balls $B(c_j^*(0), 6\delta)$ are disjoint and such that the intersections $B(c_j^*(0), 6\delta) \cap \Pi_{\mathbb{C}}(L_0)$ are homeomorphic to two n-disks intersecting at a point.

Let $W \subset \Lambda$ be a neighborhood of 0 such that $c_j^*(\lambda) \in B(c_j^*(0), \delta)$ for $\lambda \in W$. We construct a smooth Λ-family ($\lambda \in W$) of 1-parameter families of diffeomorphisms $\psi_\lambda^\sigma\colon \mathbb{C}^n \to \mathbb{C}^n$, $0 \le \sigma \le 1$, $\lambda \in W$. Note that

$$K_\lambda^1 = \Pi_{\mathbb{C}} \circ k_\lambda\colon L_0^1 = L_0 \setminus \bigcup_j U(c_j^+, 3\delta) \to \mathbb{C}^n,$$

$$K_\lambda^0 = \Pi_{\mathbb{C}} \circ k_\lambda\colon L_0^0 = L_0 \setminus \bigcup_j U(c_j^-, 3\delta) \to \mathbb{C}^n \tag{4.23}$$

are Lagrangian embeddings and that $K_\lambda^1(c_j^*(0)) = K_\lambda^0(c_j^*(0)) = c_j^*(\lambda)$, for each Reeb chord $c_j(0)$ of L_0.

Identify tubular neighborhoods of L_0^1 and L_0^0 with their respective tangent bundles so that J along the 0-section of the tangent bundles corresponds to i in $\mathbb{C}^n$ (see Section 4.3). Define for $(p,v) \in TL_0^\alpha \subset \mathbb{C}^n$, $\alpha = 0,1$,

$$\hat{K}_\lambda^\alpha(p,v) = K_\lambda^\alpha(p) + idK_\lambda^\alpha(v). \tag{4.24}$$

Then $\hat{K}_\lambda^\alpha$ is a diffeomorphism on some neighborhood of $L_0^\alpha \subset \mathbb{C}^n$, $\alpha = 0,1$. Note that the diffeomorphisms $\hat{K}_\lambda^0$ and $\hat{K}_\lambda^1$ agree outside $\bigcup_j B(c_j^*, 4\delta)$.

Extend $\hat{K}_\lambda^0$ and $\hat{K}_\lambda^1$ to diffeomorphisms on all of $\mathbb{C}^n$ in such a way that their extensions agree outside $\bigcup_j B(c_j^*, 4\delta)$. Call these extensions ψ_λ^α, $\alpha = 0,1$.

Let ψ_λ^σ, $0 \le \sigma \le 1$ be a Λ-family of 1-parameter families of diffeomorphisms which are constant in σ near $\sigma = 0$ and $\sigma = 1$ and with the following properties. First, ψ_λ^σ, $0 \le \sigma \le 1$ connects ψ_λ^0 to ψ_λ^1. Second, ψ_λ^σ is constant in σ outside $\cup_j B(c_j^*, 5\delta)$ and in $\cup_j (B(c_j^*, 5\delta) \setminus B(c_j^*, 4\delta)) \cap L_0$. Third $\psi_\lambda^\sigma(c_j^*(0)) = c_j^*(\lambda)$, $0 \le \sigma \le 1$.

For future reference we let Y_λ^σ denote the 1-parameter family of 1-forms on Λ with coefficients in smooth vector fields on $\mathbb{C}^n$ defined by

$$Y_\lambda^\sigma(x,\mu) = D_\lambda \psi_\lambda^\sigma(x) \cdot \mu, \quad \lambda \in \Lambda, \mu \in T_\lambda\Lambda, x \in \mathbb{C}^n, \sigma \in [0,1]. \tag{4.25}$$

By (4.24), $d\psi_\lambda^\alpha$, $\alpha = 0,1$ are complex linear maps when restricted to the restriction of the tangent bundle of $\mathbb{C}^n$ to L_0^α. Moreover, these maps fit together to a smooth Λ-family of maps $\hat{A}_\lambda \colon L_0 \to \mathbf{GL}(\mathbb{C}^n)$ which is obtained as follows. Pick a smooth function β on L_0 with values in $[0,1]$ which is 0 outside $U(c_j^+, 5\delta)$ and 1 inside $U(c_j^+, 4\delta)$ define

$$\hat{A}_\lambda(p) = d\psi^{\beta(p)}(d\Pi_{\mathbb{C}}(T_pL)).$$

Let $A_\lambda^\sigma \colon \mathbb{C}^n \to \mathbf{GL}(\mathbb{C}^n)$ be an s-parameter family of 1-parameter families of maps with the following properties.

- $A_\lambda^\sigma = \hat{A}_\lambda$ on $\Pi_{\mathbb{C}}(L_0) \setminus \Pi_{\mathbb{C}}(U(c_j^+, 5\delta))$
- $A_\lambda^1 = \hat{A}_\lambda$ on $\Pi_{\mathbb{C}}(U(c_j^+, 4\delta))$
- A_λ^σ is constant in σ on $B(c_j^*, 5\delta) \setminus B(c_j^*, 4\delta) \cap L_0$
- $\bar\partial A_\lambda^0 = 0$ along $\Pi_{\mathbb{C}}(L_0) \setminus \Pi_{\mathbb{C}}(U(c_j^+, 4\delta))$ and $\bar\partial A_\lambda^1 = 0$ along $\Pi_{\mathbb{C}}(L_0) \setminus \Pi_{\mathbb{C}}(U(c_j^-, 4\delta))$.
- $\|A_\lambda^\sigma - \mathrm{id}\,\|_{C^\infty} \le 2\|\hat{A}_\lambda - \mathrm{id}\,\|_{C^\infty}$.

4.5. **Local coordinates.** We consider first the chord generic case. Let $L_\lambda \subset \mathbb{C}^n \times \mathbb{R}$, $\lambda \in \Lambda$ be a family of chord generic Legendrian submanifolds. We construct local coordinates on $\mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa)$.

Let $\sigma \colon \mathbb{C}^n \times \mathbb{R} \to [0,1]$ be the function constructed from L_0, $0 \in \Lambda$. For $p \in L_\lambda$ and v a tangent vector of $\Pi_{\mathbb{C}}(L_\lambda)$ at $q = \Pi_{\mathbb{C}}(p)$, write

$$\exp_q^{g(L_\lambda,\sigma(p))} v = \exp_q^{\lambda,\sigma} v.$$

Moreover, if $\rho > 0$ is as in Lemma 4.8 and $|v| \le \rho$ we write $z(p,v)$ for the z-coordinate of the endpoint of the unique continuous lift of the path $\exp_q^{\lambda,\sigma(p)} tv$, $0 \le t \le 1$, to $L \subset \mathbb{C}^n \times \mathbb{R}$.

Let $(w,f) \in \mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa)$. Let $F \colon D_m \to \mathbb{R}$ be an extension of f such that $F \in \mathcal{H}_{2,\epsilon}(D_m,\mathbb{R})$ (in particular F is continuous) and such that F is smooth with all derivatives uniformly bounded outside a small neighborhood of ∂D_m. Then $w \times F \colon D_m \to \mathbb{C}^n \times \mathbb{R}$. In the case that w and f are smooth we take F to be smooth. Furthermore, in the case that w and f are constant close to each puncture we take F to be an affine parameterization of the corresponding Reeb-chord in a neighborhood of each puncture where w and f are constant. The purpose of this choice of F is that when we exponentiate a vector field at the disk (w,f), we need (w,F) to determine the metric.

For $r > 0$, define

$$\mathcal{B}_{2,\epsilon}((w,f),r) \subset \mathcal{H}_{2,\epsilon}(D_m,\mathbb{C}^n)$$

as the intersection of the closed subspace of $v \in \mathcal{H}_{2,\epsilon}(D_m,\mathbb{C}^n)$ which satisfies

$$v(\zeta) \in \Pi_{\mathbb{C}}\Big(T_{(f(\zeta),w(\zeta))}L\Big), \text{ for } \zeta \in \partial D_m, \tag{4.26}$$

$$\int_{\partial D_m} \langle \bar{\partial} v, a\rangle\, ds = 0, \text{ for every } a \in C_0^\infty(\partial D_m,\mathbb{C}^n) \tag{4.27}$$

and the ball $\{u \colon \|u\|_{2,\epsilon} < r\}$.

When the parameter space Λ is 0-dimensional we can define a coordinate chart around $(f,w,0) \in \mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa)$ $(0 \in \Lambda)$ by

$$\begin{aligned}
&\Phi[(w,f,0)] \colon \mathcal{B}_{2,\epsilon}((w,f),r) \times \Lambda \to \mathcal{F}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa);\\
&\Phi[(w,f,0)](v,\lambda) = (u,l,\lambda)
\end{aligned}$$

where

$$\begin{aligned}
u(\zeta) &= \exp_{w(\zeta)}^{\lambda,\sigma(\zeta)}\Big(v(\zeta)\Big),\\
l(\zeta) &= z\Big((w(\zeta),f(\zeta)),v(\zeta)\Big), \quad \zeta \in \partial D_m.
\end{aligned}$$

When Λ is not 0-dimensional we will need to use the maps A_λ^σ to move the "vector field" v from L_0 to L_λ. Moreover, to ensure our new maps are in the appropriate space of functions we will also need to cut off the original map w. To this end let $(w,f,\lambda) \in \mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa)$. Then there exists $M > 0$ and vector-valued functions ξ_j, $j = 1,\dots,m$ such that

$$w(\tau + it) = \exp_{a_j^*}^{\lambda,\omega(t)} \xi_j(\tau + it), \quad \text{for } \tau + it \in E_{p_j}[M],$$

where $\omega \colon [0,1] \to [0,1]$ is a smooth approximation of the identity, which is constant in neighborhoods of the endpoints of the interval. Define $(w[M],f[M])$ as follows. Let

$$w[M](\zeta) = \begin{cases} w(\zeta), & \text{for } \zeta \notin \cup_j E_{p_j}[M],\\ \exp_{a_j^*}^{\lambda,\omega(t)}(\alpha\xi_j), & \text{for } \zeta = \tau + it \in E_{p_j}[M], \end{cases}$$

where $\alpha \colon E_{p_j} \to \mathbb{C}$ is a smooth function which is 1 on $E_{p_j} \setminus E_{p_j}[M+1]$, 0 on $E_{p_j}[2M]$, and holomorphic on the boundary. Let $f[M]$ be the natural lift of the boundary values of $w[M]$. It is clear that $(w[M],f[M]) \to (w,f)$ as $M \to \infty$. For convenience we use the notation $(w[\infty],f[\infty])$ to denote this limit. We write $F[M]$ for the extension of $f[M]$ to D_m.

Let $(w,f,0) \in \mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa)$ $(0 \in \Lambda)$. For large $M > 0$ consider $(w[M],F[M])$. To simplify notation, write $\sigma[M](\zeta) = \sigma(w[M](\zeta),F[M](\zeta))$ and $w[M]_\lambda(\zeta) = \psi_\lambda^{\sigma[M](\zeta)}(w[M](\zeta))$. Define

$$\begin{aligned}
&\Phi[(w,f,0);M] \colon \mathcal{B}_{2,\epsilon}((w[M],f[M]),r) \times \Lambda \to \mathcal{F}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa);\\
&\Phi[(w,f,0);M](v,\lambda) = (u,l,\lambda)
\end{aligned}$$

where

$$\begin{aligned}
u(\zeta) &= \exp_{w[M]_\lambda(\zeta)}^{\lambda,\sigma[M](\zeta)}\Big(A_\lambda^{\sigma[M](\zeta)}v(\zeta)\Big),\\
l(\zeta) &= z\Big((w[M]_\lambda(\zeta),f[M]_\lambda(\zeta)),A_\lambda^{\sigma[M](\zeta)}v(\zeta)\Big), \quad \zeta \in \partial D_m.
\end{aligned}$$

In the semi-admissible case we use the above construction close to all Reeb chords except the chord c_0 at the self-tangency point. At c_0^* we utilize the fact that we have a local product structure of $\Pi_{\mathbb{C}}(L_\lambda)$ which is assumed to be preserved in a rather strong sense under $\lambda \in \Lambda$, see Section 4.1. This allows us to construct the family of metrics g_λ^σ as product metrics close to c_0^*. Once we have metrics with this property, we can apply the cut-off procedure above to the last $(n-1)$ coordinates of an element $(w,f,0) \in \mathcal{W}_{2,\epsilon,\Lambda}$ and just keep the first coordinate of w in a neighborhood of c_0^* as it is. We use the same notation $(w[M], f[M])$ for the map which results from this modified cut-off procedure from (w, f) in the semi-admissible case.

Proposition 4.9. *Let $\epsilon \in [0,\infty)^m$. Then there exists $r > 0$, $M > 0$, and a neighborhood $W \subset \Lambda$ of 0 such that the map*

$$\Phi[(w,f,0)] \colon \mathcal{B}_{2,\epsilon}((w[M], f[M]), r) \times W \to \mathcal{F}_{2,\epsilon,\Lambda}(\mathbf{a}, \kappa)$$

is C^1 and gives local coordinates on some open subset of $\mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa)$ containing $(w,f,0)$. Moreover, if Λ is 0-dimensional then we may take $M = \infty$.

Proof. Fix some small $r > 0$. Consider the auxiliary map

$$\begin{aligned}\Psi \colon \mathcal{B}_{2,\epsilon}((w[M], f[M]), r) \times \mathcal{H}_{\frac{3}{2},\epsilon}(\partial D_m, \mathbb{R}) \times \Lambda &\to \mathcal{F}_{2,\epsilon,\Lambda}(\mathbf{a}),\\ \Psi(v, r, \lambda) = \Phi[(w[M], f[M]), 0](v, \lambda) + (0,0,r)&\end{aligned}$$

where $(u,h,\mu) + (0,0,r) = (u,h,\mu + r)$.

We show in Lemma 4.11 that Ψ is C^1 with differential in a neighborhood of $(0,0,0)$ which maps injectively into the tangent space of the target and has closed images. These closed images have direct complements and hence the implicit function theorem applies and shows that the image is a submanifold. Moreover, for M large enough $(w,f,0)$ is in the image.

We finally prove in Lemma 4.13 that $\mathcal{W}_{2,\epsilon}(\mathbf{a},\kappa)$ lies inside the image and that it corresponds exactly to $r = 0$ in the given coordinates. □

Lemma 4.11 is a consequence of the following technical lemma.

Lemma 4.10. *Let Λ be an open neighborhood of 0 in a Banach space. Let $(w,f,\lambda) \in \mathcal{F}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa)$, and $v,u,q \in \mathcal{B}_{2,\epsilon}((w,f),r)$. Let ζ be a coordinate on D_m and let $\epsilon \in [0,\infty)^m$.*

(a) *Let*

$$G \colon \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times [0,1] \times \Lambda \to \mathbb{C}^n$$

be a smooth function with all derivatives uniformly bounded and let $\sigma \colon \mathbb{C}^n \times \mathbb{R} \to [0,1]$ be a smooth function with the same property. If

$$G(x,0,0,\theta,\sigma,\lambda) = 0, \tag{4.28}$$

$$G(x,\xi,0,\theta,\sigma,0) = 0, \tag{4.29}$$

then there exists a constant C (depending on $\|Dw\|_{1,\epsilon}$, $\|DF\|_{1,\epsilon}$ and r) such that $G(\zeta,\lambda) = G(w(\zeta), v(\zeta), u(\zeta), q(\zeta), \sigma(F(\zeta), w(\zeta)), \lambda)$ satisfies

$$\|G(\zeta,\lambda)\|_{2,\epsilon} \le C(\|u\|_{2,\epsilon} + \|v\|_{2,\epsilon} + |\lambda|). \tag{4.30}$$

(b) *Let*

$$G \colon \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times [0,1] \times \Lambda \to \mathbb{C}^n$$

be a smooth function with all derivatives uniformly bounded. If

$$G(x,0,0,\sigma,\lambda) = 0, \tag{4.31}$$

$$G(x,\xi,0,\sigma,0) = 0, \tag{4.32}$$

$$D_3 G(x,\xi,0,\sigma,0) = 0, \quad \text{and} \tag{4.33}$$

$$D_5 G(x,\xi,0,\sigma,0) = 0 \tag{4.34}$$

then there exists a constant C (depending on $\|Dw\|_{1,\epsilon}$, $\|DF\|_{1,\epsilon}$ and r) such that $G(\zeta,\lambda) = G(w(\zeta), v(\zeta), u(\zeta), \sigma(F(\zeta), w(\zeta)), \lambda)$ satisfies

$$\|G(\zeta,\lambda)\|_{2,\epsilon} \le C(\|u\|_{2,\epsilon}^2 + |\lambda|^2).$$

Proof. For simplicity, we suppress intermediate functions in the notation, e.g., we write $\sigma(\zeta)$ for $\sigma(w(\zeta), F(\zeta))$. Consider (a). Assume that w, v, u, q, F are smooth functions. By (4.28)

$$|G(\zeta,\lambda)| \le C(|v| + |u|), \tag{4.35}$$

since the derivatives of G are uniformly bounded.

(For simplicity, we will use the letter C to denote many different constants in this proof. This (constant!) change of notation will not be pointed out each time.)

Let $\hat{G}(\zeta) = G(\zeta, \lambda)$. We write $(w, v, u, q, \sigma) = (x_1, x_2, x_3, x_4, x_5)$ and use the Einstein summation convention. The derivative of $\hat{G}(\zeta)$ is

$$D\hat{G}(\zeta) = D_j\hat{G} \cdot Dx_j,$$

where D without subscript refers to derivatives with respect to ζ, and $D_j\hat{G}$ refers to the derivative of $\hat{G}$ with respect to its j-th argument. We use the following notation for functions $(y_1, \dots, y_l)$,

$$|D^{j_1}y|^{k_1}\dots|D^{j_m}y|^{k_m} = \sum_{\alpha\in A}\Pi_{k=1}^{l}|D^{j_1}y_k|^{\alpha_k^1}\dots|D^{j_m}y_k|^{\alpha_k^m}$$

where $A = \{\alpha \in (\mathbb{Z}_{\ge 0})^{lm} : \alpha_1^r + \dots + \alpha_l^r = k_r\}$.

Let $(w, F, q) = (y_1, y_2, y_3)$ and $(v, u) = (z_1, z_2)$ then, by (4.28)

$$\begin{aligned}
&|D_j\hat{G}| \le C|z|, \quad j \in \{1, 4, 5\}\\
&|D_j\hat{G}| \le C, \quad j \in \{2, 3\}\\
&D\sigma = D_1\sigma \cdot DF + D_2\sigma \cdot Dw, \text{ hence } |D\sigma| \le C|Dy|.
\end{aligned}$$

Then

$$|D\hat{G}(\zeta)|^2 \le C\Big(|z|^2|Dy|^2 + |z||Dz||Dy| + |Dz|^2\Big). \tag{4.36}$$

The second derivative of $\hat{G}(\zeta)$ is

$$D^2\hat{G}(\zeta) = D_iD_j\hat{G} \cdot Dx_i \cdot Dx_j + D_j\hat{G} \cdot D^2x_j$$

By (4.28),

$$\begin{aligned}
&|D_iD_j\hat{G}| \le C|z|, \quad i, j \in \{1, 4, 5\}\\
&|D_iD_j\hat{G}| \le C, \quad j \in \{2, 3\}\\
&\qquad D^2\sigma = D_1^2\sigma \cdot DF \cdot DF + 2D_2D_1\sigma \cdot Dw \cdot DF\\
&\qquad\qquad + D_2^2\sigma Dw \cdot Dw + D_1\sigma \cdot D^2F + D_2\sigma \cdot D^2w,\\
&\quad \text{hence } |D^2\sigma| \le C(|Dy|^2 + |D^2y|).
\end{aligned}$$

Thus

$$\begin{aligned}
|D^2\hat{G}(\zeta)|^2 \le C\Big(&|z^2|(|Dy|^4 + |Dy|^2|D^2y|) + |z||Dz||Dy||D^2y|\\
&+|Dz|^4 + |Dz||D^2z||Dy| + |Dz|^2|D^2z| + |D^2z|^2\Big).
\end{aligned} \tag{4.37}$$

Note that by (4.31) and (4.33), r, which the constant C absorbs, controls the q (or y_3) norms. Moreover, the remaining y_1 and y_2 norms are also absorbed by C. Thus, using (4.35), (4.36), and (4.37) we derive the estimate

$$\|\hat{G}(\zeta)\|_{2,\epsilon} \le C(\|u\|_{2,\epsilon} + \|v\|_{2,\epsilon}) \tag{4.38}$$

as follows. The Sobolev-Gagliardo-Nirenberg theorem implies $\|Dy\|_{L^4} \le C\|Dy\|_{1,2}$ (and the corresponding statement for u and v). Morrey's theorem implies that $\|u\|_{2,\epsilon}$ controls the sup-norm of u (and the corresponding statement for v). These facts together with Hölder's inequality gives (4.38).

It is now straightforward to prove (a). Let $\Omega = (x, \xi, \eta, \theta, \sigma)$ then

$$G(\Omega, \lambda) = G(\Omega, 0) + D_6 G(\Omega, 0) \cdot \lambda + R(\Omega, \lambda) \cdot \lambda \cdot \lambda. \tag{4.39}$$

Differentiating (4.39) twice with respect to λ of and applying (4.28) we find $R(x, 0, 0, \theta, \sigma, \lambda) = 0$. Applying the argument above to $D_6 G$ and R, and to $G(\Omega, 0)$ but using (4.29) and u instead of (4.28) and (u, v), (4.30) follows.

The proof of (b) is similar. We first use (4.32) and (4.33) to conclude

$$\hat{G} = G(w, v, u, \sigma, 0) \leq C|u|^2. \tag{4.40}$$

The derivative of $\hat{G}(\zeta)$ is

$$D\hat{G}(\zeta) = D_j \hat{G} \cdot Dx_j,$$

and with $(w, F, v) = (y_1, y_2, y_3)$

$$\begin{aligned} &|D_j \hat{G}| \leq C|u|^2, \quad j \in \{1, 2, 4\} \\ &|D_3 \hat{G}| \leq C|u|, \\ &D\sigma = D_1\sigma \cdot DF + D_2\sigma \cdot Dw, \text{ hence } |D\sigma| \leq C|Dy|. \end{aligned}$$

Thus

$$|D\hat{G}(\zeta)|^2 \leq C\Big(|u|^4|Dy|^2 + |u|^3|Du||Dy| + |u|^2|Du|^2\Big). \tag{4.41}$$

The second derivative of $\hat{G}(\zeta)$ is

$$D^2\hat{G}(\zeta) = D_i D_j \hat{G} \cdot Dx_i \cdot Dx_j + D_j \hat{G} \cdot D^2 x_j$$

We have

$$\begin{aligned} &|D_i D_j \hat{G}| \leq C|u|^2, \quad i, j \in \{1, 2, 4\} \\ &|D_i D_3 \hat{G}| \leq C|u|, \quad i \in \{1, 2, 4\}, \\ &|D_3^2 \hat{G}| \leq C. \end{aligned}$$

This implies

$$\begin{aligned} |D^2\hat{G}(\zeta)|^2 \leq C\Big(&|u|^4(|Dy|^4 + |Dy|^2|D^2y| + |D^2y|^2) \\ &+ |u|^3(|Du||Dy|^3 + |Du||Dy||D^2y|) + |u|^2|Du|^2|Dy|^2 \\ &+ |u|^2|Du|^4 + |Du||Dy||D^2y| + |D^2u||D^2y| + |u|^2|D^2u|^2\Big). \end{aligned} \tag{4.42}$$

In the same way as above we derive from (4.40), (4.41), and (4.42) the estimate

$$\|\hat{G}(\zeta)\|_{2,\epsilon} \leq C\|u\|_{2,\epsilon}^2. \tag{4.43}$$

The proof of (b) can now be completed as follows. Write $\Omega = (x, \xi, \eta, \sigma)$ then

$$G(\Omega, \lambda) = G(\Omega, 0) + D_5 G(\Omega, 0) \cdot \lambda + D_5^2 G(\Omega, 0) \cdot \lambda \cdot \lambda + R(\Omega, \lambda) \cdot \lambda \cdot \lambda \cdot \lambda,$$

and differentiation gives $R(x, 0, 0, \sigma, \lambda) = 0$. For $G(\Omega, 0)$, we use (4.43). The term $D_5\hat{G}(\zeta)$ can be estimated as in (a) by $C\|u\|_{2,\epsilon}$. The two remaining terms are also estimated as in (a) by $C(\|u\|_{2,\epsilon} + \|v\|_{2,\epsilon})$. □

In order to express the derivative of Ψ we will use the function $K\colon \mathbb{C}^n \times \mathbb{C}^n \times [0,1] \times \Lambda \to \mathbb{C}^n$ defined by

$$K(x,\xi,\sigma,\lambda) = \exp^{\lambda,\sigma}_{\psi^\sigma_\lambda(x)} A^\sigma_\lambda \xi - \psi^\sigma_\lambda(x). \tag{4.44}$$

We will need to lift K (at least on part of its domain) so that it maps to $\mathbb{C}^n \times \mathbb{R}$. We describe this lift.

Consider $L_\lambda \subset \mathbb{C}^n \times \mathbb{R}$, $\lambda \in \Lambda$. Let $K_\lambda\colon TL_0 \to \mathbb{C}^n \times \mathbb{R}$ be an embedding extension of k_λ (see Section 4.4). Consider the immersion $P_\lambda\colon V \subset TL_0 \to \mathbb{C}^n$ which extends $\Pi_{\mathbb{C}} \circ k_\lambda$, where V is a neighborhood of the 0-section in TL_0. Choose V and a neighborhood $W \subset \Lambda$ of 0, so small that the self-intersection of P_λ is contained inside $\bigcup_j B(c^*_j(0), 2\delta)$. Consider the following subset N of the product $\mathbb{C}^n \times [0,1]$.

$$\begin{aligned} N =& P(V) \setminus \bigcup_j B(c^*_j(0), 3\delta) \times [0,1] \;\cup\; \bigcup_j P(V|L_\lambda \cap U(c^+_j, 4\delta)) \times [1-\epsilon, 1] \\ &\cup \bigcup_j P(V|L_\lambda \cap U(c^-_j, 4\delta)) \times [0,\epsilon]. \end{aligned}$$

We define a map $\psi_\lambda\colon N \to \mathbb{C}^n \times \mathbb{R}$ in the natural way, $\psi_\lambda(q,\sigma) = K_\lambda(p_\sigma, v_\sigma)$ where (p_σ, v_σ) is the preimage of q under P with $p \in U(c^\pm_j, 4\delta)$ where the sign is determined by σ.

Using this construction we may do the following. If $W \subset \mathbb{C}^n \times \mathbb{C}^n \times [0,1] \times \Lambda$ and $G\colon W \to \mathbb{C}^n$ is a function such that $(G,\sigma)(W) \in N$ then we may define a lift $\tilde{G}\colon W \to \mathbb{C}^n \times \mathbb{R}$.

We now use this construction to lift the function K defined in (4.44). For x sufficiently close to L_0, ξ sufficiently small and σ sufficiently close to 0 or 1 when x is close to double points of $\Pi_{\mathbb{C}}(L_0)$ the lift $\tilde{K}$ of K can be defined. Let $K_{\mathbb{R}}$ denote the $\mathbb{R}$-coordinate of $\tilde{K}$.

Lemma 4.11. *If $\dim \Lambda > 0$, let $M < \infty$. If $\dim \Lambda = 0$, let $M = \infty$. The map*

$$\Psi\colon \mathcal{B}_{2,\epsilon}((w[M], f[M]), r) \times \mathcal{H}_{\frac{3}{2},\epsilon}(\partial D_m, \mathbb{R}) \times \Lambda \to \mathcal{F}_{2,\epsilon,\Lambda}(\mathbf{a}, \kappa)$$

is C^1. Its derivative at (v,h,μ) is the map

$$(u,l,\lambda) \mapsto \Big(D_2K \cdot v + D_4K \cdot \lambda, \langle D_2K_{\mathbb{R}} \cdot v + D_4K_{\mathbb{R}} \cdot \lambda\rangle + l, \lambda\Big),$$

where all derivatives of K and $\tilde{K}$ are evaluated at $(w[M]_\lambda, v, \sigma[M], \lambda)$ and where $\langle u\rangle$ denotes restriction of $u\colon D_m \to \mathbb{R}$ to the boundary.

Proof. Using local coordinates on $\mathcal{F}_{2,\epsilon,\Lambda}$ as described in Section 4.1, we write $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ Statements concerning Ψ_3 are trivial. Note that $\Psi_1 = K + \Psi^\sigma_\lambda(x)$. So to see that Ψ_1 is continuous we note that $K(x,0,\sigma,\lambda) = 0$ and apply Lemma 4.10 (a) to get that K is Lipschitz in v and λ and hence continuous. To see that Ψ_1 is differentiable we note that if

$$\begin{aligned} G(x,\xi,\eta,\sigma,\lambda) &= K(x,\xi+\eta,\sigma,\mu+\lambda) - K(x,\xi,\sigma,\mu) \\ &\quad - \Big(D_2K(x,\xi,\sigma,\mu)\cdot\eta + D_4K(x,\xi,\sigma,\mu)\cdot\lambda\Big), \end{aligned}$$

then the conditions (4.31)–(4.34) are fulfilled and Lemma 4.10 (b) implies Ψ_1 is differentiable and has differential as claimed. Finally, applying Lemma 4.10 (a) to the map

$$G(x,\xi,\eta,\sigma,\lambda) = D_2K(x,\xi,\sigma,\mu)\cdot\eta + D_4K(x,\xi,\sigma,\mu)\cdot\lambda$$

shows Ψ_1 is C^1.

Using $\tilde{K}$, we can extend the $\mathbb{R}$-valued function $z((w[M]_\lambda, f[M]_\lambda), A^\sigma_\lambda v)$ to a small neighborhood of ∂D_m in D_m. With this done the (non-trivial part) of the derivative of Ψ_2 can be handled exactly as above. □

Let $\zeta = x_1 + ix_2$ be a complex local coordinate in D_m. Then, if $u\colon D_m \to \mathbb{C}^n$, we may view $\bar\partial u$ as $\partial_1 u + i\partial_2 u$. As in the proof of Lemma 4.11 we use local coordinates on $\mathcal{F}_{2,\epsilon,\Lambda}$ and write $\Psi = (\Psi_1, \Psi_2, \Psi_3)$.

Lemma 4.12. *Assume that $w\colon D_m \to \mathbb{C}^n$ and $v\colon D_m \to \mathbb{C}^n$ are smooth functions and let g be any metric on $\mathbb{C}^n$. If $u(\zeta) = \exp_{w(\zeta)}(v(\zeta))$ then $\bar\partial u = X_1(1) + iX_2(1)$, where X_j, $j = 1, 2$ are the Jacobi-fields along the geodesic $\exp_{w(\zeta)}(tv(\zeta))$, $0 \le t \le 1$, with $X(0) = \partial_j w(\zeta)$ and $X'(0) = \partial_j v(\zeta)$.*

In particular, there exists $r > 0$ such that if $(v(\zeta), \lambda) \in \mathcal{H}_{2,\epsilon}(D_m, \mathbb{C}^n) \times \Lambda$, $\|v\|_{2,\epsilon} \le r$ then the restriction of $\bar\partial\Psi_1(v, \lambda)$ to ∂D_m equals 0 if and only if the restriction of $\bar\partial v = 0$.

Proof. Consider

$$\alpha(s, t) = \exp_{w(\zeta+s)}(tv(\zeta + s)), \quad 0 \le t \le 1, -\epsilon \le s \le \epsilon.$$

Since for fixed s, $t \mapsto \alpha(s, t)$ is a geodesic we find that $\partial_s\alpha(0, t) = X_1(t)$ is a Jacobi field along the geodesic $t \mapsto \exp_{w(\zeta)}(tv(\zeta))$ with initial conditions

$$X_1(0) = \partial_s \exp_{w(\zeta+s)}(0 \cdot v) = \partial_1 w(\zeta),$$
$$X_1'(0) = \partial_t\partial_s\alpha(0, 0) = \partial_s\partial_t\alpha(0, 0) = \partial_1 v(\zeta).$$

Moreover,

$$\exp_{w(\zeta+s)}(v(\zeta + s)) = \alpha(s, 1)$$

and hence

$$\partial_1 \exp_{w(\zeta)}(v(\zeta)) = \partial_s\alpha(0, 1) = X_1(1).$$

A similar analysis shows that

$$\partial_2 \exp_{w(\zeta)}(v(\zeta)) = X_2(1).$$

This proves the first statement.

Consider the second statement. Note that the metrics $g(L_\lambda, \sigma(w[M](\zeta), F[M](\zeta)))$ are constant in ζ for ζ in a neighborhood of ∂D_m. Consider first the case that $w[M]$ and v are smooth. Then the above result together with the Jacobi-field property of the metric $\hat g$ (see Lemma 4.6), from which $g(L_\lambda, \sigma)$ is constructed implies that for $\zeta \in \partial D_m$, $\bar\partial(\Psi_1(v, \lambda)) = X_1(1)+iX_2(1)$ equals the value of the Jacobi-field $X_1+JX_2 = Y$ with initial condition $Y(0) = 0$, $Y'(0) = \bar\partial A^\sigma_\lambda v$. (Note $Y(0) = 0$ since along the boundary $\bar\partial w = 0$.) Hence $\bar\partial(\Psi_1(v, \lambda)) = 0$ for $\zeta \in \partial D_m$ if and only if the same is true for v provided v is shorter than the minimum of injectivity radii of $g(L_\lambda, \sigma)$. An approximation argument together with the continuity of Ψ_1 (also in $w[M]$, see the proof of Lemma 4.10 (a)), $\bar\partial$, and of restriction to the boundary gives the second statement in full generality. □

Lemma 4.13. *For $r > 0$ small enough the image of Ψ is a submanifold of $\mathcal{F}_{2,\epsilon,\Lambda}$. Moreover, there exists $M > 0$, $r > 0$, and a neighborhood U of $(w[M], f[M], 0)$ in $\mathcal{F}_{2,\epsilon,\Lambda}$ such that $(w, f) \in U$ and $U \cap \mathcal{W}_{2,\epsilon,\Lambda}$ is contained in the image of Ψ and corresponds to the subset $h = 0$ in the coordinates*

$$(v, h, \lambda) \in B_{2,\epsilon}((w[M], f[M]), r) \times \mathcal{H}_{\frac{3}{2},\epsilon}(\partial D_m, \mathbb{R}) \times \Lambda.$$

Proof. Let $(w, f) \in \mathcal{W}_{2,\epsilon}(\mathbf{a}, \kappa)$. Let K be as in Lemma 4.11. Then $D_2K(x, 0, \sigma, 0) \cdot \eta = \eta$ and $D_4K(x, 0, \sigma, 0) = 0$. Hence the differential of Ψ at $(0, 0, 0)$ is

$$d\Psi(0, 0, 0) = \begin{pmatrix} \iota & 0 & 0 \\ \langle \iota_{\mathbb{R}} \rangle & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix},$$

where ι denotes the inclusion of the tangent space of $B_{2,\epsilon}((f,w),r)$ into $\mathcal{H}_{2,\epsilon}(D_m,\mathbb{C}^n)$ and $\langle\iota_{\mathbb{R}}\rangle v$ denotes the $\mathbb{R}$-component of the vector field $\tilde{v}$ which maps to v under $\Pi_{\mathbb{C}}$ and is tangent to L_0. Note that the tangent space of $B_{2,\epsilon}((w,f),r)$ is a closed subspace of $\mathcal{H}_{2,\epsilon}(D_m,\mathbb{C}^n)$.

Thus, $d\Psi(0,0,0)$ is an injective map with closed image. Since the first component of $\mathcal{F}_{2,\epsilon}$ is modeled on a Banach space which allow a Hilbert-space structure we see that the image of the differential admits a direct complement. Moreover, applying Lemma 4.10 to the explicit differential in Lemma 4.11 we conclude that the norm of the differential of Ψ is Lipschitz in v and λ with Lipschitz constant depending only on $\|Dw[M]\|_{1,\epsilon}$ and $\|DF[M]\|_{1,\epsilon}$. Hence the the implicit function theorem shows that there exists $r>0$ and $W\subset\Lambda$ (independent of M) such that the image of $B((w[M],f[M]),r)\times W$ is a submanifold. From the norm-estimates on the differential it follows that for M large enough (w,f) lies in this image.

The statement about surjectivity onto $U\cap\mathcal{W}_{2,\epsilon,\Lambda}$ follows from the fact that $\Pi_{\mathbb{C}}(L_\lambda)$ is totally geodesic in the metric $g(L_\lambda,\sigma)$ and Lemma 4.12. The statement on coordinates is trivial. □

4.6. **Bundle over conformal structures.** The constructions above all depend on the conformal structure κ on D_m. This conformal structure is unique if $m\le 3$. Assume that $m>3$ and recall that we identified the space of conformal structures $\mathcal{C}_m$ on D_m with an open simplex of dimension $m-3$.

The space

$$\mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a})=\bigcup_{\kappa\in\mathcal{C}_m}\mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa),$$

has a natural structure of a locally trivial Banach manifold bundle over $\mathcal{C}_m$. To see this we must present local trivializations.

Let Δ denote the unit disk in the complex plane and let Δ_m denote the same disk with m punctures $p_1,\dots,p_m$ on the boundary and conformal structure κ. Fixing the positions of p_1,p_2,p_3, this structure is determined by the positions of the remaining $m-3$ punctures. We coordinatize a neighborhood of the conformal structure κ in $\mathcal{C}_m$ as follows. Pick $m-3$ vector fields $v_1,\dots,v_{m-3}$, with v_k supported in a neighborhood of p_{k+3}, $k=1,\dots,m-3$ in such a way that v_k generate a 1-parameter family of diffeomorphism $\phi^{\tau_k}_{p_{k+3}}\colon\Delta\to\Delta$, $\tau_k\in\mathbb{R}$ which is a rigid rotation around p_{k+3} and which is holomorphic on the boundary. Let the supports of v_k be sufficiently small so that the supports of $\phi^{\tau_k}_{p_{k+3}}$, $k=1,\dots,m-3$ are disjoint. Then the diffeomorphisms $\phi^{\tau_1}_{p_3},\dots,\phi^{\tau_{m-3}}_{p_m}$ all commute. Define, for $\tau=(\tau_1,\dots,\tau_{m-3})\in\mathbb{R}^{m-3}$, $\phi^\tau=\phi^{\tau_1}_{p_3}\circ\dots\circ\phi^{\tau_{m-3}}_{p_m}$ and a local coordinate system around κ in $\mathcal{C}_m$ by

$$\tau\mapsto\Big(d\phi^\tau\Big)^{-1}\circ j_\kappa\circ d\phi^\tau.$$

These local coordinate systems give an atlas on $\mathcal{C}_m$.

Using this family we define the trivialization over $\mathbb{R}^{m-3}\approx U\subset\mathcal{C}_m$ by composition with $\phi^{-\tau}$. That is, a local trivialization over U is given by

$$\Phi\colon\mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a},\kappa)\times U\to\mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a});$$

$$\Phi(w,f,\lambda,\tau)=\Big(w\circ\phi^{-\tau},f\circ\phi^{-\tau},\lambda,\theta\Big)$$

In a similar way we endow the space

$$\mathcal{H}_{1,\epsilon}(D_m,T^*D_m)=\bigcup_{\kappa\in\mathcal{C}_m}\mathcal{H}_{1,\epsilon}(D_m,T^*D_m,g(\kappa)),$$

with its natural structure as a locally trivial Banach space bundle over $\mathcal{C}_m$.

Representing the space of conformal structures $\mathcal{C}_m$ in this way we are led to consider its tangent space $T_\kappa\mathcal{C}_m$ as generated by $\gamma_1,\dots,\gamma_{m-3}$, where $\gamma_k=i\cdot\bar\partial v_k$, in the following sense.

If γ is any section of $\mathrm{End}(TD_m)$ which anti-commutes with j_κ and which vanishes on the boundary then there exists unique numbers $a_1, \dots, a_{m-3}$ and a unique vector field v on Δ_m which is holomorphic on the boundary and which vanish at p_k, $k = 1, \dots, m$ such that

$$\gamma = \sum_k a_k \gamma_k + i\bar{\partial}v. \tag{4.45}$$

The existence of such v is a consequence of the fact that the classical Riemann-Hilbert problem for the $\bar{\partial}$-operator on the unit disk with tangential boundary conditions has index 3 and is surjective (the kernel being spanned by the vector fields $z \mapsto iz, z \mapsto i(z^2+1), z \mapsto z^2 - 1$).

Going from the punctured disk Δ_m to D_m with our standard metric, the behavior of the vector fields v_j close to punctures where they are supported is easily described. In fact the vector fields can be taken as ∂_x in coordinates $z = x+iy \in (\mathbb{C}_+, \mathbb{R}, 0)$ in a neighborhood of the puncture p on $\partial\Delta_m$. The change of coordinates taking us to the standard end $[0,\infty)\times[0,1]$ is $\tau+it = \zeta = -\frac{1}{\pi}\log z$ and we see the corresponding vector field on $[0,\infty)\times[0,1]$ is $\frac{1}{\pi}e^{\pi\zeta}$ (where we identify vector fields with complex valued functions). As in Proposition 5.13, we see that equation (4.45) holds on D_m with v in a Sobolev space with (small) negative exponential weights at the punctures.

4.7. **The $\bar{\partial}$-map and its linearization.** Consider the space $\mathcal{H}_{1,\epsilon}(D_m, T^{*0,1}D_m \otimes \mathbb{C}^n)$ and the closed subspace $\mathcal{H}_{1,\epsilon}[0](D_m, T^{*0,1}D_m \otimes \mathbb{C}^n)$ consisting of elements whose trace (restriction to the boundary) is 0. Note that this space depends on the metric on D_m. In our case it thus depends on the conformal structure. For simplicity we keep the notation and consider $\mathcal{H}_{1,\epsilon}[0](D_m, T^{*0,1}D_m \otimes \mathbb{C}^n)$ as a bundle over $\mathcal{C}_m$. We extend this bundle to a bundle over Λ making it trivial in the Λ directions and denote the result $\mathcal{H}_{1,\epsilon,\Lambda}[0](D_m, T^{*0,1}D_m \otimes \mathbb{C}^n)$.

The $\bar{\partial}$-map is the map

$$\begin{aligned}&\hat{\Gamma}\colon \mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a}) \to \mathcal{H}_{1,\epsilon,\Lambda}[0](D_m, T^{*0,1}D_m \otimes \mathbb{C}^n);\\ &\hat{\Gamma}(w,f,\kappa,\lambda) = \Big(dw + i\circ dw \circ j_\kappa, \kappa, \lambda\Big).\end{aligned}$$

We will denote the first component of this map simply Γ. An element (w,f,κ,λ) is thus holomorphic with respect to the complex structure j_κ if and only if $\hat{\Gamma}(w,f,\kappa,\lambda) = (0,\kappa,\lambda)$. Hence, if L_λ, $\lambda \in \Lambda$ is a family of chord generic Legendrian submanifolds then the (parameterized) moduli-space of holomorphic disks with boundary on L_λ, positive puncture at a_1, and negative punctures at $a_2, \dots, a_m$ is naturally identified with the preimage under $\hat{\Gamma}$ of the 0-section in $\mathcal{H}_{1,\epsilon,\Lambda}[0](D_m, T^{*0,1}D_m \otimes \mathbb{C}^n)$ for sufficiently small $\epsilon \in [0,\infty)$.

We compute the linearization of the $\bar{\partial}$-map. As in Section 4.6 we think of tangent vectors γ to $\mathcal{C}_m$ at κ as sections of $\mathrm{End}(TD_m)$. For $\kappa \in \mathcal{C}_m$ and $u\colon D_m \to \mathbb{C}^n$, let $\bar{\partial}_\kappa u = du + i\circ du \circ j_\kappa$.

Let $(w,f,\kappa,0) \in \mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{a})$. Identify the tangent space of $\mathcal{W}_{2,\epsilon,\lambda}(\mathbf{a})$ at $(w,f,\kappa,0)$ with $T\mathcal{B}_{2,\epsilon}((w,f),r) \times T_\kappa\mathcal{C}_m \times T_0\Lambda$.

Lemma 4.14. *The differential of* Γ *at* $(w,f,\kappa,0)$ *is the map*

$$d\Gamma(v,\gamma,\lambda) = \bar{\partial}_\kappa v + \bar{\partial}_\kappa\big(Y_0^\sigma(w,\lambda)\big) + i\circ dw \circ \gamma. \tag{4.46}$$

Recall Y_0^σ *was defined in* (4.25).

Proof. Assume first w and f are constant close to punctures. Let $\mathcal{B}_{2,\epsilon}((w,f),r) \times \mathcal{C}_m \times \Lambda$ be a local coordinates around $(w,f,\kappa,0)$.

Let $K(x,\xi,\sigma,\lambda) = \psi_\lambda^\sigma(x) + \xi$. Then

$$R(x,\xi,\sigma,\lambda) = \exp^{\lambda,\sigma}_{\psi^\sigma_\lambda(x)} A^\sigma_\lambda \xi - K(x,\xi,\sigma,\lambda)$$

satisfies

$$R(x,0,\sigma,\lambda) = 0, \quad D_2R(x,0,\sigma,0) = 0, \quad D_4R(x,0,\sigma,0) = 0;$$

thus, Lemma 4.10 (b) implies that

$$\|R(w,v,\sigma,\lambda)\|_{2,\epsilon} \le C(\|v\|_{2,\epsilon}^2 + |\lambda|^2).$$

Continuity of the linear operators

$$\bar{\partial}_{\kappa+\gamma}\colon \mathcal{H}_{2,\epsilon}(D_m,\mathbb{C}^n) \to \mathcal{H}_{1,\epsilon}(D_m, T^*D_m \otimes \mathbb{C}^n),$$

where we use local coordinates $\mathbb{R}^{m-3}$ on $\mathcal{C}_m$ and $\kappa+\gamma \in \mathbb{R}^{m-3} \subset \mathcal{C}_m$, shows that

$$\|\bar{\partial}_{\kappa+\gamma} R(w,v,\sigma,\lambda)\|_{1,\epsilon} \le C(\|v\|_{2,\epsilon}^2 + |\lambda|^2). \tag{4.47}$$

It is straightforward to check that

$$\begin{aligned}&\left\|\bar{\partial}_{\kappa+\gamma} K(w,v,\kappa+\mu,\lambda) - \bar{\partial}_\kappa w - \Big(\bar{\partial}_\kappa v + \bar{\partial}_\kappa\big(Y_0(w,\lambda)\big) + i\circ dw\circ\gamma\Big)\right\|_{1,\epsilon}\\ &\le C(\|v\|_{2,\epsilon}^2 + |\lambda|^2 + |\gamma|^2)\end{aligned} \tag{4.48}$$

Equations (4.48) and (4.47) imply the lemma in the special case when (w,f) is constant close to punctures (and in the general case if $\dim(\Lambda)=0$).

If (w,f) is not constant close to punctures consider the maps $(w[M],f[M])$ which are constant close to punctures. We have $(w[M],f[M]) \to (w,f)$ as $M\to\infty$. Since the local coordinates are C^1 a limiting argument proves (4.46) in the general case. □

4.8. **Auxiliary spaces in the semi-admissible case.** In Section 6.9 we show that for a dense open set of semi-admissible Legendrian submanifolds L no rigid holomorphic disks with boundary on L have exponential decay at their degenerate corners. Once this has been shown we know that if 0 is the degenerate corner and L has the form (2.4) around 0 then for any rigid holomorphic disk $u\colon D_m \to \mathbb{C}^n$ with puncture p mapping to 0 there exists $M>0$ and $c\in\mathbb{R}$ such that

$$u(\zeta) = \Big(-2(\zeta+c)^{-1},0,\dots,0\Big) + \mathcal{O}(e^{-\theta|\zeta|}),\ \text{for } \zeta\in E_p[\pm M],$$

where $\theta>0$ is the smallest non-zero complex angle of the Reeb chord at 0. (Here we implicitly assume that P_2 in our standard self tangency model lies above P_1 in the z-direction, and that neighborhoods of positive (negative) punctures are parameterized by $[1,\infty)\times[0,1]$ ($(-\infty,-1]\times[0,1]$).) To study disks of this type we use the following construction.

Let a_0 denote the Reeb-chord at 0. Assume that $\mathbf{a}$ has the Reeb-chord a_0 in k positions. For $C=(c_1,\dots,c_k)\in\mathbb{R}^k$ fix a smooth reference function which equals

$$u^C_{\text{ref}}(\zeta) = \Big(-2(\zeta+c_j)^{-1},0,\dots,0\Big),$$

in a neighborhood of the j^{th} puncture mapping to a_0.

Let L_λ, $\lambda\in\Lambda$ be a family of semi-admissible Legendrian submanifolds. We construct for $\epsilon\in[0,\infty)^m$, with those components ϵ_j of ϵ which correspond to punctures mapping to the degenerate corner satisfying $0<\epsilon_j<\theta$ and for fixed $C\in\mathbb{R}^k$, the spaces

$$\mathcal{F}^C_{2,\epsilon}(\mathbf{a})$$

by using reference functions looking like u^C_{ref} for $C\in\mathbb{R}^k$ in neighborhoods of punctures mapping to $a_0^*=0\in\mathbb{C}^n$. We construct local coordinates as in Section 4.1 taking advantage of the fact that $\lambda\in\Lambda$ fixes a_0^*. Also we consider the space

$$\mathcal{W}^C_{2,\epsilon,\Lambda}(\mathbf{a}),$$

which is defined in the same way as before. We note that the construction giving local coordinates on this space in Section 4.5 can still be used since in the semi-admissible case we need not cut-off the first component of w in (w,f) close to punctures mapping to c_0 since $\lambda\in\Lambda$ are assumed to preserve the product structure and γ_1 and γ_2.

With this done we consider the bundle

$$\widetilde{\mathcal{W}}_{2,\epsilon,\Lambda} = \bigcup_{C\in\mathbb{R}^k} \mathcal{W}^C_{2,\epsilon,\Lambda}(\mathbf{a}), \tag{4.49}$$

which we present as a locally trivial bundle over $\mathbb{R}^k$ as follows.

In the case that $\mathbf{a}$ has ≥ 3 elements we fix for $C \in \mathbb{R}^k$ the diffeomorphism $\phi^C : D_m \to D_m$ which equals to $\zeta \to \zeta + c_j$ in $E_{p_j}[M]$ for any puncture p_j mapping to c_0, equals the identity on $D_m \setminus \bigcup_j E_{p_j}[M-2]$, and is holomorphic on the boundary.

(Since we often reduce the few punctured cases to the many punctured case, see Section 7.7, the following two constructions will not be used in the sequel, we add them here for completeness.) In case $\mathbf{a}$ has length 1 we think of D_1 as of the upper half-plane $\mathbb{C}_+$ with the puncture at ∞. The map $z \mapsto -\frac{1}{\pi}\log z$ identifies the region $\{z \in \mathbb{C}_+ : |z| > R\}$ with the strip $[\frac{1}{\pi}\log R, \infty) \times [0,1]$ where we think of the latter space as a part of E_p, where p is the puncture of D_1. Also, this map takes the conformal reparameterization $z \mapsto e^{\pi C} z$ to $\phi^C : \zeta \mapsto \zeta + C$ in E_p and we identify $\mathbb{R}$ with this set of conformal reparameterizations $\{\phi^C\}_{C\in\mathbb{R}}$. In case $\mathbf{a}$ has length 2 we think of D_2 as the strip $\mathbb{R}\times[0,1]$ and identify $\mathbb{R}$ with the conformal reparameterizations $\phi^C(\zeta) = \zeta \mapsto \zeta + C$.

Using composition with the maps ϕ^C we construct local trivializations of the bundle in (4.49). We then find local coordinates $B_{2,\epsilon}(0,r)\times\mathcal{C}_m\times\mathbb{R}^k\times\Lambda$ on $\widetilde{\mathcal{W}}_{2,\epsilon,\Lambda}$ and the linearization of the $\bar\partial$-map Γ at $(w,f,\kappa,0,0)$ is

$$\begin{aligned} d\Gamma(v,\gamma,c,\lambda) &= \bar\partial_\kappa v + \bar\partial_\kappa\big((Y_0(w,\lambda))\big) \\ &\quad + i\circ dw\circ\gamma + \bar\partial_\kappa\left(dw\cdot\left(\frac{\partial\phi^C}{\partial C}|_{C=0}\right)\cdot c\right). \end{aligned} \tag{4.50}$$

Here $c = (c_1,\dots,c_k)$ is a tangent vector to $\mathbb{R}^k$ written in the basis $\{\hat C_1,\dots,\hat C_k\}$ where $\hat C_j$ is a unit vector in the tangent space to $C_j \in \mathbb{R}$. We notice that the second term in (4.50) lies in $\mathcal{H}_{1,\epsilon}[0](D_m, T^{*0,1}D_m)$ because of the special assumptions on L_λ in a neighborhood of c_0^* and that the last term does as well since the difference of w and the holomorphic function u^C_{ref} lies in a Sobolev space weighted by $e^{\epsilon\tau}$ in E_{p_j} and for a holomorphic function the last term vanishes in the region where ϕ^C is just a translation.

4.9. **Homology decomposition.** Let $L \subset \mathbb{C}^n\times\mathbb{R}$ be a (semi-)admissible Legendrian submanifold. Let $\mathbf{c} = c_0c_1\dots c_m$ be a word of Reeb chords of L. If $(u,f)\in\mathcal{W}_{2,\epsilon}(\mathbf{c})$ then the homotopy classes of the paths induced by $(u|\partial D_m, f)$ in L connecting the Reeb chord endpoints determines the path component of $(u,f)\in\mathcal{W}_{2,\epsilon}(\mathbf{c})$.

Let $A \in H_1(L)$ and let $\mathcal{W}_{2,\epsilon}(\mathbf{c};A) \subset \mathcal{W}_{2,\epsilon}(\mathbf{c})$ be the union of those path components of $\mathcal{W}_{2,\epsilon}(\mathbf{c})$ such that the homology class of the loop $f(\partial D_m)\cup(\bigcup_j\gamma_j)$ equals A, where γ_j is the capping path chosen for the Reeb chord c_j endowed with the appropriate orientation, see Section 1.3. For fixed conformal structure κ we write $\mathcal{W}_{2,\epsilon}(\mathbf{c},\kappa;A)$ and in the chord semi generic case $\widetilde{\mathcal{W}}_{2,\epsilon}(\mathbf{c};A)$ and interpret these notions in a similar way.

5. Fredholm properties of the linearized equation

In this section we study properties of the linearized $\bar\partial$-equation. In particular we determine the index of the $\bar\partial$-operator with Legendrian boundary conditions. It will be essential for our geometric applications to use weighted Sobolev spaces and to understand how constants in certain elliptic estimates depend on the weights.

Our presentation has two parts: the "model" case where the domain is a strip or half-plane; and the harder case where the domain is D_m.

In Section 5.1, we show that an element of the cokernel has a smooth representative. In Section 5.2, we derive expansions for the kernel and cokernel elements. We use these two subsections in Sections 5.3 through 5.5, to prove the elliptic estimate for the model problem, as well as derive a formula for the index. In Sections 5.6, we set up the boundary conditions for the linearized problem with domain D_m. In Sections 5.7 through 5.10, we prove the Fredholm properties for the D_m case. In Sections 5.10 and 5.11, we connect the index formula to the Conley-Zehnder index of Section 1.

5.1. **Cokernel regularity.** To control the cokernels of the operators studied below we use the following regularity lemma.

For this subsection only, we use coordinates (x, y) for the half-plane $\mathbb{R}^2_+ = \{(x, y) : y \geq 0\}$. Let $A : \mathbb{R} \to \mathbf{GL}(\mathbb{C}^n)$ be a smooth map with $\det(A)$ uniformly bounded away from 0 and all derivatives uniformly bounded. We also simplify notation for this subsection only and define the following Sobolev spaces: let $\mathcal{H}_k = \mathcal{H}_k(\mathbb{R}^2, \mathbb{C}^n)$; let $\bar{\mathcal{H}}_k$ denote the space of restrictions of elements in $\mathcal{H}_k$ to $\mathrm{int}(\mathbb{R}^2_+)$; let $\dot{\mathcal{H}}_k$ denote the subspace of elements in $\mathcal{H}_k$ with support in $\mathbb{R}^2_+$; let $\bar{\mathcal{H}}_1[0]$ denote the subspace of all elements in $\bar{\mathcal{H}}_1$ which vanish on the boundary; and let $\bar{\mathcal{H}}_2[A]$ denote the subspace of elements u in $\bar{\mathcal{H}}_2$ such that $u(x, 0) \in A(x)\mathbb{R}^n$ and such that the trace of $\bar{\partial}u$ (its restriction to the boundary) equals 0 in $\mathcal{H}_{\frac{3}{2}}(\mathbb{R}, \mathbb{C}^n)$.

An element ξ in the cokernel of $\bar{\partial}$ will be in the dual space of $\mathcal{H}_1[0]$. The dual of $\mathcal{H}_1$ is $\mathcal{H}_{-1}$ and thus the dual of $\mathcal{H}_1[0]$ is the quotient space

$$\mathcal{H}_{-1}/\mathcal{H}_1[0]^{\perp}, \tag{5.1}$$

where $\mathcal{H}_1[0]^{\perp}$ denotes the annihilator of $\mathcal{H}_1[0]$ in $\mathcal{H}_{-1}$. As usual, let $\langle\, , \rangle$ denote the standard Riemannian inner product on $\mathbb{C}^n \approx \mathbb{R}^n$.

Lemma 5.1. *Fix $h > 0$ and assume that $v \in \dot{\mathcal{H}}_{-1}$ satisfies*

$$\int_{\mathbb{R}\times[0,h)} \langle \bar{\partial}u, v\rangle \, dx \wedge dy = 0, \tag{5.2}$$

for all $u \in \bar{\mathcal{H}}_2[A]$ with compact support in $\mathbb{R} \times [0, h)$. Then, for every ϵ with $0 < \epsilon < h$ and every $k > 0$, the class $[v] \in \dot{\mathcal{H}}_{-1}/\bar{\mathcal{H}}_1[0]^{\perp}$ of v contains an element v_0 which is C^k in $\mathbb{R} \times [0, \epsilon)$, up to and including the boundary.

Proof. Let $m > 2$ be such that $\mathcal{H}_m \subset C^k$ and $\epsilon < h$ be given. Extend A to a smooth map $\hat{A}$ on $\mathbb{R}^2_+$ with uniformly bounded derivatives as follows. Let $0 < \delta < \frac{\epsilon}{4}$ and let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth non-decreasing function such that $\eta(y) = y$ for $0 \leq y \leq \frac{\delta}{2}$ and $\eta(y) = \delta$ for $y \geq \delta$. Define

$$\hat{A}(x, y) = A(x) + \sum_{k=1}^{m+1} \frac{i^k \eta(y)^k}{k!} \partial_x^k A(x),$$

and choose $\delta > 0$ so small that $|\det(\hat{A}(x, y))| > \rho > 0$. Note that $\bar{\partial}\hat{A}(x, 0)$ vanishes to order m. Therefore multiplication with $\hat{A}$ and $\hat{A}^{-1}$ gives the following commutative diagram where horizontal arrows are isomorphisms:

$$\begin{array}{ccc} \bar{\mathcal{H}}_2[\mathrm{id}] & \xrightarrow{\times \hat{A}} & \bar{\mathcal{H}}_2[A] \\ {\scriptstyle \bar{\partial}+\hat{A}^{-1}\bar{\partial}\hat{A}}\downarrow & & \downarrow{\scriptstyle \bar{\partial}} \\ \bar{\mathcal{H}}_1[0] & \xleftarrow[\times \hat{A}^{-1}]{} & \bar{\mathcal{H}}_1[0] \end{array} .$$

The lemma follows once its analogue with the vertical operator on the left in the diagram replacing $\bar{\partial}$ is proved. Let $B = \hat{A}^{-1}\bar{\partial}\hat{A}$.

Consider smooth functions u with compact support in $\mathbb{R} \times (0,h)$. For such u we have (recall $\dot{\mathcal{H}}_k \subset \mathcal{H}_k$, all k)

$$0 = \int \langle (\bar\partial + B)u, v\rangle \, dx \wedge dy = \int \langle u, (\partial + B^*)v\rangle \, dx \wedge dy. \tag{5.3}$$

Hence $(\partial + B^*)v = 0$ in $\mathbb{R} \times (0,h)$. Let α and $\hat\alpha$ be smooth functions with all derivatives uniformly bounded, with support in $\mathbb{R} \times (0,h)$, and such that $\hat\alpha = 1$ on the support of α. Then the elliptic estimate for ∂ on $\mathbb{R}^2$ implies (with $K > 0$ a sufficiently large constant and k any integer)

$$\begin{aligned}
\|\alpha v\|_k &\le C(\|\alpha v\|_{k-1} + \|\partial(\alpha v)\|_{k-1}) \\
&\le C(\|\alpha v\|_{k-1} + \|(\partial + B^*)\alpha v\|_{k-1} + \|B^*\alpha v\|_{k-1}) \\
&\le C'(\|\alpha v\|_{k-1} + \|(\partial + B^*)\alpha v\|_{k-1}) \\
&\le C'(\|\alpha v\|_{k-1} + \|((\partial + B^*)\alpha)v\|_{k-1} + \|\alpha(\partial + B^*)v\|_{k-1}) \\
&\le C''\|\hat\alpha v\|_{k-1} \qquad (5.4)
\end{aligned}$$

since $(\partial + B^*)v = 0$ in $\mathbb{R} \times (0,h)$. It follows from (5.4) that v is smooth in $\mathbb{R} \times (0,h)$. Thus, $(\partial + B^*)v \in \dot{\mathcal{H}}_{-2}$ is a distribution such that $\mathrm{supp}((\partial + B^*)v) \cap \mathbb{R} \times [0,h) \subset \mathbb{R} \times \{0\}$ and hence there exist distributions f and g on $\mathbb{R}$ such that

$$(\partial + B^*)v = \delta(y) \otimes f(x) + \delta'(y) \otimes g(x) \tag{5.5}$$

in $\mathbb{R} \times [0,h)$, where δ denotes the Dirac distribution and δ' its derivative. Since any function $\phi \in \mathcal{H}_2(\mathbb{R}, \mathbb{C}^n)$ can be extended constantly in the y-direction in a neighborhood of the real axis so that it lies in $\mathcal{H}_2$ we find that $f(x) \in \mathcal{H}_{-2}(\mathbb{R}, \mathbb{C}^n)$. Thus, $\delta(y) \otimes f(x)$ lies in $\dot{\mathcal{H}}_{-2}$ and therefore so does $\delta'(y) \otimes g(x)$.

Let $u \in \bar{\mathcal{H}}_2[\mathrm{id}]$ have support in $\mathbb{R} \times [0,h)$ and let $\hat u$ be an extension of u to some neighborhood of $\mathbb{R}^2_+$ in $\mathbb{R}^2$. Let $\tilde B$ and $\tilde B^*$ denote the extensions of B and B^* to $\mathbb{R}^2$ by defining them to be 0 on $\mathrm{int}(\mathbb{R}^2_-)$. Note that by the definition of $\hat A$, $\tilde B$ and $\tilde B^*$ are C^m-functions with uniformly bounded derivatives. Let γ be a smooth function which equals 1 on $\mathbb{R}^2_+$ and equals 0 outside a neighborhood of $\mathbb{R}^2_+$ in $\mathbb{R}^2$. Then

$$\begin{aligned}
\int \langle u, (\partial + \tilde B^*)v\rangle \, dx \wedge dy &= \int \langle \gamma\hat u, (\partial + \tilde B^*)v\rangle \, dx \wedge dy \\
&= \int \langle (\bar\partial\gamma)\hat u, v\rangle \, dx \wedge dy + \int \langle (\gamma)(\bar\partial + \tilde B)\hat u, v\rangle \, dx \wedge dy = 0 + 0. \qquad (5.6)
\end{aligned}$$

Let $\phi\colon \mathbb{R} \to \mathbb{R}^n$ be any smooth compactly supported function. Let β be a function with $\beta(0) = 0$ and $\beta'(0) = 1$ and $\beta = 0$ outside a small neighborhood of 0. Then

$$u(x,y) = \phi(x) + i\beta(y)\phi'(x),$$

lies in $\mathcal{H}_2[\mathrm{id}]$ and by (5.6)

$$0 = \int \langle u, (\partial + \tilde B^*)v\rangle \, dx \wedge dy = \int (\mathrm{Re}\, f - \mathrm{Im}\, g') \cdot \phi \, dx.$$

Hence $\mathrm{Re}\, f = \mathrm{Im}\, g'$. Since $\mathrm{Re}\, f \in \mathcal{H}_{-2}(\mathbb{R}, \mathbb{C}^n)$ we find that if $v^{\mathrm{I}} = \delta(y) \otimes \mathrm{Im}\, g$ then

$$\bar\partial v^{\mathrm{I}} = \delta(y) \otimes \mathrm{Re}\, f + i\delta'(y) \otimes \mathrm{Im}\, g \in \mathcal{H}_{-2}$$

and hence

$$\|v^{\mathrm{I}}\|_{-1} \le C(\|v^{\mathrm{I}}\|_{-2} + \|\bar\partial v^{\mathrm{I}}\|_{-2}) < \infty.$$

Let $a_j \in C_0^\infty(\mathbb{R}^2_+)$ be a sequence such that $a_j \to v^{\mathrm{I}}$ in $\mathcal{H}_{-1}$ as $j \to \infty$. Define $a_j^-(x,y) = a_j(x,-y)$. Then a_j^- are supported in the lower half plane and $\bar\partial a_j(x,-y) = \partial a_j^-(x,y)$.

Hence a_j^- approaches a distribution v^{II} with support on the boundary, $\partial v^{\mathrm{II}} = \bar\partial v^{\mathrm{I}}$, and since $v^{\mathrm{I}} \in \bar{\mathcal{H}}_{-1}[0]$ also $v^{\mathrm{II}} \in \bar{\mathcal{H}}_1[0]^{\perp}$. Let $v_{\mathrm{I}} = v - v^{\mathrm{II}}$ then

$$(\partial + \tilde{B}^*)v_{\mathrm{I}} = i\delta(y) \otimes \operatorname{Im} f(x) + \delta'(y) \otimes \operatorname{Re} g(x),$$

in $\mathbb{R} \times [0,h)$.

Let $b_j \in C_0^\infty(\mathbb{R}^2_+)$ be a sequence such that $b_j \to v_{\mathrm{I}}$ as $j \to \infty$ and define $b_j^*(x,y) = \overline{b_j(x,-y)}$. Then $\partial b^*(x,y) = \overline{\partial b_j(x,y)}$. Define

$$B^d(x,y) = \begin{cases} B(x,y), & \text{if } y \geq 0 \\ \overline{B(x,-y)}, & \text{if } y \leq 0 \end{cases}$$

Then B^d is a C^m function. If $b_j^d = \frac{1}{2}(b_j + b_j^*)$ then b_j^d approaches a distribution v_{I}^d with

$$(\partial + (B^d)^*)v_{\mathrm{I}}^d = \delta'(y) \otimes \operatorname{Re} g(x),$$

in $\mathbb{R} \times [0,h)$. Again, let $\phi\colon \mathbb{R} \to \mathbb{R}$ be any smooth function with compact support. Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be an odd smooth function with $\theta(0) = 0$, $\theta'(0) = 1$ and $\theta = 0$ outside a small neighborhood of 0. Then with $u(x,y) = \phi(x)\theta(y)$

$$\int \langle (\bar\partial + B^d)u, v_{\mathrm{I}}^d \rangle \, dx \wedge dy = -\int \langle u, (\partial + (B^d)^*)v_{\mathrm{I}}^d \rangle \, dx \wedge dy = \int_{\mathbb{R}} \phi(x) \operatorname{Re} g(x) \, dx.$$

But, writing $B^d = B^d_{\mathrm{Re}} + iB^d_{\mathrm{Im}}$,

$$\begin{aligned} &\int \langle (\bar\partial + B^d)u, v_{\mathrm{I}}^d \rangle \, dx \wedge dy = \\ &\int \Big(\operatorname{Re} v_{\mathrm{I}}^d \cdot (B^d_{\mathrm{Re}}(x,y) + \phi'(x)\theta(y) - B^d_{\mathrm{Im}}(x,y) - \phi(x)\theta'(y)) \\ &+ \operatorname{Im} v_{\mathrm{I}}^d \cdot (B^d_{\mathrm{Re}}(x,y) + \phi(x)\theta'(y) + B^d_{\mathrm{Im}}(x,y) + \phi'(x)\theta(y)) \Big) \, dx \wedge dy = \\ \lim_{j\to\infty} &\int \Big(\operatorname{Re} b_j^d(x,y) \cdot (B^d_{\mathrm{Re}}(x,y) + \phi'(x)\theta(y) - B^d_{\mathrm{Im}}(x,y) - \phi(x)\theta'(y)) \\ &+ \operatorname{Im} b_j^d(x,y) \cdot (B^d_{\mathrm{Re}}(x,y) + \phi(x)\theta'(y) + B^d_{\mathrm{Im}}(x,y) + \phi'(x)\theta(y)) \Big) \, dx \wedge dy = 0, \end{aligned} \tag{5.7}$$

since both summands in the last integral are odd in y. Hence $\operatorname{Re} g(x) = 0$ and $(\partial + (B^d)^*)v_{\mathrm{I}}^d = 0$ in $\mathbb{R} \times [0,h)$ and therefore by the elliptic estimate for ∂, v_{I}^d lies in $\mathcal{H}_{m+1}$ in $\mathbb{R} \times [0,h)$. Let $\eta\colon \mathbb{R}^2 \to [0,1]$ be a smooth function which is 1 on $\mathbb{R} \times [0,\epsilon)$ and 0 outside $[0,h)$. Let $v_{\mathrm{II}} = \eta v_{\mathrm{I}}^d | \mathbb{R}^2_+$. Then $v_{\mathrm{II}} \in \dot{\mathcal{H}}_0 \subset \dot{\mathcal{H}}_{-1}$ and $v^{\mathrm{III}} = v_{\mathrm{II}} - \eta v_{\mathrm{I}} \in \dot{\mathcal{H}}_{-1}$ is a distribution with support on the boundary. Hence $v^{\mathrm{III}} = \delta(y) \otimes h(x)$ and thus $v^{\mathrm{III}} \in \mathcal{H}_1[0]^{\perp}$. Since

$$v = v_{\mathrm{I}} + v^{\mathrm{II}} = \eta v_{\mathrm{I}} + (1-\eta)v_{\mathrm{I}} + v^{\mathrm{II}} = v_{\mathrm{II}} + (1-\eta)v_{\mathrm{I}} - v^{\mathrm{III}} + v^{\mathrm{II}}, \tag{5.8}$$

where $v^{\mathrm{III}} + v^{\mathrm{II}} \in \bar{\mathcal{H}}_1[0]^{\perp}$ we find that $v_0 = v_{\mathrm{II}} + (1-\eta)v_{\mathrm{I}}$ is a representative of $[v]$ which is as smooth as required in $\mathbb{R} \times [0,\epsilon)$. □

5.2. **Kernel and cokernel elements.** Consider the strip $\mathbb{R} \times [0,1] \subset \mathbb{C}$ endowed with the standard flat metric, the corresponding complex structure and coordinates $\zeta = \tau + it$. For $k \geq 0$, let

$$\mathcal{H}_k = \mathcal{H}_k(\mathbb{R} \times [0,1], \mathbb{C}^n),$$

and for $k \leq 0$, let $\mathcal{H}_k$ denote the L^2-dual of $\mathcal{H}_{-k}$. We also use the notions $\mathcal{H}_k^{\mathrm{loc}}$ which are to be understood in the corresponding way.

If $u \in \mathcal{H}_k^{\mathrm{loc}}$ then the restriction of u to $\partial(\mathbb{R}\times[0,1]) = \mathbb{R}\cup\mathbb{R}+i$ lies in $\mathcal{H}^{\mathrm{loc}}_{k-\frac{1}{2}}(\mathbb{R}\cup\mathbb{R}+i,\mathbb{C}^n)$. For $u \in \mathcal{H}_1^{\mathrm{loc}}$ consider the boundary conditions

$$\int_{\mathbb{R}} \langle u, v\rangle \, d\tau = 0 \quad \text{ for all } v \in C_0^\infty(\mathbb{R}, i\mathbb{R}^n), \tag{5.9}$$

$$\int_{\mathbb{R}+i} \langle u, v\rangle \, d\tau = 0 \quad \text{ for all } v \in C_0^\infty(\mathbb{R}+i, \mathbb{R}^n), \tag{5.10}$$

Let $f\colon \mathbb{R}\times[0,1]\to\mathbb{C}^n$ be a smooth function satisfying (5.9) and (5.10). Define the function $f^d\colon \mathbb{R}\times[0,2]\to\mathbb{C}^n$ as

$$f^d(\tau+it) = \begin{cases} f(\tau+it) & \text{for } 0\le t\le 1,\\ -\overline{f}(\tau+i(2-t)) & \text{for } 1 < t \le 2,\end{cases}$$

where $\overline{w}$ denotes the complex conjugate of $w \in \mathbb{C}^n$. Then f^d and $\partial_\tau f^d$ are continuous, $\partial_t f^d$ may have a jump discontinuity over the line $\mathbb{R}+i$, $f^d(\tau+0i) = -f^d(\tau+2i)$, and $\|f^d\|_1 = 2\|f\|_1$. Hence we can define the double $u^d \in \mathcal{H}_1^{\mathrm{loc}}(\mathbb{R}\times[0,2])$ of any $u\in\mathcal{H}_1^{\mathrm{loc}}$ which satisfies (5.9) and (5.10). For $u\in\mathcal{H}_k^{\mathrm{loc}}$, let $\bar\partial u = (\partial_\tau + i\partial_t)u$ and $\partial u = (\partial_\tau - i\partial_t)u$.

Lemma 5.2. *If $u \in \mathcal{H}_1^{\mathrm{loc}}$ satisfies* (5.9) *and* (5.10) *and*

(a) $\bar\partial u = 0$ *in the interior of* $\mathbb{R}\times[0,1]$ *then*

$$u(\zeta) = \sum_{n\in\mathbb{Z}} C_n \exp\left((\tfrac{\pi}{2}+n\pi)\zeta\right),$$

where $C_n \in \mathbb{R}$.

(b) $\partial u = 0$ *in the interior of* $\mathbb{R}\times[0,1]$ *then*

$$u(\zeta) = \sum_{n\in\mathbb{Z}} C_n \exp\left((\tfrac{\pi}{2}+n\pi)\bar\zeta\right),$$

where $C_n \in \mathbb{R}$.

Moreover, if u satisfies (a) *or* (b) *and $u\in\mathcal{H}_k$ for some $k\in\mathbb{Z}$ then $u=0$.*

Proof. We prove (a), (b) is proved in the same way. Clearly it is enough to consider one coordinate at a time. So assume the target is $\mathbb{C}$ and let u be as in the statement.

Consider u^d, then $\bar\partial u^d$ is an element of $\mathcal{H}_0^{\mathrm{loc}}(\mathbb{R}\times[0,2],\mathbb{C})$ with support on $\mathbb{R}+i\cup\partial(\mathbb{R}\times[0,2])$. Such a distribution is a three-term linear combination of tensor products of a Dirac-delta in the t-variable and a distribution on $\mathbb{R}$ and hence lies in $\mathcal{H}_0(\mathbb{R}\times[0,2],\mathbb{C})$ only if it is zero. Thus $\bar\partial u = 0$ and we may use elliptic regularity to conclude that u is smooth in the interior of $\mathbb{R}\times[0,2]$. (In fact, doubling again and using the same argument, we find that u is smooth also on the boundary.)

We may now Fourier expand $u^d(\tau,\cdot)$ in the eigenfunctions ϕ of the operator $i\partial_t$ which satisfy the boundary condition $\phi(0) = -\phi(2)$. These eigenfunctions are

$$t \mapsto \exp\left(i(\tfrac{\pi}{2}+n\pi)t\right), \text{ for } n\in\mathbb{Z}.$$

We find

$$u^d = \sum_n c_n(\tau)\exp\left(i(\tfrac{\pi}{2}+n\pi)t\right)$$

where, by the definition of u^d, $c_n(\tau)$ are real valued functions and

$$\bar\partial u^d = \sum_n \left(c_n'(\tau) - (\tfrac{\pi}{2}+n\pi)c_n(\tau)\right)\exp\left(i(\tfrac{\pi}{2}+n\pi)t\right).$$

Hence,

$$u(\zeta) = \sum_n C_n \exp\left((\tfrac{\pi}{2} + n\pi)\zeta\right).$$

Assume that $u \in \mathcal{H}_k$ for some $k \in \mathbb{Z}$. Then, since for $j \geq 0$ the restriction of any $v \in \mathcal{H}_j(\mathbb{R}\times[0,2],\mathbb{C})$ to $\mathbb{R}\times[0,1]$ lies in $\mathcal{H}_j$,

$$\lambda_u(v) = \int_{\mathbb{R}\times[0,2]} \langle v, u^d\rangle\, d\tau \wedge dt,$$

is a continuous linear functional on $\mathcal{H}_j(\mathbb{R}\times[0,2],\mathbb{C})$ for $j = k$ if $k \geq 0$ or $j = -k$ if $k < 0$.

Let $\psi\colon \mathbb{R} \to [0,1]$ be a smooth function equal to 1 on $[0,1]$ and 0 outside $[-1,2]$. For $n, r \in \mathbb{Z}$ let

$$\alpha_{n,r}(\tau + it) = \psi(\tau + r)\exp(i(\tfrac{\pi}{2} + n\pi)t).$$

Then $\alpha_{n,r} \in \mathcal{H}_j(\mathbb{R}\times[0,2],\mathbb{C})$ and $\|\alpha_{n,r}\|_j = K(n)$ for some constant $K(n)$ and all r. It is straightforward to see that

$$\lambda_u(\alpha_{n,r}) = 2C_n \int_{r-1}^{r+2} \psi(\tau + r)\exp((\tfrac{\pi}{2} + n\pi)\tau)\, d\tau = l_{n,r}.$$

The set $\{l_{n,r}\}_{r\in\mathbb{Z}}$ is unbounded unless $C_n = 0$. Hence λ_u is continuous only if each $C_n = 0$. □

5.3. **The right angle model problem.** As mentioned we will use weighted Sobolev spaces. The weight functions are functions on $\mathbb{R}\times[0,1]$ which are independent of t and have the following properties.

For $a = (a^+, a^-) \in \mathbb{R}^2$ and $\theta \in [0, \pi)$, let

$$m(\theta, a) = \min\Big\{|n\pi + \theta + a^+|, |n\pi + \theta + a^-|\Big\}_{n\in\mathbb{Z}}. \tag{5.11}$$

For $a \in \mathbb{R}^2$ with $m(\frac{\pi}{2}, a) > 0$, let $e_a\colon \mathbb{R} \to \mathbb{R}$ be a smooth positive function with the following properties:

- **P1** There exists $M > 0$ such that $e_a(\tau) = e^{a^+\tau}$ for $\tau \geq M$ and $e_a(\tau) = e^{a^-\tau}$ for $\tau \leq -M$.
- **P2** The logarithmic derivative of e_a, $\alpha(\tau) = \frac{e_a'(\tau)}{e_a(\tau)}$, is (weakly) monotone and $\alpha'(\tau) = 0$ if and only if $\alpha(\tau)$ equals the global maximum or minimum of α.
- **P3** The derivative of α satisfies $|\alpha'(\tau)| < \frac{1}{5} m(\frac{\pi}{2}, a)^2$ for all $\tau \in \mathbb{R}$.

Let

$$\mu = (\mu_1, \dots, \mu_n) = (\mu_1^+, \mu_1^-, \dots, \mu_n^+, \mu_n^-) \in \mathbb{R}^{2n},$$

be such that $m(\frac{\pi}{2}, \mu_j) > 0$, for $j = 1, \dots, n$. Define the $(n\times n)$-matrix valued function $\mathbf{e}_\mu$ on $\mathbb{R}$ as

$$\mathbf{e}_\mu(\tau) = \mathrm{Diag}(e_{\mu_1}(\tau), \dots, e_{\mu_n}(\tau)).$$

Define the weighted Sobolev spaces

$$\mathcal{H}_{k,\mu} = \left\{u \in \mathcal{H}_k^{\mathrm{loc}}\colon \mathbf{e}_\mu u \in \mathcal{H}_k\right\}, \text{ with norm } \|u\|_{k,\mu} = \|\mathbf{e}_\mu u\|_k.$$

To make the doubling operation used in Section 5.2 work on $\mathcal{H}_2$, we impose further boundary conditions. If $u \in \mathcal{H}_1^{\mathrm{loc}}$ then its trace lies in $\mathcal{H}_{\frac{1}{2}}^{\mathrm{loc}}(\mathbb{R}\cup\mathbb{R}+i, \mathbb{C}^n)$. We say that *u vanishes on the boundary if*

$$\int_{\mathbb{R}\cup\mathbb{R}+i} \langle u, v\rangle\, d\tau = 0 \text{ for every } v \in \mathbb{C}_0^\infty(\mathbb{R}\cup\mathbb{R}+i, \mathbb{C}^n). \tag{5.12}$$

Define

$$\mathcal{H}_{2,\mu}(\underbrace{\tfrac{\pi}{2},\dots,\tfrac{\pi}{2}}_{n}) = \left\{u\in\mathcal{H}_{2,\mu}\colon u \text{ satisfies (5.9), (5.10), and } \bar\partial u \text{ satisfies (5.12)}\right\},$$
$$\mathcal{H}_{1,\mu}[0] = \{u\in\mathcal{H}_{1,\mu}\colon u \text{ satisfies (5.12)}\}.$$

Proposition 5.3. *If $m(\frac{\pi}{2},\mu_j)>0$ for $j=1,\dots,n$ then the operator*

$$\bar\partial\colon \mathcal{H}_{2,\mu}(\tfrac{\pi}{2},\dots,\tfrac{\pi}{2})\to\mathcal{H}_{1,\mu}[0]$$

is Fredholm with index

$$\sum_{j=1}^{n}\sharp\left(-\frac{\mu_j^-}{\pi}-\frac12,-\frac{\mu_j^+}{\pi}-\frac12\right)-\sharp\left(\frac{\mu_j^-}{\pi}-\frac12,\frac{\mu_j^+}{\pi}-\frac12\right)$$

where $\sharp(a,b)$ denotes the number of integers in the interval (a,b).

Moreover, if $\mu_j^+=\mu_j^-$ for all j and $M(\mu)=\min\{m(\frac{\pi}{2},\mu_1),\dots,m(\frac{\pi}{2},\mu_n)\}$ then $u\in\mathcal{H}_{2,\mu}(\frac{\pi}{2},\dots,\frac{\pi}{2})$ satisfies

$$\|u\|_{2,\mu}\le C(\mu)\|\bar\partial u\|_{1,\mu}, \tag{5.13}$$

where $C(\mu)\le\frac{K}{M(\mu)}$, for some constant K.

Proof. The problem studied is split and it is clearly sufficient to consider the case $n=1$. We first determine the dimensions of the kernel and cokernel. It is immediate from Lemma 5.2 that the kernel of $\bar\partial$ is finite dimensional on $\mathcal{H}_{2,\mu}(\frac{\pi}{2})$ and that the number of linearly independent solutions is exactly $\sharp\left(-\frac{\mu^-}{\pi}-\frac12,-\frac{\mu^+}{\pi}-\frac12\right)$.

Recall that an element in the cokernel of $\bar\partial$ is an element ξ in the dual space of $\mathcal{H}_{1,\mu}[0]$. The dual of $\mathcal{H}_{1,\mu}$ is $\mathcal{H}_{-1,-\mu}$ and thus, as in (5.1), the dual of $\mathcal{H}_{1,\mu}[0]$ is the quotient space

$$\mathcal{H}_{-1,-\mu}/\mathcal{H}_{1,\mu}[0]^{\perp}.$$

Lemma 5.1 implies that any element in the cokernel has a smooth representative. Let v be a smooth representative. Then

$$\int_{\mathbb{R}\times[0,1]}\langle\bar\partial u,v\rangle\,d\tau\wedge dt=0,$$

for any smooth compactly supported function u which meets the boundary conditions (5.9), (5.10), and (5.12). Using partial integration we conclude

$$\int_{\mathbb{R}\times[0,1]}\langle u,\partial v\rangle\,d\tau\wedge dt=0. \tag{5.14}$$

Thus $\partial v=0$ in the interior. Noting that for any two functions $\phi_0\in C_0^\infty(\mathbb{R},\mathbb{R})$ and $\phi_1\in C_0^\infty(\mathbb{R},i\mathbb{R})$ there exists a function $u\in\mathbb{C}_0^\infty(\mathbb{R}\times[0,1],\mathbb{C})$ such that $\bar\partial u|\partial(\mathbb{R}\times[0,1])=0$, $u|\mathbb{R}=\phi_0$, and $u|\mathbb{R}+i=\phi_1$ we find that iv satisfies (5.9) and (5.10). Lemma 5.2 then implies that the cokernel has dimension $\sharp\left(\frac{\mu_j^-}{\pi}-\frac12,\frac{\mu_j^+}{\pi}-\frac12\right)$.

We now prove that the image of $\bar\partial$ is closed, and in doing so also establish (5.13). Let

$$A(\tau)=\exp\left(\int_0^\tau\alpha(\sigma)\,d\sigma\right).$$

Then multiplication with A defines a Banach space isomorphism $A\colon\mathcal{H}_{k,\mu}\to\mathcal{H}_k$. The inverse A^{-1} of A is multiplication with $A(\tau)^{-1}$. These isomorphisms gives the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}_{2,\mu}(\frac{\pi}{2}) & \xleftarrow{A^{-1}} & \mathcal{H}_2(\frac{\pi}{2}^*) \\ \bar\partial \downarrow & & \downarrow \bar\partial-\alpha \\ \mathcal{H}_{1,\mu}[0] & \xrightarrow[A]{} & \mathcal{H}_1[0], \end{array}$$

where $\mathcal{H}_2(\frac{\pi}{2}^*)$ is defined as $\mathcal{H}_2(\frac{\pi}{2})$ except that instead of requiring that $\bar\partial u$ vanishes on the boundary we require that $(\bar\partial-\alpha)u$ does. We prove that the operator $\bar\partial-\alpha$ on the right in the above diagram has closed range and conclude the corresponding statement for the operator on the left. Note that if $u\in\mathcal{H}_2(\frac{\pi}{2}^*)$ then both $\partial_\tau u$ and $\partial_t u$ satisfy (5.9) and (5.10). Hence the doubling operation described in Section 5.2 induces a map $\mathcal{H}_2(\frac{\pi}{2}^*)\to\mathcal{H}_2(\mathbb{R}\times[0,2])$ with $\|u^d\|_2=2\|u\|_2$.

Let

$$S(\mu)=\{n\in\mathbb{Z}\colon -\tfrac{\mu^-}{\pi}-\tfrac12<n<-\tfrac{\mu^+}{\pi}-\tfrac12\}. \tag{5.15}$$

(Note that $S(\mu)=\emptyset$ if $\mu^+\geq\mu^-$.) The map $\gamma_n\colon\mathcal{H}_2(\frac{\pi}{2}^*)\to\mathcal{H}_2(\mathbb{R},\mathbb{R})$,

$$u\mapsto c_n(\tau)=\int_0^2 u^d(\tau,t)\exp\left(-i(\tfrac{\pi}{2}+n\pi)t\right)\,dt \tag{5.16}$$

is continuous. Let $W_2\subset\mathcal{H}_2(\frac{\pi}{2}^*)$ be the closed subspace

$$W_2=\bigcap_{n\in S(\mu)}\ker(\gamma_n). \tag{5.17}$$

Using the Fourier expansion of u^d we see that W_2 has a direct complement

$$V_2=\bigcap_{n\notin S(\mu)}\ker(\gamma_n). \tag{5.18}$$

(Note that if $\mu^+>\mu^-$ then $W_2=\mathcal{H}_2(\frac{\pi}{2}^*)$ and $V_2=\emptyset$.)

Similarly, we view the maps γ_n defined by (5.16) as maps $\mathcal{H}_1[0]\to\mathcal{H}_1(\mathbb{R},\mathbb{R})$ and get the corresponding direct sum decomposition $\mathcal{H}_1[0]=W_1\oplus V_1$. If $u\in\mathcal{H}_2(\frac{\pi}{2}^*)$ then the Fourier expansion of u^d is

$$u^d(\tau+it)=\sum_n c_n(\tau)e^{i(\frac{\pi}{2}+n\pi)t}.$$

Hence

$$(\bar\partial-\alpha)u^d(\tau+it)=\sum_n(c_n'(\tau)-(\alpha(\tau)+\tfrac{\pi}{2}+n))e^{i(\frac{\pi}{2}+n\pi)t}. \tag{5.19}$$

It follows that $\bar\partial(W_2)\subset W_1$ and $\bar\partial(V_2)\subset V_1$.

Let $w\in W_2$. Fourier expansion of w^d gives

$$\begin{aligned} &2\|(\bar\partial-\alpha)w\|_0^2=\\ &=\sum_{n\notin S(\mu)}\int_{\mathbb{R}}\left(|c_n'|^2+\left((\tfrac{\pi}{2}+n\pi+\alpha(\tau))^2+\alpha'\right)|c_n|^2\right)\,d\tau\\ &\geq 2C\|w\|_1^2, \end{aligned} \tag{5.20}$$

where the constant C is obtained as follows. If $\mu^+>\mu^-$ then **P2** implies that the coefficients of $|c_n|^2$ are strictly positive, and if $\mu^+<\mu^-$ then **P3** implies that the coefficients in front of $|c_n|^2$ are larger than $\frac45 m(\frac{\pi}{2},\mu)^2$ since $n\notin S(\mu)$. Finally, if $\mu^-=\mu^+$ then $\alpha'=0$ and the coefficients in front of $|c_n|^2$ are larger than $m(\frac{\pi}{2},\mu)^2$ for all n.

If $w \in \mathcal{H}_2(\frac{\pi}{2}^*)$, then $\partial_\tau w$ and $i\partial_t w$ satisfies (5.9) and (5.10) and the Fourier coefficients $c_n(\tau)$ of their doubles vanish for $n \in S(\mu)$. Thus, the same argument applies to these functions and the following estimates are obtained

$$\|(\bar\partial - \alpha)\partial_\tau w\|_0 \geq C\|\partial_\tau w\|_1,$$
$$\|(\bar\partial - \alpha)\partial_t w\|_0 \geq C\|\partial_t w\|_1.$$

If $\mu_+ = \mu_-$ then $\alpha' = 0$ and $\bar\partial - \alpha$ commutes with both ∂_t and ∂_τ. Hence

$$\begin{aligned}\|(\bar\partial - \alpha)w\|_1 &\geq \frac{1}{2}(\|(\bar\partial - \alpha)w\|_0 + \|\partial_\tau(\bar\partial - \alpha)w\|_0 + \|\partial_t(\bar\partial - \alpha)w\|_0)\\ &\geq C(\|w\|_1 + \|\partial_\tau w\|_1 + \|\partial_t w\|_1) \geq C\|w\|_2,\end{aligned}$$

where $C = Km(\mu, \frac{\pi}{2})$. This proves (5.13).

If $\mu_+ \neq \mu_-$ then $\partial_\tau(\bar\partial - \alpha)w = (\bar\partial - \alpha)\partial_\tau w - \alpha' w$, and with $K > 0$ we conclude from the triangle inequality

$$\begin{aligned}&K\|(\bar\partial - \alpha)w\|_0 + \|\partial_\tau(\bar\partial - \alpha)w\|_0 + \|\partial_t(\bar\partial - \alpha)w\|_0\\ &\geq KC\|w\|_1 + C\|\partial_\tau w\|_1 - \|\alpha' w\|_0 + C\|\partial_t w\|_1\\ &\geq \left(KC - \frac{m(\frac{\pi}{2},\mu)^2}{5}\right)\|w\|_1 + C\|\partial_\tau w\|_1 + C\|\partial_t w\|_1,\end{aligned}$$

since $|\alpha'| < \frac{m(\frac{\pi}{2},\mu)^2}{5}$. Thus choosing K sufficiently large we find that there exist a constant K_1 such that for $w \in W$

$$\|w\|_2 \leq K_1\|(\bar\partial - \alpha)w\|_1. \tag{5.21}$$

Thus, if $\mu^+ > \mu^-$ we conclude that the range of $\bar\partial - \alpha$ is closed. If $\mu^+ < \mu^-$ we need to consider also V_2.

For $v \in V_2$ we have

$$v^d(\tau, t) = \sum_{n \in S(\mu)} c_n(\tau)\exp\left(i(\tfrac{\pi}{2} + n\pi)t\right).$$

Let $V_2^\perp$ be the space of functions in V_2 which, under doubling, map to the orthogonal complement of the doubles ϕ_n^d of the functions $\phi_n(\zeta) = \exp((\frac{\pi}{2} + n\pi)\zeta + \int \alpha\, d\tau)$, $n \in S(\mu)$ with respect to the L^2-pairing on $\mathcal{H}_2(\mathbb{R} \times [0,2], \mathbb{C})$. Then $V_2^\perp$ is a closed subspace of finite codimension in V_2.

We claim there exists a constant K_2 such that for all $v^\perp \in V_2^\perp$

$$\|v^\perp\|_2 \leq K_2\|(\bar\partial - \alpha)v^\perp\|_1. \tag{5.22}$$

Assume that this is not the case. Then there exists a sequence $v_j^\perp$ of elements in $V_2^\perp$ such that

$$\|v_j^\perp\|_2 = 1, \tag{5.23}$$
$$\|(\bar\partial - \alpha)v_j^\perp\|_1 \to 0. \tag{5.24}$$

Let $P > M$ be an integer (see condition **P1**) and let $v^\perp \in V^\perp$. Consider the restriction of $v^\perp$ and $\bar\partial v^\perp$ to $\Theta_P = \{\tau + it\colon |\tau| \geq P\}$. Using Fourier expansion as in (5.20), partial integration, and the fact that $\alpha'(\tau) = 0$ for $|\tau| > M$ we find

$$\begin{aligned}&2\|(\bar\partial - \alpha)v^\perp|\Theta_P\|_1 \geq\\ &C\left(\|v^\perp|\Theta_P\|_2 + \sum_{n \in S(\mu)} \mu^+(|c_n(P)|^2 + |c_n'(P)|^2) - \mu^-(|c_n(-P)|^2 + |c_n'(-P)|^2)\right).\end{aligned} \tag{5.25}$$

By a compact Sobolev embedding we find for each positive integer P a subsequence $\{v^{\perp}_{j(P)}\}$ which converges in $\mathcal{H}_1([-P,P]\times[0,1],\mathbb{C})$. Moreover, we may assume that these subsequences satisfies $\{v^{\perp}_{j(P)}\}\supset\{v^{\perp}_{j(Q)}\}$ if $P<Q$.

Let $(c_n)_j$ be the sequence of Fourier coefficient functions associated to the sequence $v^{\perp}_j$. The estimates

$$\|c\|_k \le C(\|c\|_{k-1} + \|(\frac{d}{d\tau} - (\tfrac{\pi}{2} + n\pi + \alpha))c\|_{k-1}), \tag{5.26}$$

and (5.24) implies that $(c_n)_{j(P)}$ converges to a smooth solution of the equation $(\frac{d}{d\tau} - (\frac{\pi}{2} + n\pi + \alpha))c = 0$ on $[-P,P]$. Hence, $v^{\perp}_{j(P)}$ converges to a smooth solution of $(\bar\partial - \alpha)u = 0$ on Θ_P satisfying the boundary conditions (5.9) and (5.10). Such a solution has the form

$$\sum_{n\in S(\mu)} k_n\phi_n(\zeta),$$

where k_n are real constants.

We next show that in fact all k_n must be zero. Note that by Morrey's theorem and (5.23) we get a uniform C^0-bound $|v^{\perp}_j|\le K$. Therefore, $|(c_n)_j|\le 2K$ and hence

$$\begin{aligned}&\int\langle (v^{\perp}_j)^d, (\phi_n)^d\rangle\, d\tau\wedge dt = \\ &\int_{\mathbb{R}} (c_n)_j \exp\left((\tfrac{\pi}{2}+n\pi)\tau + \int \alpha\, d\tau\right) d\tau = \int_{-P}^{P} (c_n)_j \exp\left((\tfrac{\pi}{2}+n\pi)\tau + \int \alpha\, d\tau\right) d\tau \\ &+ \int_P^{\infty} (c_n)_j \exp((\tfrac{\pi}{2}+n\pi+\mu^+)\tau)\, d\tau + \int_{-\infty}^{-P} (c_n)_j \exp((\tfrac{\pi}{2}+n\pi+\mu^+)\tau)\, d\tau.\end{aligned}$$

But

$$\begin{aligned}&\left|\int_P^{\infty} (c_n)_j \exp((\tfrac{\pi}{2}+n\pi+\mu^+)\tau)\, d\tau\right| + \left|\int_{-\infty}^{-P} (c_n)_j \exp((\tfrac{\pi}{2}+n\pi+\mu^-)\tau)\, d\tau\right| \le \\ &\frac{2K}{m(\frac{\pi}{2},\mu)}\big(\exp((\tfrac{\pi}{2}+n\pi+\mu^+)P) + \exp(-(\tfrac{\pi}{2}+n\pi+\mu^-)P)\big) \to 0 \text{ as } P\to\infty.\end{aligned}$$

We conclude from this that unless $k_n = 0$, $v^{\perp}_{j(P)}$ violates the orthogonality conditions for P and $j(P)$ sufficiently large.

Consider (5.25) applied to elements in the sequence $\{v^{\perp}_j\}$. As $j\to\infty$ the term on the left hand side and the sum in the right hand side tends to 0. Hence $\|v^{\perp}_j|\Theta_P\|_2\to 0$. Applying (5.26) to $(c_n)_j$ and noting that both terms on the right hand side goes to 0 we conclude that also $\|v^{\perp}_j|[-P\times P]\times[0,1]\|_2\to 0$. This contradicts (5.23) and hence (5.22) holds.

The estimates (5.21) and (5.22) together with the direct sum decompositions $\mathcal{H}_2(\frac{\pi}{2}^*) = W_2\oplus V_2$ and $\mathcal{H}_1[0] = W_1\oplus V_1$, and the fact that $\bar\partial - \alpha$ respects this decomposition shows that the image of $\bar\partial-\alpha$ is closed also in the case $\mu^+<\mu^-$. □

Remark 5.4. In many cases, the first statement in Proposition 5.3 still holds with weaker assumptions on the weight function than **P1–P3**. For example, if $\mu_+<\mu_-$ then we need only know that $\max\{\alpha',0\}$ is sufficiently small compared to $(\frac{\pi}{2}+n\pi+\alpha)^2$ for $n\notin S(\mu)$ to derive (5.21) and the derivation of (5.22) is quite independent of α' as long as α eventually becomes constant.

5.4. **The model problem with angles.** We study more general boundary conditions than those in Section 5.3. Recall $(x_1+iy_1,\dots,x_n+iy_n)$ are coordinates on $\mathbb{C}^n$. Let ∂_j denote the unit tangent vector in the x_j-direction, for $j=1,\dots,n$. For $\theta=(\theta_1,\dots,\theta_n)\in[0,\pi)^n$, let

$\Lambda(\theta)$ be the Lagrangian subspace of $\mathbb{C}^n$ spanned by the vectors $e^{i\theta_1}\partial_1, \dots, e^{i\theta_n}\partial_n$. Consider the following boundary conditions for $u \in \mathcal{H}_1^{\mathrm{loc}}$.

$$\int_{\mathbb{R}} \langle u, v\rangle \, d\tau = 0 \text{ for all } v \in C_0^\infty(\mathbb{R}, i\mathbb{R}^n), \tag{5.27}$$

$$\int_{\mathbb{R}+i} \langle u, v\rangle \, d\tau = 0 \text{ for all } v \in C_0^\infty(\mathbb{R}+i, i\Lambda(\theta)). \tag{5.28}$$

If $m(\theta_j, \mu_j) > 0$ (see (5.11)) for all j then define

$$\mathcal{H}_{2,\mu}(\theta) = \left\{u \in \mathcal{H}_{2,\mu} \colon u \text{ satisfies (5.27), (5.28), and } \bar\partial u \text{ satisfies (5.12)}\right\},$$
$$\mathcal{H}_{1,\mu}[0] = \{u \in \mathcal{H}_{1,\mu} \colon u \text{ satisfies (5.12)}\}.$$

Proposition 5.5. *If $m(\theta_j, \mu_j) > 0$ for $j = 1, \dots, n$ then the operator*

$$\bar\partial \colon \mathcal{H}_{2,\mu}(\theta) \to \mathcal{H}_{1,\mu}[0]$$

is Fredholm of index

$$\sum_{j=1}^{n} \sharp\left(-\tfrac{\mu_j^- + \theta_j}{\pi}, -\tfrac{\mu_j^+ + \theta_j}{\pi}\right) - \sharp\left(\tfrac{\mu_j^- + \theta_j}{\pi} - 1, \tfrac{\mu_j^+ + \theta_j}{\pi} - 1\right). \tag{5.29}$$

Moreover, if $\mu_j^+ = \mu_j^-$ for all j and $M(\mu) = \min\{m(\mu_1, \theta_1), \dots, m(\mu_n, \theta_n)\}$ then $u \in \mathcal{H}_{2,\mu}(\theta_1, \dots, \theta_n)$ satisfies

$$\|u\|_{2,\mu} \le C(\mu)\|\bar\partial u\|_{1,\mu}, \tag{5.30}$$

where $C(\mu) \le \frac{K}{M(\mu)}$, for some constant K.

Proof. Consider the holomorphic $(n \times n)$-matrix

$$\mathbf{g}_\theta(\zeta) = \mathrm{Diag}\left((\exp(\tfrac{\pi}{2} - \theta_1)\zeta), \dots, (\exp(\tfrac{\pi}{2} - \theta_n)\zeta)\right).$$

Multiplication with $\mathbf{g}_\theta$ defines isomorphisms

$$\mathcal{H}_{2,\mu}(\theta) \to \mathcal{H}_{2,\lambda}(\tfrac{\pi}{2}, \dots, \tfrac{\pi}{2}) \quad \text{and}$$
$$\mathcal{H}_{1,\mu}[0] \to \mathcal{H}_{1,\lambda}[0]$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\lambda_j^\pm = \mu_j^\pm - \frac{\pi}{2} + \theta_j$. Since $\mathbf{g}_\theta$ is holomorphic it commutes with $\bar\partial$. The proposition now follows from Proposition 5.3. □

5.5. **Smooth perturbations of the model problem with angles.** Let $B \colon \mathbb{R} \times [0,1] \to \mathbf{U}(n)$ be a smooth map such that

$$\bar\partial B | \partial\mathbb{R} \times [0,1] = 0. \tag{5.31}$$

Let $\theta \in [0, \pi)$ and consider the following boundary conditions for $u \in \mathcal{H}_1^{\mathrm{loc}}$:

$$\int_{\mathbb{R}} \langle u, v\rangle \, d\tau \wedge dt = 0, \quad \text{for all } v \in C_0^\infty(\mathbb{R}, \mathbb{C}^n) \text{ such that } v(\tau) \in iB(\tau)\mathbb{R}^n, \tag{5.32}$$

$$\int_{\mathbb{R}+i} \langle u, v\rangle \, d\tau \wedge dt = 0, \quad \text{for all } v \in C_0^\infty(\mathbb{R}+i, \mathbb{C}^n) \text{ such that } v(\tau + i) \in iB(\tau)\Lambda(\theta). \tag{5.33}$$

For $\mu = (\mu^+, \mu^-) \in \mathbb{R}^2$ let $\lambda(\mu) = (\mu^+, \mu^-, \mu^+, \mu^-, \dots, \mu^+, \mu^-) \in \mathbb{R}^{2n}$ define

$$\mathcal{H}_{2,\mu}(\theta, B) = \{u \in \mathcal{H}_2^{\mathrm{loc}} \colon u \text{ satisfies (5.32) and (5.33)}, \bar\partial u \text{ satisfies (5.12) and } \mathbf{e}_{\lambda(\mu)} u \in \mathcal{H}_2\}.$$

Proposition 5.6. *If $m(\theta_j,\mu)>0$ for $j=1,\dots,n$ then there exists $\delta>0$ such that for all B satisfying* (5.31) *with $\|B-\mathrm{id}\|_{C^2}<\delta$, the operator*

$$\bar\partial\colon \mathcal{H}_{2,\mu}(\theta,B)\to\mathcal{H}_{1,\mu}[0]$$

is Fredholm of index

$$\sum_{j=1}^{n}\sharp\left(-\tfrac{\mu^-+\theta_j}{\pi},-\tfrac{\mu^++\theta_j}{\pi}\right)-\sharp\left(\tfrac{\mu^-+\theta_j}{\pi}-1,\tfrac{\mu^++\theta_j}{\pi}-1\right). \tag{5.34}$$

Proof. Multiplication with B and B^{-1} defines Banach space isomorphisms

$$\mathcal{H}_{2,\mu}(\theta)\xrightarrow{\times B}\mathcal{H}_{2,\mu}(\theta,B),$$

and

$$\mathcal{H}_{1,\mu}[0]\xrightarrow{\times B^{-1}}\mathcal{H}_{1,\mu}[0].$$

Thus up to conjugation the operator considered is the same as

$$\bar\partial+B^{-1}\bar\partial B\colon\mathcal{H}_{2,\mu}(\theta)\to\mathcal{H}_{1,\mu}[0].$$

The theorem now follows from Proposition 5.5, and the fact that the subspace of Fredholm operators is open and that the index is constant on path components of this subspace. □

5.6. **Boundary conditions.** In the upcoming subsections we study the linearized $\bar\partial$-problem on a disk D_m with m punctures. Refer back to Section 3.4 for notation concerning D_m.

Definition 5.7. A smooth map $A\colon\partial D_m\to\mathbf{U}(n)$ will be called *small at infinity* if there exists $M>1$ such that for each $j=1,\dots,m$ the restriction of A to $\partial E_{p_j}[M]$ approaches a constant map in the C^2-norm on each component of $\partial E_{p_j}[M']$ as $M'\to\infty$. It will be called *constant at infinity* if there exists $M>1$ such that for each $j=1,\dots,m$ the restriction of A to each component component of $\partial E_{p_j}[M]$ is constant.

Let $A\colon\partial D_m\to\mathbf{U}(n)$ be small at infinity. For $u\in\mathcal{H}_1^{\mathrm{loc}}(D_m,\mathbb{C}^n)$, consider the boundary condition:

$$\int_{\partial D_m}\langle u,v\rangle\,ds=0,$$
$$\text{for all } v\in C_0^0(\partial D_m,\mathbb{C}^n)\text{ such that } v(\zeta)\in iA(\zeta)\mathbb{R}^n \text{ for all }\zeta\in\partial D_m. \tag{5.35}$$

In previous subsections coordinates $\zeta=\tau+it$ on $\mathbb{R}\times[0,1]$ were used and we implicitly considered the bundle $T^{*0,1}\mathbb{R}\times[0,1]$ as trivialized by the form $d\bar\zeta$, and sections in this bundle as $\mathbb{C}^n$-valued functions. We do not want to specify any trivialization of $T^{*0,1}D_m$ and so we view the $\bar\partial$-operator as a map from $\mathcal{H}_2$-functions into $\mathcal{H}_1$-sections of $T^{*0,1}D_m\otimes\mathbb{C}^n$. Consider, for $u\in\mathcal{H}_1^{\mathrm{loc}}(D_m,T^{*0,1}D_m\otimes\mathbb{C}^n)$, the boundary condition

$$\int_{\partial D_m}\langle u,v\rangle\,ds=0,\ \text{ for all } v\in C_0^0(\partial D_m,T^{0,1}D_m\otimes\mathbb{C}^n). \tag{5.36}$$

Henceforth, to simplify notation, if the source space X in a Sobolev space $\mathcal{H}_k(X,Y)$ is D_m we will drop it from the notation. If $u\in\mathcal{H}_2^{\mathrm{loc}}(\mathbb{C}^n)$ then $\bar\partial u\in\mathcal{H}_1^{\mathrm{loc}}(T^{*0,1}D_m\otimes\mathbb{C}^n)$. Define

$$\mathcal{H}_2(\mathbb{C}^n;A)=\left\{u\in\mathcal{H}_2(\mathbb{C}^n)\colon u\text{ satisfies (5.35) and }\bar\partial u\text{ satisfies (5.36)}\right\},$$

and

$$\mathcal{H}_1(T^{*0,1}D_m\otimes\mathbb{C}^n;[0])=\left\{u\in\mathcal{H}_1(T^{*0,1}D_m\otimes\mathbb{C}^n)\colon u\text{ satisfies (5.36)}\right\},$$

Define

$$\mathcal{H}_{2,\mu}(\mathbb{C}^n; A) = \\ \left\{u \in \mathcal{H}_2^{\mathrm{loc}}(\mathbb{C}^n) \colon u \text{ satisfies (5.35)}, \bar{\partial} u \text{ satisfies (5.36), and } \mathbf{e}_\mu u \in \mathcal{H}_2(\mathbb{C}^n)\right\}.$$

and

$$\mathcal{H}_{1,\mu}(T^{*0,1}D_m \otimes \mathbb{C}^n; [0]) = \\ \left\{u \in \mathcal{H}_1^{\mathrm{loc}}(T^{*0,1}D_m \otimes \mathbb{C}^n) \colon u \text{ satisfies (5.36) and } \mathbf{e}_\mu u \in \mathcal{H}_1(T^{*0,1}D_m \otimes \mathbb{C}^n)\right\}.$$

Let p_j be a puncture of D_m. The orientation of D_m induces an orientation of ∂D_m. Let A_j^0 and A_j^1 denote the constant maps to which A converges on the component of ∂E_{p_j} close to p_j corresponding to $\mathbb{R}$ and $\mathbb{R} + i$, respectively. Define

$$\theta(j) = \theta(A_j^0 \mathbb{R}^n, A_j^1 \mathbb{R}^n).$$

Then there are unique unitary complex coordinates

$$z(j) = (x(j)_1 + iy(j)_1, \dots, x(j)_n + iy(j)_n)$$

in $\mathbb{C}^n$ such that

$$A_j^0 \mathbb{R}^n = \mathrm{Span}\left\langle \partial(j)_1, \dots, \partial(j)_n \right\rangle, \\ A_j^1 \mathbb{R}^n = \mathrm{Span}\left\langle e^{i\theta(j)_1}\partial(j)_1, \dots, e^{i\theta(j)_n}\partial(j)_n \right\rangle.$$

Proposition 5.8. *Let $A \colon \partial D_m \to \mathbf{U}(n)$ be small at infinity. If μ satisfies $\mu_j \neq -\theta(j)_r + k\pi$ for $j = 1, \dots, m$, $r = 1, \dots, n$, and every $k \in \mathbb{Z}$, then the operator*

$$\bar{\partial} \colon \mathcal{H}_{2,\mu}(\mathbb{C}^n; A) \to \mathcal{H}_{1,\mu}(T^{*0,1}D_m \otimes \mathbb{C}^n; [0]) \tag{5.37}$$

is Fredholm.

Proof. Assume that for $M > 0$, $A|\partial E_{p_j}[M-1]$ is sufficiently close to a constant map (see Proposition 5.6). Choose smooth complex-valued functions $\alpha_0, \alpha_1, \dots, \alpha_m$ with the following properties: α_j is constantly 1 on $E_{p_j}[M+2]$; the sum $\sum_j \alpha_j$ is close to the constant function 1, $\bar{\partial}\alpha_j = 0$ on ∂D_m; and α_j is constantly equal to 0 on $D_m - E_{p_j}[M+1]$, for $j = 1, \dots, m$.

Glue to each $E_{p_j}[M]$ a half-infinite strip $(-\infty, M] \times [0,1]$ and denote the result $\bar{E}_{p_j}$. Extend the boundary conditions from $E_{p_j}[M]$ to $\bar{E}_{p_j}$ keeping them close to constant. Let the weight in the weight function remain constant. Glue to $D_m - \bigcup_j E_{p_j}[M+2]$, m half disks and extend the boundary conditions smoothly. Denote the result $\bar{D}_m$. Note that the boundary value problem on $\bar{D}_m$ is the vector-Riemann-Hilbert problem, which is known to be Fredholm, and that the weighted norm on this compact disk is equivalent to the standard norm.

Now let $u \in \mathcal{H}_{2,\mu}(\mathbb{C}^n; A)$. Then $\alpha_j u$ is in the appropriate Sobolev space for the extended boundary value problem on $\bar{E}_{p_j}$ ($\bar{D}_m$ if $j = 0$) and because the elliptic estimate holds for all of these problems and since all of them except possibly the one on $\bar{D}_m$ has no kernel, there exists a constant C such that

$$\begin{aligned}\|u\|_{2,\mu} \leq & \|\alpha_0 u\|_{2,\mu} + \sum_{j=1}^{n} \|\alpha_j u\|_{2,\mu} \\ \leq & C \left(\|\alpha_0 u\|_{1,\mu} + \sum_{j=0}^{n} \|\bar\partial(\alpha_j u)\|_{1,\mu} \right) \\ \leq & C \left(\sum_{j=0}^{n} \|\bar\partial\alpha_j u\|_{1,\mu} + \|\alpha_0 u\|_{1,\mu} + \sum_{j=0}^{n} \|\alpha_j \bar\partial u\|_{1,\mu} \right). \end{aligned} \tag{5.38}$$

We shall show that (5.38) implies that every bounded sequence u_r such that $\bar\partial u_r$ converges has a convergent subsequence. This implies that $\bar\partial$ has a closed image and a finite dimensional kernel ([20] Proposition 19.1.3). Clearly it is sufficient to consider the case $\bar\partial u_r \to 0$. Consider the restrictions of u_r to a compact subset K of D_m such that

$$\operatorname{supp}(\alpha_0) \cup \operatorname{supp}(\bar\partial\alpha_0) \cup \dots \cup \operatorname{supp}(\bar\partial\alpha_m) \subset K.$$

A compact Sobolev embedding argument gives a subsequence $\{u_{r'}\}$ which converges in $\mathcal{H}_1(K, \mathbb{C}^n)$. Thus, (5.38) implies that $\{u_{r'}\}$ is a Cauchy sequence in $\mathcal{H}_{2,\mu}(A; \mathbb{C}^n)$ and hence it converges.

It remains to prove that the cokernel is finite dimensional. Lemma 5.1 shows that any element in the cokernel of $\bar\partial$ can be represented by a smooth function v on D_m. Partial integration implies this function satisfies $\partial v = 0$ with boundary conditions given by the matrix function iA. Assume first that A is constant at infinity. Then, Lemma 5.2 and conjugation with the holomorphic $(n \times n)$-matrix $\mathbf{g}_\theta$ as in the proof of Proposition 5.5 gives explicit formulas for the restrictions of these smooth functions to $E_{p_j}[M]$, for each j. It is straightforward to check from these local formulas that v lies in $\mathcal{H}_{2,-\mu}(\mathbb{C}^n, iA)$. Thus, repeating the argument above with ∂ replacing $\bar\partial$ shows that the cokernel is finite dimensional. The lemma follows in the case when A is constant at infinity. The general case then follows by an approximation argument as in the proof of Proposition 5.6. □

5.7. **Index-preserving deformations.** We compute the index of the operator in (5.37). Using approximations it is easy to see that it is sufficient to consider the case when $A\colon \partial D_m \to \mathbf{U}(n)$ is constant at infinity. Thus, let A be such a map which is constant on $\partial E_{p_j}[M]$ for every j and consider the Fredholm operator

$$\bar\partial\colon \mathcal{H}_{2,\mu}(\mathbb{C}^n; A) \to \mathcal{H}_{1,\mu}(T^{*0,1}D_m \otimes \mathbb{C}^n; [0]), \tag{5.39}$$

where $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$ satisfies

$$\mu_j \neq -\theta(j)_r + n\pi \text{ for every } j, r, n. \tag{5.40}$$

Lemma 5.9. *Let $B_s\colon D_m \to \mathbf{U}(n)$, $s \in [0, 1]$, be a continuous family of smooth maps such that*

$$\begin{aligned} & B_s \textit{ is bounded in the } C^2\textit{-norm,} \\ & B_s|\partial E_{p_j}[M] \textit{ is constant in } \tau + it, \\ & \bar\partial B_s|\partial D_m = 0, \textit{ and} \\ & B_0 \equiv \mathrm{id}. \end{aligned} \tag{5.41}$$

Let $\lambda\colon [0, 1] \to \mathbb{R}^m$ be a continuous map such that $\lambda(0) = \mu$ and $\lambda(s)$ satisfies (5.40) *for every $s \in [0, 1]$. Then the operator*

$$\bar\partial\colon \mathcal{H}_{2,\lambda(1)}(\mathbb{C}^n; B_1 A) \to \mathcal{H}_{1,\lambda(1)}(T^{*0,1}D_m \otimes \mathbb{C}^n; [0])$$

has the same Fredholm index as the operator in (5.39).

Proof. The Fredholm operator

$$\bar{\partial}\colon \mathcal{H}_{2,\lambda(s)}(\mathbb{C}^n; B_sA) \to \mathcal{H}_{1,\lambda(s)}(T^{*0,1}D_m \otimes \mathbb{C}^n; [0])$$

is conjugate to

$$\bar{\partial} - B_s\bar{\partial}B_s^{-1}\colon \mathcal{H}_{2,\mu}(\mathbb{C}^n; A) \to \mathcal{H}_{1,\mu}(T^{*0,1}D_m \otimes \mathbb{C}^n; [0]).$$

The family $\bar{\partial} - B_s\bar{\partial}B_s^{-1}$ is then a continuous family of Fredholm operators. □

In order to apply Lemma 5.9 we shall show how to deform given weights and boundary conditions into other boundary conditions and weights keeping the Fredholm index constant using the conditions in Lemma 5.9. We accomplish this in two steps: first deform the problem so that the boundary value matrix is diagonal; then change the weights and angles at the ends into a special form where compactification is possible.

Lemma 5.10. *Let $A\colon \partial D_m \to \mathbf{U}(n)$ be constant at infinity. Then there exists a continuous family $B_s\colon D_m \to \mathbf{U}(n)$, $0 \le s \le 1$, of maps satisfying* (5.41) *such that*

$$B_1(\zeta)A(\zeta) = \mathrm{Diag}(b_1(\zeta), \dots, b_n(\zeta)), \zeta \in \partial D_m.$$

Proof. We first make A diagonal on the ends where it is constant. Note that in canonical coordinates $z(j)$ on the end $E_{p_j}[M]$ the matrix A is diagonal. Let $B_j \in \mathbf{U}(n)$ be the matrix which transforms the complex basis $\partial(j)_1, \dots, \partial(j)_n$ to the standard basis. Let $B_j(s)$ be a smooth path in $\mathbf{U}(n)$, starting at id and ending at B_j. Define $B_s = B_j(s)$ on $E_{p_j}[M]$ for each j.

We need to extend this map to all of D_m. To this end consider the loop on the boundary of $S = D_m - E_{p_j}[M]$. There exists a 1-parameter family of functions $B_s\colon S \to \mathbf{U}(n)$ such that $B_0 = \mathrm{id}$ and B_1A is diagonal, since any loop is homotopic to a loop of diagonal matrices. The loops B_s can be smoothly extended to all of D_m

Finally, we need that $\bar{\partial}B_s = 0$ on the boundary. We get this as follows: let C be a collar on the boundary with coordinates τ along the boundary and t orthogonal to the boundary, $0 \le t \le \epsilon$ and let $\phi\colon [0,\epsilon] \to \mathbb{R}$ be a smooth function which equals the identity on $[0, \frac{\epsilon}{4}]$ and 0 for $t \ge \frac{\epsilon}{2}$. Redefine B_s on the collar as

$$\tilde{B}_s = B_s(\zeta)\exp(i\phi(t)B_s^{-1}(\zeta)\bar{\partial}B_s(\zeta)).$$

Then $\tilde{B}_s$ satisfies the boundary conditions and equals B_s on the boundary and in the complement of the collar.

Consider the loop on the boundary of $S = D_m - \partial E_{p_j}[M]$. There exists a 1-parameter family of functions $B_s\colon S \to \mathbf{U}(n)$ such that $B_0 = \mathrm{id}$ and B_1A is diagonal, since any loop is homotopic to a loop of diagonal matrices. The loops B_s can be smoothly extended to all of D_m and the above trick makes B_s satisfy the boundary conditions. □

Now let $A\colon D_m \to \mathbf{U}(n)$ take values in diagonal matrices. Assume that A is constant near the punctures and that $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$ satisfies (5.40).

Lemma 5.11. *There are continuous families of smooth maps $B_s\colon D_m \to \mathrm{Diag} \subset \mathbf{U}(n)$ and $\lambda\colon [0,1] \to \mathbb{R}$ which satisfy* (5.41) *and* (5.40) *(where the $\theta(j)$ are computed w.r.t. B_s) respectively such that*

$$B_1A = \mathrm{id}$$

in a neighborhood of each puncture.

Proof. Let $M > 0$ be such that A is constant in $E_{p_j}[M]$ for each j. Let $\phi\colon [0,1] \to [0,1]$ be an approximation of the identity which is constant near the endpoints of the interval. Let $\psi\colon [M,\infty) \to [0,1]$ be a smooth increasing function which is identically 0 on $[M, M+1]$ and identically 1 on $[M+2,\infty)$. For $\alpha = (\alpha_1,\dots,\alpha_m) \in (-\pi,\pi)^m$ let

$$\tilde{g}_\alpha(\zeta) = \begin{cases} 1 & \text{for } \zeta \in D_m - \bigcup_j E_{p_j}[M], \\ e^{i\psi(\tau)\alpha_j\phi(t)} & \text{for } \zeta = \tau + it \in E_{p_j}[M], \end{cases}$$

and let g_α be a function which agrees with $\tilde{g}_\alpha$ except on $E_{p_j}[M] - E_{p_j}[M+2]$ and which satisfies $\bar{\partial} g_\alpha | \partial D_m = 0$.

Consider the complex angle $\theta(j) \in [0,\pi)^n$ and the weight μ_j. Assume first that $\mu_j \neq k\pi$ for all $k \in \mathbb{Z}$ and $j = 1,\dots,m$. Let m_j be the unique number $0 \le m_j \le \pi$ such that $m_j = k\pi - \mu_j$ for some $k \in \mathbb{Z}$. By (5.40) $\theta(j)_r \neq m_j$ for all r. If $\theta(j)_r > m_j$ define $\alpha_r = \pi - \theta(j)_r$, and if $\theta(j)_r < m_j^*$ define $\alpha(j)_r = -\theta(j)_r$. Define

$$B_s = \operatorname{Diag}\left(g_{s\alpha_1},\dots,g_{s\alpha_n}\right)$$

and let $\lambda(s) \equiv \mu$.

Assume now that $\mu_j = k\pi$ for some j. For $0 \le s \le \frac{1}{2}$, let $B_s = \mathrm{id}$ and take $\lambda_j = \mu_j - \epsilon s$ for some sufficiently small $\epsilon > 0$. Repeat the above construction to construct B_s for $s \le \frac{1}{2} \le 1$. □

5.8. **The Fredholm index of the standardized problem.** Consider D_m with m punctures on the boundary, conformal structure κ and metric $g(\kappa)$ as above and neighborhoods E_{p_j} of the punctures $p_1,\dots,p_m$.

Let Δ_m denote the representative of the conformal structure κ on D_m which is the unit disk in $\mathbb{C}$ with m punctures at $1, i, -1, q_3,\dots,q_m$ with the flat metric. Then there exists a conformal and therefore holomorphic map $\Gamma\colon D_m \to \Delta_m$. We study the behavior of Γ on E_{p_j}. Let $p = p_j$ and let q be the puncture on Δ_m to which p maps. After translation and rotation in $\mathbb{C}$ we may assume that the point $q = 0$ and that Δ_m is the disk of radius 1 centered at i. We may then find a holomorphic function on a neighborhood $U \subset \Delta_m$ of $q = 0$ which fixes 0 and maps $\partial\Delta_m \cap U$ to the real line. Composing with this map we find that Γ maps ∞ to 0, $\tau + 0i$ to the negative real axis and $\tau + i$ to the positive real axis for $\tau > M$ for some M. Thus this composition equals $C\exp(-\pi\zeta)$ where $C < 0$ is some negative real constant. Thus, up to a bounded holomorphic change of coordinates on a neighborhood of q the map Γ on E_{p_j} looks like $\Gamma(\zeta) = \exp(-\pi\zeta)$ and its inverse Γ^{-1} in these coordinates satisfies $\Gamma^{-1}(z) = -\frac{1}{\pi}\log(z)$.

Let $A\colon \partial D_m \to \operatorname{Diag} \subset \mathbf{U}(n)$ be a smooth function which is constantly equal to id close to each puncture. We may now think of A as being defined on $\partial\Delta_m$. We extend A smoothly to $\partial\Delta$ by defining its extension $\hat{A}$ at the punctures as $\hat{A}(p_j) = \mathrm{id}$ for each j.

Consider the following boundary condition for $u \in \mathcal{H}_2(\Delta,\mathbb{C}^n)$:

$$\int_{\partial\Delta} \langle u, v\rangle\, ds = 0 \text{ for all } v \in C_0^\infty(\partial\Delta,\mathbb{C}^n) \text{ with } v(z) \in i\hat{A}(z)\mathbb{R}^n \text{ for all } z \in \partial\Delta. \tag{5.42}$$

For $u \in \mathcal{H}_1(\Delta, T^{*0,1}\Delta \otimes \mathbb{C}^n)$ consider the boundary conditions

$$\int_{\partial\Delta} \langle u, v\rangle\, ds = 0 \text{ for all } v \in C_0^\infty(\partial\Delta, T^{0,1}\Delta \otimes \mathbb{C}^n). \tag{5.43}$$

Define

$$\mathcal{H}_2(\Delta,\mathbb{C}^n;\hat{A}) = \left\{u \in \mathcal{H}_2(\Delta,\mathbb{C})\colon u \text{ satisfies (5.42) and } \bar{\partial}u \text{ satisfies (5.43)}\right\}$$
$$\mathcal{H}_1(\Delta, T^{*0,1}\Delta;[0]) = \left\{u \in \mathcal{H}_2(\Delta,\mathbb{C})\colon u \text{ satisfies (5.43)}\right\}.$$

Lemma 5.12. *The operator*

$$\bar\partial\colon \mathcal{H}_2(\Delta,\mathbb{C}^n;\hat A)\to\mathcal{H}_1(\Delta,T^{*0,1}\Delta;[0])$$

is Fredholm of index $n+\mu(\hat A)$, *where* $\mu(\hat A)$ *denotes the Maslov index of the loop* $z\mapsto A(z)\mathbb{R}^n$, $z\in\partial\Delta$, *of Lagrangian subspaces in* $\mathbb{C}^n$.

Proof. This is (a direct sum of) classical Riemann-Hilbert problems. □

Let $\lambda(a)=(a,\dots,a)\in\mathbb{R}^m$.

Proposition 5.13. *For* $-\pi<a<0$ *the Fredholm index of the operator*

$$\bar\partial\colon \mathcal{H}_{2,\lambda(a)}(\mathbb{C}^n;A)\to\mathcal{H}_{1,\lambda(a)}(T^{*0,1}D_m\otimes\mathbb{C}^n;[0])$$

equals $n+\mu(A)$.

Proof. The holomorphic map $\Gamma\colon D_m\to\Delta_m$ and its holomorphic inverse commute with the $\bar\partial$ operator. Any solution on D_m must look like $\sum_{n\le 0}c_ne^{\pi n\zeta}$ (the negative weights allows for $c_0\neq 0$) in canonical coordinates close to each puncture. Thus Γ^{-1} pulls back solutions on D_m to solutions on Δ_m. Using also Γ we see that the kernels are isomorphic.

Elements in the cokernel on D_m are of the form $(\sum_{n<0}c_ne^{n\pi\bar\zeta})d\bar\zeta$ (the positive weight implies $c_0=0$). Pulling back with Γ^{-1} gives elements of the form $(\sum_{n>0}\bar z^n)\frac{d\bar z}{\bar z}$ which are in the cokernel of the $\bar\partial$ on Δ. So the cokernels are also isomorphic. □

5.9. **The index of the linearized problem.** In this subsection we determine the Fredholm indices of the problems which are important in our applications to contact geometry.

Let $A\colon D_m\to\mathbf{U}(n))$ be a map which is small at infinity. Assume that $A^0_j\mathbb{R}^n$ and $A^1_j\mathbb{R}^n$ are transverse for all j. For $0\le s\le 1$, let $\mathbf{f}_j(s)\in\mathbf{U}(n)$ be the matrix which in the canonical coordinates $z(j)$ is represented by the matrix

$$\mathrm{Diag}(e^{-i(\pi-\theta(j)_1)s},\dots,e^{-i(\pi-\theta(j)_n)s}).$$

If p and q are consecutive punctures on ∂D_m then let $I(a,b)$ denote the (oriented) path in ∂D_m which connects them. Define the loop Γ_A of Lagrangian subspaces in $\mathbb{C}^n$ by letting the loop

$$(A|I(p_1,p_2))*\mathbf{f}_2*(A|I(p_2,p_3))*\mathbf{f}_3*\dots*(A|I(p_m,p_1))*\mathbf{f}_1$$

of elements of $\mathbf{U}(n)$ act on $\mathbb{R}^n\subset\mathbb{C}^n$.

Proposition 5.14. *For* A *as above the index of the operator*

$$\bar\partial\colon \mathcal{H}_2(\mathbb{C}^n;A)\to\mathcal{H}_1(T^{*0,1}D_m\otimes\mathbb{C}^n;[0])$$

equals $n+\mu(\Gamma_A)$ *where* μ *is the Maslov index.*

Proof. Using Lemmas 5.10 and 5.11 we deform A to put the problem into standardized form with weight $-\epsilon$ at each corner, without changing the index. Call the new matrix B. We need to consider how B is constructed from A. The key step to understand is the point where we make B equal the identity on the ends. This is achieved by first introducing a small *negative* weight and then rotating the space

$$A^1_j\mathbb{R}^n=\mathrm{Span}\left\langle e^{i\theta(j)_1}\partial_1,\dots,e^{i\theta(j)_n}\partial_n\right\rangle$$

to A^0_j according to

$$\mathrm{Span}\left\langle e^{i(\theta(j)_1+s\phi(\tau)(\pi-\theta(j)_1)}\partial_1,\dots,e^{i(\theta(j)_n+s\phi(\tau)(\pi-\theta(j)_n)}\partial_n\right\rangle, \tag{5.44}$$

where $0\le s\le 1$ and $\phi\colon[M,\infty)\to[0,1]$ equals 1 on $[M+2,\infty)$ and 0 on $[M,M+1]$.

We now calculate the Maslov-index $\mu(B)$. Since as we follow $\mathbb{R}+i$ along the negative τ-direction from $M+1$ to M, B experiences the inverse of the rotation (5.44), the proposition follows. □

We now consider the simplest degeneration at a corner. Compare this with Theorem 4.A of [13] or the appendix of [30]. Let $\epsilon>0$ be a small number. Let $A_s\colon D_m\to \mathbf{U}(n)$, $0\le s\le 1$ be a family matrices which are small at infinity and constant in s near each puncture in $S\subset\{1,\dots,m\}$, where each component of the complex angle is assumed to be positive. At p_r, $r\notin S$, assume that $\theta(r)_s=(\pi-s,\theta_2(r),\dots,\theta_n(r))$, where $\theta_j(r)\neq 0$, $j=2,\dots,n$.. Let $\beta(\epsilon)\in\mathbb{R}^m$ satisfy $\beta(\epsilon)_r=0$ if $r\in S$ and $\beta(\epsilon)_r=-\epsilon$ if $r\notin S$.

Proposition 5.15. *The index of the operators*

$$\bar\partial\colon \mathcal{H}_2(\mathbb{C}^n;A_s)\to\mathcal{H}_1(T^{0,1}D_m;[0])$$

for $s>0$ and of the operator

$$\bar\partial\colon \mathcal{H}_{2,\beta(\epsilon)}(\mathbb{C}^n;A_0)\to\mathcal{H}_{1,(-\epsilon,0,\dots,0)}(T^{0,1}D_m;[0]),$$

are the same.

Proof. This is a consequence of Lemma 5.9. □

Finally, we show how the index is affected if the weight is changed.

Proposition 5.16. *Let $A\colon D_m\to\mathbf{U}(n)$ be constant at infinity and suppose that the complex angle at each puncture except possibly p_1 has positive components. Assume that $0\le\pi-\theta(1)_1<\pi-\theta(1)_2<\dots<\pi-\theta(1)_n$. Let $\epsilon>0$ be smaller than $\min_r(\pi-\theta(r)_r)$, and let $\pi-\theta(1)_j<\delta<\pi-\theta(1)_{j-1}$ Then the index of the problem*

$$\bar\partial\colon \mathcal{H}_{2,(-\epsilon,0,\dots,0)}(\mathbb{C}^n;A)\to\mathcal{H}_{1,(-\epsilon,0,\dots,0)}(T^{*0,1}D_m;[0])$$

is j larger than that of

$$\bar\partial\colon \mathcal{H}_{2,(\delta,0,\dots,0)}(\mathbb{C}^n;A)\to\mathcal{H}_{1,(\delta,0,\dots,0)}(T^{*0,1}D_m;[0])$$

Proof. First deform the matrix into diagonal form without changing the weights. If $n>1$ this can be done in such a way that the index corresponding to the first component is positive. Then put the first component in standardized form. We must consider the index difference arising from the first component as the weight changes from negative to positive. The condition that a solution lies in $\mathcal{H}_{2,\delta}$ means that the corresponding solution on Δ vanishes at p_1. Thus the dimension of the kernel increases by 1. The cokernel remains zero-dimensional. This argument can then be repeated for other components. To handle the 1-dimensional case one may either use similar arguments for cokernels or reduce to the higher dimensional case by adding extra dimensions. □

5.10. **The index and the Conley-Zehnder index.** We translate Proposition 5.14 into a more invariant language. Recall from Section 1.2 that we denote by $\nu_\gamma(c)$ the Conley-Zehnder index of Reeb chord c with capping path γ. In the following proposition we suppress γ from the notation.

Proposition 5.17. *Let $(u,f)\in\mathcal{W}_2(\mathbf{c},\kappa;B)$ be a holomorphic disk with boundary on an admissible L, and with j positive punctures at Reeb chords $a_1,\dots,a_j$ and k negative punctures at Reeb chords $b_1,\dots,b_k$. Then the index of $d\Gamma_{(u,f)}$ equals*

$$\mu(B)+(1-j)n+\sum_{r=1}^{j}\nu(a_r)-\sum_{r=1}^{k}\nu(b_r). \tag{5.45}$$

Remark 5.18. Note that (5.45) is independent of the choices of capping paths.

Proof. We simply translate the result of Proposition 5.14. At a positive puncture p, the tangent space corresponding to $\mathbb{R} + 0i$ ($\mathbb{R} + i$) in ∂E_p is the lower (upper) one and at a negative puncture the situation is reversed. We must compare the rotation path $\lambda(V_1, V_0)$ used in the definition of the Conley–Zehnder index with the rotation used in the construction of the arcs $\mathbf{f}_i$ in Proposition 5.14. At a negative puncture, the path $\mathbf{f}_i$ is the inverse path of $\lambda(V_1, V_0)$. Hence the contribution to the Maslov index of $\mathbf{f}_i$ at a negative corner equals minus the contribution from λ. Consider the situation at positive puncture mapping to a^*. Let $\lambda(V_1, V_0)$ be the path used in the definition of the Conley–Zehnder index. Then $\lambda(V_1, V_0)$ rotates the lower tangent space V_1 of $\Pi_{\mathbb{C}}(L)$ at a^* to the upper V_0 according to e^{sI}, $0 \le s \le \frac{\pi}{2}$, where I is a complex structure compatible with ω. Let $\lambda(V_0, V_1)$ be the path which rotates V_0 to V_1 in the same fashion. Then the path $\mathbf{f}_j$ is the inverse path of $\lambda(V_0, V_1)$ and hence the contribution to the Maslov index of $\mathbf{f}_j$ equals the contribution from $\lambda(V_1, V_0)$ minus n.

To get the loop B from Γ_A (see Proposition 5.14) the arcs $\mathbf{f}_i$, must be removed and replaced by the arcs Γ_i, induced from the capping paths of the Reeb chords. A straightforward calculation gives

$$\mu(\Gamma_A) = \mu(B) + \sum_{r=1}^{j} \nu(a_r) - nj - \sum_{s=1}^{k} \nu(b_s).$$

Hence,

$$n + \mu(\Gamma_A) = \mu(B) + (1 - j)n + \sum_{r=1}^{j} \nu(a_r) - \sum_{s=1}^{k} \nu(b_s).$$

□

5.11. **The index and the Conley–Zehnder index at a self tangency.** In this section we prove the analog of Proposition 5.17 for semi-admissible submanifolds. First we need a definition of the Conley-Zehnder index of a degenerate Reeb chord. Let $L \subset \mathbb{R} \times \mathbb{C}^n$ be a chord semi generic Legendrian submanifold. Let c be the Reeb chord of L such that $\Pi_{\mathbb{C}}(L)$ has a double point with self tangency along one direction at c^*. Let a and b be the end points of c, $z(a) > z(b)$. Let $V_0 = d\Pi_{\mathbb{C}}(T_a L)$ and $V_1 = d\Pi_{\mathbb{C}}(T_b L)$. Then V_0 and V_1 are Lagrangian subspaces of $\mathbb{C}^n$ such that $\dim_{\mathbb{R}}(V_0 \cap V_1) = 1$. Let $W \subset \mathbb{C}^n$ be the 1-dimensional complex linear subspace containing $V_0 \cap V_1$ and let $\mathbb{C}^{n-1}$ be the Hermitian orthogonal complement of W. Then $V_0' = V_0 \cap \mathbb{C}^{n-1}$ and $V_1' = V_1 \cap \mathbb{C}^{n-1}$ are transverse Lagrangian subspaces in $\mathbb{C}^{n-1}$. Pick a complex structure I' on $\mathbb{C}^{n-1}$ compatible with $\omega|\mathbb{C}^{n-1}$ such that $I'V_1 = V_0$. Define $\lambda(V_1, V_0)$ to be the path of Lagrangian planes $s \mapsto V_0 \cap V_1 \times e^{sI'} V_1'$. Also pick a capping path $\gamma\colon [0, 1] \to L$ with $\gamma(0) = a$ and $\gamma(1) = b$. Then γ induces a path Γ of Lagrangian subspaces of $\mathbb{C}^n$. Define the Conley–Zehnder index of c as

$$\nu_\gamma(c) = \mu(\Gamma * \lambda(V_1, V_0)).$$

Let $0 < \epsilon < \theta$, where θ is the smallest non-zero complex angle of L at c. Let $\mathcal{W}_{2,\epsilon}(\mathbf{c}; \kappa)$ denote the space of maps with boundary conditions constructed from the Sobolev space with weight ϵ at each puncture mapping to c and define $\widetilde{\mathcal{W}}_{2,\epsilon}(\mathbf{c}; \kappa)$ as in Section 4.8. If a is a Reeb chord of L then let $\delta(a, c) = 0$ if $a \neq c$ and $\delta(c, c) = 1$. Again we suppress capping paths from the notation.

Proposition 5.19. *Let $(u, f) \in \mathcal{W}_{2,\epsilon}(\mathbf{c}, \kappa; B)$ and $(v, g) \in \widetilde{\mathcal{W}}_{2,\epsilon}(\mathbf{c}, \kappa; B)$ be holomorphic disks. If $\mathbf{c} = (a; b_1, \dots, b_m)$ where $a \neq c$ then the index of $d\Gamma_{(u,f)}$ equals*

$$\mu(B) + \nu(a) - \sum_{r=1}^{k} (\nu(b_j) + \delta(b_j, c)),$$

and the index of $d\Gamma_{(v,g)}$ *equals*

$$\mu(B) + \nu(a) - \sum_{r=1}^{k} \nu(b_j).$$

If $\mathbf{c} = (c; b_1, \dots, b_m)$ *then the index of* $d\Gamma_{(u,f)}$ *equals*

$$\mu(B) + \nu(c) - \sum_{r=1}^{k} (\nu(b_j) + \delta(b_j, c)),$$

and the index of $d\Gamma_{(v,g)}$ *equals*

$$\mu(B) + (\nu(c) + 1) - \sum_{r=1}^{k} \nu(b_j).$$

Remark 5.20. Note again that the index computations are independent of the choices of capping paths.

Proof. The proof is similar to the proof of Proposition 5.17. Consider first the $\bar{\partial}$-operator with boundary conditions determined by (u, f) and acting on a Sobolev space with small negative weight. Again we need to compute the Maslov index contributions from the paths $\mathbf{f}_i$ in the loop Γ_A, where $\mathbf{f}_i$ fixes the common direction in the tangent spaces at a self tangency double point. Note that at a positive puncture c the contribution is now the contribution of $\lambda(V_1, V_0)$ minus $(n-1)$. At a negative puncture it is again minus the contribution of $\lambda(V_1, V_0)$. Applying Proposition 5.16 the first and third index calculations above follow. Noting that the tangent space of $\widetilde{\mathcal{W}}_{2,\epsilon}(\mathbf{c}, \kappa, B)$ is obtained from that of $\mathcal{W}_{2,\epsilon}(\mathbf{c}, \kappa; B)$ by adding one $\mathbb{R}$-direction for each puncture mapping to c the other index formulas follow as well. □

6. Transversality

In this section we show how to achieve transversality (or "surjectivity") for the linearized $\bar{\partial}$ equation by perturbing the Lagrangian boundary condition. When proving transversality for some Floer-type theory, it is customary to show that solution-maps are "somewhere injective" (see [22, 16], for example). One then constructs a small perturbation, usually of the almost complex structure or the Hamiltonian term, which is supported near points where the map is injective. With a partial integration argument, these perturbations eliminate non-zero elements of the cokernel of $\bar{\partial}$.

For our set-up, we perturb the Lagrangian boundary condition. In Sections 6.1 through 6.4, we describe the space of perturbations for the chord generic, one-parameter chord generic, and chord semi-generic cases. Although we do not have an injective (boundary) point, we exploit the fact that there is only one positive puncture, and hence, by Lemma 1.1, the corresponding double point can represent a corner only once. Of course other parts of the boundary can map to this corner elsewhere, but not at other boundary punctures. With this observation, we prove transversality in Sections 6.7 and 6.9 first for the open set of non-exceptional maps, defined in Section 6.6 and from this for all maps provided the expected kernel has sufficiently low dimension. We also prove some results in Sections 6.10 and 6.12 which will be useful later for the degenerate gluing of Section 7.

6.1. **Perturbations of admissible Legendrian submanifolds.** Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be an admissible Legendrian submanifold. Let $a(L)$ denote the minimal distance between the images under $\Pi_{\mathbb{C}}$ of two distinct Reeb chords of L and let $A(L)$ be such that $\Pi_{\mathbb{C}}(L)$ is contained in the ball $B(0, A(L)) \subset \mathbb{C}^n$. Fix $\delta > 0$ and $R > 0$ such that $\delta \ll a(L)$ and such that $R \gg A(L)$.

Definition 6.1. Let $\mathrm{Ham}(L,\delta,R)$ be the linear space of smooth functions $h\colon \mathbb{C}^n \to \mathbb{R}$ with support in $B(0,R)$ and satisfying the following two conditions for any Reeb chord c.

(i) The restriction of h to $B(c^*,\delta)$ is real analytic,
(ii) The differential of h satisfies $Dh(c^*) = 0$ and also $h(c^*) = 0$.

We are going to use Hamiltonian vector fields of elements in $\mathrm{Ham}(L,\delta,R)$ to perturb L. Condition (i) ensures that L stays admissible, and (ii) that the set of Reeb chords $\{c_0,\dots,c_m\}$ of L remains fixed.

Lemma 6.2. *The space* $\mathrm{Ham}(L,\delta,R)$ *with the* C^∞*-norm is a Banach space.*

Proof. Using the characterization of real analytic functions as smooth functions the derivatives of which satisfy certain uniform growth restrictions one sees that the limit of a C^∞-convergent sequence of real analytic functions on an open set is real analytic. □

Lemma 6.3. *If L is admissible and $h \in \mathrm{Ham}(L,\delta,R)$ then $\tilde\Phi_h(L)$ (see Section 2) is admissible.*

Proof. For each Reeb chord c, the Hamiltonian vector field is real analytic in $B(c^*,\delta)$. Also, $\Phi_h(c^*) = c^*$ and hence there exists a neighborhood W of c^* such that $\Phi_h^t(W) \subset B(c^*,\delta)$ for $0 \le t \le 1$. A well-known ODE-result implies that the flow of a real analytic vector field depends in a real analytic way on its initial data. This shows that $\tilde\Phi_h(L)$ is admissible. □

6.2. **Perturbations of 1-parameter families of admissible submanifolds.** Let L_t, $t \in [0,1]$ be an admissible 1-parameter family of Legendrian submanifolds without self-tangencies. Let $a = \min_{0\le t\le 1} a(L_t)$ and $A = \max_{0\le t\le 1} A(L_t)$. Fix $\delta > 0$ and $R > 0$ such that $\delta \ll a$ and $R \gg A$.

We define a continuous family of isomorphisms $\mathrm{Ham}(\delta,R,L_0) \to \mathrm{Ham}(\delta,R,L_t)$, $0 \le t \le 1$. Let $(c_1(t),\dots,c_m(t))$ be the Reeb chords of L_t. Then $(c_1^*(t),\dots,c_m^*(t))$, $0 \le t \le 1$ is a continuous curve in $(\mathbb{C}^n)^m$. Let $\psi^t\colon B(0,R) \to B(0,R)$ be a continuous family of compactly supported diffeomorphisms which when restricted to $B(c_j^*(0),\delta)$, $j = 1,\dots,m$ agree with the map

$$z \mapsto z + (c_j^*(t) - c_j^*(0)).$$

Composition with ψ^t can be used to give the space

$$\mathrm{pHam}(L_t,\delta,R) = \bigcup_{0\le t\le 1} \mathrm{Ham}(L_t,\delta,R)$$

the structure of a Banach manifold which is a trivial bundle over $[0,1]$. We note that if (h,t) in $\mathrm{pHam}(L_t,\delta,R)$ then Lemma 6.3 implies that $\tilde\Phi_a(L_t)$ is admissible.

6.3. **Bundles over perturbations.** Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be an admissible chord generic Legendrian submanifold. Above we constructed a smooth map of the Banach space $\mathrm{Ham}(L,\delta,R)$ into the space of admissible chord generic Legendrian embeddings of L into $\mathbb{C}^n \times \mathbb{R}$.

Let $\mathbf{c} = (c_0, c_1,\dots,c_m)$ be Reeb chords of L and let $\epsilon \in [0,\infty)^m$ and consider as in Section 4.1 the space

$$\mathcal{W}_{2,\epsilon,\mathrm{Ham}(L,\delta,R)}(\mathbf{c}) \tag{6.1}$$

and its tangent space

$$T_{(w,f,\kappa,a)}\mathcal{W}_{2,\epsilon,\mathrm{Ham}(L,\delta,R)} \approx T_{(w,f)}\mathcal{W}_{2,\epsilon} \oplus T_\kappa\mathcal{C}_m \oplus \mathrm{Ham}(L,\delta,R). \tag{6.2}$$

In a similar way we consider for a 1-parameter family L_t the space

$$\mathcal{W}_{2,\epsilon,\mathrm{pHam}(L_t,\delta,R)}(\mathbf{c}) \tag{6.3}$$

and its tangent space.

For $\Lambda = \mathrm{Ham}(L,\delta,R)$ or $\Lambda = \mathrm{pHam}(L_t,\delta,R)$, consider also the bundle map $\Gamma\colon \mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{c}) \to \mathcal{H}_{1,\epsilon,\Lambda}[0](T^{*0,1}D_m \otimes \mathbb{C}^n)$ here we are thinking of the spaces as bundles over $\mathrm{Ham}(L,\delta,R)$ and we denote projection onto this space by pr. To emphasize this we will be write (Γ, pr) instead of just Γ in the sequel. The differential $d\Gamma$ was calculated in Lemma 4.14.

6.4. **Perturbations in the semi-admissible case.** Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a semi-admissible Legendrian submanifold. Let $(c_0,\dots,c_m)$ be the Reeb chords of L. Assume that the self tangency Reeb chord is c_0, that $c_0^* = 0$, and that L has standard form in a neighborhood of 0, see Definition 2.3.

Let $a(L)$ denote the minimal distance between the images under $\Pi_{\mathbb{C}}$ of two distinct Reeb chords of L. Fix $\delta > 0$ such that $\delta \ll a(L)$. For $r > 0$ let $C(r) = \mathbb{C} \times B'(0,r) \subset \mathbb{C}^n$, where $B'(0,r)$ is the r-ball in $\mathbb{C}^{n-1} \approx \{z_1 = 0\}$, where as always $(z_1,\dots,z_n) = (x_1+iy_1,\dots,x_n+iy_n)$ are coordinates on $\mathbb{C}^n$.

Definition 6.4. Let $\mathrm{Ham}_0(L,\delta)$ be the linear space of smooth functions $h\colon \mathbb{C}^n \to \mathbb{R}$ with support in $C(10\delta) \cup \bigcup_{j\ge 1} B(c_j^*, 10\delta)$ and satisfying the following conditions.

(i) The restriction of h to $B(c_j^*,\delta)$ $1 \le j \le m$ is real analytic,

(ii) In $C(10\delta)$, $\frac{\partial h}{\partial x_1} = 0 = \frac{\partial h}{\partial y_1}$ and the restriction of h to $C(\delta)$ is real analytic.

(iii) The differential of h satisfies $Dh(c_j^*) = 0$ and also $h(c_j^*) = 0$, for all j.

Lemma 6.5. *The space* $\mathrm{Ham}_0(L,\delta)$ *with the* C^∞*-norm is a Banach space.*

Proof. See Lemma 6.2 and note that the restriction of h to $C(10\delta)$ can be identified with a function of $(n-1)$-complex variables supported in $B'(0,10\delta)$. □

Let $\tilde{\Phi}_h$ be the Legendrian isotopy which is defined by using the flow of h locally around the Reeb chords of L. This is well-defined for h sufficiently small. Let $\mathrm{Ham}_0(L,\delta,s)$ denote the s-ball around 0 in $\mathrm{Ham}_0(L,\delta)$.

Lemma 6.6. *There exists* $s > 0$ *such that for* $h \in \mathrm{Ham}_0(L,\delta,s)$*,* $\tilde{\Phi}_h(L)$ *is an admissible chord semi-generic Legendrian submanifold.*

Proof. Note that the product structure in $C(10\delta)$ is preserved since h does not depend on (x_1,y_1). Moreover, the isotopy is fixed in the region $B(0,2+\epsilon)\setminus B(0,2)$ for s and δ sufficiently small. □

We have defined a smooth map of $\mathrm{Ham}_0(L,\delta,s)$ into the space of admissible chord semi-generic Legendrian submanifolds and this maps fulfills the conditions on Λ in Section 4.8. We can therefore construct the spaces

$$\mathcal{W}_{2,\epsilon,\mathrm{Ham}_0(L,\delta,s)}, \quad \text{and} \quad \widetilde{\mathcal{W}}_{2,\epsilon,\mathrm{Ham}_0(L,\delta)}, \tag{6.4}$$

see Sections 4.1 and 4.8, respectively. Moreover, as there we will consider the $\bar{\partial}$-map and its linearization.

6.5. **Consequences of real analytic boundary conditions.** For $r > 0$, let $E_+ = \{z \in \mathbb{C}\colon |z| < r, \mathrm{Im}(z) \ge 0\}$. If $w\colon (E_+, \partial E_+) \to (\mathbb{C}^n, M)$ where M is a real analytic Lagrangian submanifold and w is holomorphic in the interior and continuous on the boundary, then by Schwartz-reflection principle, w extends in a unique way to a holomorphic map $w^d\colon E \to \mathbb{C}^n$ mapping $\mathrm{Im}(z) = 0$ to M, where $E = \{z \in \mathbb{C}\colon |\zeta| < r\}$. We call w^d the double of w.

Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a chord (semi-)generic Legendrian submanifold.

Lemma 6.7. *Let* p *be a point in* $U \subset L$ *such that* $\Pi_{\mathbb{C}}(U)$ *is real analytic, where* $U \subset L$ *is a neighborhood of* p *on which* $\Pi_{\mathbb{C}}$ *is injective. Assume that*

$$w\colon (E_+, \partial E_+, 0) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(U), p),$$

is holomorphic. Then there is a holomorphic function u with Taylor expansion at 0,

$$u(z) = a_0 + a_1 z + \dots, a_0 \neq 0$$

such that $w(z) = p + z^k u(z)$ for some integer $k > 0$.

Proof. The double w^d has a Taylor expansion. □

Lemma 6.8. *Let p, U and L satisfy the conditions of Lemma 6.7. Assume that $w\colon D_m \to \mathbb{C}^n$ is holomorphic with boundary on $\Pi_{\mathbb{C}}(L)$. Then $w^{-1}(p) \cap \partial D_m$ is a finite set.*

Proof. Using Lemma 3.6, we may find $M > 0$ such that there are no preimages of p in $\cup_j E_{p_j}[M]$. Since the complement of $\cup_j E_{p_j}[M]$ is compact, the lemma now follows from Lemma 6.7. □

Lemma 6.9. *Let $p \in L$ satisfy the conditions of Lemma 6.7, and let*

$$w_1, w_2\colon (E_+, \partial E_+, 0) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(L), p), \tag{6.5}$$

be holomorphic maps such that w_2 maps one of the components I of $\partial E_+ \setminus \{0\}$ to $w_1(I)$. Then there exists a map $\hat{w}\colon E \to \mathbb{C}^n$ and integers $k_j \geq 1$ such that $w_j^d(z) = \hat{w}(z^{k_j})$, $j = 1, 2$.

Proof. As above we may reduce to the case when $\Pi_{\mathbb{C}}(L) = \mathbb{R}^n \subset \mathbb{C}^n$. The images $C_j = w_j^d(E)$, $j = 1, 2$ are analytic subvarieties of complex dimension 1 which intersects in a set of real dimension 1. Hence they agree. Projection of $C = C_1 = C_2$ onto a generic complex line through p identifies C (locally) with the standard cover of the disk possibly branched at 0. This gives the map $\hat{w}$. □

6.6. **Exceptional holomorphic maps.** Let Λ be one of the spaces $\mathrm{Ham}(L, \delta, R)$, $\mathrm{pHam}(L, \delta, R)$, or $\mathrm{Ham}_0(L, \delta, s)$. Let $(w, f, \lambda) \in \mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{c})$ (or $\widetilde{\mathcal{W}}_{2,\epsilon,\Lambda}(\mathbf{c})$) be a holomorphic disk and let q be a point on ∂D_m such that $w(q)$ lies in a region where $\Pi_{\mathbb{C}}(L_\lambda)$ is real analytic. Assume that $dw(q) = 0$. Since w has a Taylor expansion around q in this case we know there exists a half-disk neighborhood E of q in D_m such that q is the only critical point of w in E. The boundary ∂E is subdivided by q into two arcs $\partial E \setminus \{q\} = I_+ \cup I_-$. We say that q is *an exceptional point of* (w, f) if there exists a neighborhood E as above such that $w(I_+) = w(I_-)$.

Definition 6.10. Let $(w, f, \lambda) \in \mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{c})$, where $\mathbf{c} = (c_0(\lambda), c_1(\lambda), \dots, c_m(\lambda))$ and $c_0(\lambda)$ is the Reeb chord on L_λ of the positive puncture of D_{m+1}. Let $B_1(\lambda)$ and $B_2(\lambda)$ be the two local branches of $\Pi_{\mathbb{C}}(L_\lambda)$ at $c_0^*(\lambda)$. Then (w, f) is *exceptional holomorphic* if it has two exceptional points q_1 and q_2 with $w(q_1) = w(q_2) = c_0^*(\lambda)$ and if a neighborhood in ∂D_m of q_j maps to $B_j(\lambda)$, $j = 1, 2$.

Definition 6.11. Let $\mathcal{W}'_{2,\epsilon,\Lambda}(\mathbf{c})$ ($\widetilde{\mathcal{W}}'_{2,\epsilon,\Lambda}(\mathbf{c})$) denote the complement of the closure of the set of all exceptional holomorphic maps in $\mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{c})$ ($\widetilde{\mathcal{W}}_{2,\epsilon,\Lambda}(\mathbf{c})$).

We note that $\mathcal{W}'_{2,\epsilon,\Lambda}(\mathbf{c})$ is an open subspace of a Banach manifold and hence a Banach manifold itself.

6.7. Transversality on the complement of exceptional holomorphic maps in the admissible case.

Lemma 6.12. *For L admissible (respectively L_t a 1-parameter family of admissible submanifolds) the bundle map, see Section 4.7*

$$(\Gamma, \mathrm{pr})\colon \mathcal{W}'_{2,\epsilon,\Lambda}(\mathbf{c}) \to \mathcal{H}_{1,\epsilon,\Lambda}[0](D_m, T^*D_m \otimes \mathbb{C}^n),$$

where $\Lambda = \mathrm{Ham}(L, \delta, R)$ (respectively $\Lambda = \mathrm{pHam}(L_t, \delta, R)$) is transverse to the 0-section.

Proof. The proof for 1-parameter families L_t is only notationally more difficult. We give the proof in the stationary case. We must show that if $w\colon D_m \to \mathbb{C}^n$ is a (non-exceptional) holomorphic map (in the conformal structure κ on D_m) which represents a holomorphic disk (w, f) with boundary on $L = L_\lambda$ (without loss of generality we take $\lambda = 0$ below) then

$$d\Gamma\Big(T_{((w,f),\kappa,0)}\mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{c})\Big) = \mathcal{H}_{1,\epsilon}(D_m, T^*D_m \otimes \mathbb{C}^n),$$

i.e., $d\Gamma$ is surjective. To show this it is enough to show that

$$\Big\{d\Gamma\Big(T_{((w,f),\kappa,0)}\mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{c})\Big)\Big\}^{\perp} = \{0\},$$

where $V^\perp$ denotes the annihilator with respect to the L^2-pairing of $V \subset \mathcal{H}_{1,\epsilon}[0](D_m, T^*D_m \otimes \mathbb{C}^n)$ in its dual space.

An element u in this annihilator satisfies

$$\int_{D_m} \langle \bar\partial v, u\rangle \, dA = 0, \tag{6.6}$$

for all $v \in T_w\mathcal{B}_{2,\epsilon}(0, r)$. Here dA is the area form on D_m. Lemma 5.1 implies that u can be represented by a C^2-function which is anti-holomorphic.

We note that integrals of the form

$$\int_{D_m} \langle \phi, \psi\rangle \, dA, \tag{6.7}$$

where $\langle\,,\rangle$ is the inner product on T^*D_m and where ϕ and ψ are sections, are conformally invariant. We may therefore compute integrals of this form in any conformal coordinate system on the disk D_m.

Restrict attention to a small neighborhood of the image of the positive corner at c_0^*. Recall that w is assumed non-exceptional and consider a branch B of $\Pi_{\mathbb{C}}(L)$ at c_0^* such that w does not have an exceptional point mapping to $c_0^* \in B$. Since B is real analytic we may biholomorphically identify $(\mathbb{C}^n, B, c_0^*)$ with $(\mathbb{C}^n, \mathbb{R}^n, 0)$.

Let p be the positive puncture on D_m. For M large enough, by Lemma 3.6, the image of the component of $\partial E_p[M]$ which lies in B is a regular oriented curve. Denote it by γ. For simplicity we assume that the component mapping to γ is $[M, \infty) \times \{0\} \subset E_{p_0}[M]$ and we let $E_0 = [M, \infty) \times [0, \frac{1}{2})$.

Let $p_1, \dots, p_r$ be the preimages under w of c_0^* with the property that one of the components of a punctured neighborhood of p_j in ∂D_m maps to γ. Note that $r < \infty$ by Lemma 6.8 and that by shrinking γ we may assume that all these images are exactly γ.

We say that a point p_j is positive if close to p_j, w and the natural orientation on the boundary of ∂D_m induce the positive orientation on γ otherwise we say it is negative.

The image of the other half of the punctured neighborhood of p_1 in ∂D_m maps to a curve γ' under w. Our assumption that w is non-exceptional guarantees that γ and γ' are distinct.

Let w_j denote the restriction of w to a small neighborhood of p_j. Let $E = \{z \in \mathbb{C}\colon |z| < r\}$, let $E_+ = \{z \in E\colon \operatorname{Im}(z) \geq 0\}$, and let $E_- = \{z \in E\colon \operatorname{Im}(z) \leq 0\}$. Lemma 6.9 implies that we can find a map $\hat w\colon E \to \mathbb{C}^n$ and coordinate neighborhoods $(E_\pm(j), \partial E_\pm(j))$ of p_j (where the sign $\pm$ is that of p_j) such that $w_j^d(z) = \hat w(z^{k_j})$ for each j. Note that w non-exceptional implies all k_j are odd.

Let $k = k_1 k_2 \dots k_r$ and let $\hat k_j = \frac{k}{k_j}$. Let $\phi_j\colon E \to E(j)$ be the map $z \mapsto z^{\hat k_j}$. Consider the restrictions u_j of the anti-holomorphic map u to the neighborhoods $(E_\pm(j), \partial E_\pm(j))$. Because of the real analytic boundary conditions (recall that $(B, \mathbb{C}^n)$ is biholomorphically identified with $(\mathbb{R}^n, \mathbb{C}^n)$), these maps can be doubled using Schwartz reflection principle. Use ϕ_j to pull-back the maps u_j and w_j to E.

Let $a\colon \mathbb{C}^n \to \mathbb{R}$ be any smooth function with support in a small ball around a point $q' \in \gamma'$, where q' is chosen so that no point outside $\bigcup_j E_\pm(j)$ in ∂D_m maps to q'. (There exists such a point because of the asymptotics of w at punctures and Lemma 6.7.) Let Y_a is the Hamiltonian vector field associated with a, see Section 2.3.

If v is a smooth function with support in $\bigcup_j E_\pm(j)$ which is real and holomorphic on $\bigcup_j \partial E_\pm(j)$, if $\xi + i\eta$ are coordinates on $E_\pm(j)$, and if the support of a is sufficiently small then

$$0 = \int_{D_m} \langle \bar\partial(Y_a + v), u\rangle\, dA \tag{6.8}$$

$$= -\sum_j \int_{E_\pm(j)} \langle Y_a + v, \partial u\rangle\, d\xi \wedge d\eta + \sum_j \int_{\partial E_\pm(j)} \langle -i(Y_a+v), u\rangle\, d\xi \tag{6.9}$$

$$= \sum_j \int_{\partial E_\pm(j)} \langle -i(Y_a+v), u\rangle\, d\xi. \tag{6.10}$$

The equality in (6.8) follows since u is an element of the annihilator and since a can be arbitrarily well C^2-approximated by elements in $\mathrm{Ham}(L,\delta,R)$. The equality in (6.9) follows by partial integration and the restrictions on the supports of a and v. The equality in (6.10) follows from $\partial u = 0$. Taking $a = 0$ we see, since we are free to choose v, that u must be real valued on $\partial E_\pm(j)$ for every j.

We then take $v = 0$ and express the integral in (6.10) as an integral over $I_+ = \{x + 0i\colon x > 0\} \subset E$. Note that if $\xi + i\eta$ are coordinates on $E(j)$ then under the identification by ϕ_j, $d\xi = dx^{\hat k_j} = \hat k_j x^{\hat k_j - 1}\, dx$ and

$$\sum_j \int_{\partial E_\pm(j)} \langle -iY_a, u\rangle\, d\xi \qquad = \int_{I_+} \langle -iY_a(\hat w(z^k)), \sum_j \sigma(j)\hat k_j \bar z^{\hat k_j - 1} u_j(z^{\hat k_j})\rangle\, dx, \tag{6.11}$$

where $\sigma(j) = \pm 1$ equals the sign of p_j. Thus, if $\alpha(z) = \sum_j \sigma(j)\hat k_j \bar z^{\hat k_j-1} u_j(z^{\hat k_j})$ then α is antiholomorphic and by varying a we see that α vanishes in the $i\mathbb{R}^n$-directions along an arc in I_+. Therefore α vanishes identically on E.

Pick now instead a supported in a small ball around q in γ. With the same arguments as above we find

$$\begin{aligned}
0 &= \int_{D_m} \langle (\bar\partial Y_a + v), u\rangle\, dA\\
&= -\int_{E_0} \langle Y_a + v, \partial u\rangle\, d\tau \wedge dt - \sum_j \int_{E_\pm(j)} \langle Y_a + v, \partial u\rangle\, d\xi \wedge d\eta\\
&\quad + \int_{[M,\infty)} \langle -i(Y_a+v), u\rangle\, d\tau + \sum_j \int_{\partial E_\pm(j)} \langle -i(Y_a+v), u\rangle\, d\xi
\end{aligned}$$

$$= \int_{[M,\infty)} \langle -i(Y_a+v), u\rangle\, d\tau + \sum_j \int_{\partial E_\pm(j)} \langle -i(Y_a+v), u\rangle\, d\xi. \tag{6.12}$$

and conclude that $u(\tau, 0) \in \mathbb{R}^n$ for $\tau \in [M,\infty)$ as well.

Again taking $v = 0$ we get for the last integral in (6.12)

$$\sum_j \int_{\partial E_\pm(j)} \langle -iY_a, u\rangle\, d\xi = \int_{I_-} \langle -iY_a(\hat w(z^k)), \alpha(z)\rangle\rangle\, dx = 0, \tag{6.13}$$

where $I_- = \{x + 0i\colon x < 0\} \subset E$, and where the last equality follows since $\alpha = 0$. Equations (6.13) and (6.12) together implies (by varying a) that u must vanish along an arc in $[M,\infty)$.

Since u is antiholomorphic it must then vanish everywhere. This proves the annihilator is 0 and the lemma follows. □

Remark 6.13. In the case that w has an injective point on the boundary, the above argument can be shortened. Namely, under this condition there is an arc A on the boundary of D_m where w is injective and varying v and a there we see that u must vanish along A and therefore everywhere. Oh achieves transversality using boundary perturbations assuming an injective point [25].

Corollary 6.14. *Let* $\mathbf{c} = ab_1 \dots b_m$*. For a Baire set of* $h \in \mathrm{Ham}(L, \delta, R) = \Lambda$*,* $\Gamma^{-1}(0) \cap \mathrm{pr}^{-1}(h) \cap \mathcal{W}'_{2,\epsilon,\Lambda}(\mathbf{c}; A)$ *is a finite dimensional smooth manifold of dimension*

$$\mu(A) + \nu_\gamma(a) - \sum_j \nu_\gamma(b_j) + \max(0, m-2).$$

For a Baire set of $h \in \mathrm{pHam}(L, \delta, R) = \Lambda$ $\Gamma^{-1}(0) \cap \mathrm{pr}^{-1}(h) \cap \mathcal{W}'_{2,\epsilon,\Lambda}(\mathbf{c}; A)$ *is a finite dimensional smooth manifold of dimension*

$$\mu(A) + \nu_\gamma(a) - \sum_j \nu_\gamma(b_j) + \max(0, m-2) + 1.$$

Proof. Let $Z \subset \mathcal{W}'_{2,\epsilon,\Lambda}(\mathbf{c}; A)$ denote the inverse image of the 0-section in $\mathcal{H}_{1,\epsilon,\Lambda}[0](T^{*0,1}D_m \otimes \mathbb{C}^n)$ under (Γ, pr). By the implicit function theorem and Lemma 6.12, Z is a submanifold. Consider the restriction of the projection $\pi \colon Z \to \Lambda$. Then π is a Fredholm map of index equal to the index of the Fredholm section Γ. An application of the Sard-Smale theorem shows that for generic $\lambda \in \Lambda$, $\pi^{-1}(\lambda)$ is a submanifold of dimension given by the Fredholm index of Γ. Note that in the first case, the restriction of $d\Gamma$ to the complement of the $\max(0, m-2)$-dimensional subspace $T\mathcal{C}_m \subset T\mathcal{W}_{2,\epsilon}(\mathbf{c}; A)$ is an operator of the type considered in Proposition 5.17. Thus, the proposition follows in this first case. In the second case, we restrict to a $(\max(0, m-2) + 1)$-codimensional subspace instead. □

6.8. **General transversality in the admissible case.** If $\mathbf{c}$ is a collection of Reeb chords we define $l(\mathbf{c})$ as the number of elements in $\mathbf{c}$. We note that if (f, w) is a holomorphic disk with boundary on L with r punctures, then, if $r \leq 2$, the kernel of $d\Gamma$ at (f, w) is at least $(3 - r)$-dimensional. This is a consequence of the existence of conformal reparameterizations in this case.

Theorem 6.15. *For a dense open set of* $h \in \mathrm{Ham}(L, \delta, R)$ *(*$h \in \mathrm{pHam}(L, \delta, R)$*),* $\Gamma^{-1}(0) \cap \mathrm{pr}^{-1}(h) \subset \mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{c})$ *is a finite dimensional* C^1*-smooth manifold of dimension as in Corollary 6.14, provided this dimension is* ≤ 1 *if* $l(\mathbf{c}) \geq 3$ *and* $\leq 1 + (3 - l(\mathbf{c}))$ *otherwise.*

Proof. After Corollary 6.14 we need only exclude holomorphic disks in the closure of exceptional holomorphic disks. Let $a \in \mathrm{Ham}$ ($a \in \mathrm{pHam}$) be such that $\Gamma^{-1}(0)$ is regular. Then the same is true for $\tilde{a}$ in a neighborhood of a. Now assume there exists a holomorphic disk in the closure of exceptional holomorphic disks at a. Then there must exists an exceptional holomorphic disk for some $\tilde{a}$ in the neighborhood. However, such a disk w has $k \geq 2$ points mapping to the image of the positive puncture and with $w(I_+) = w(I_-)$. It is then easy to construct (by "moving the branch point") a k-parameter ($k + (3 - l(\mathbf{c}))$-parameter if $l(\mathbf{c}) \leq 2$) family of distinct (since the location of the branch point changes) non-exceptional holomorphic disks with boundary on $L(\tilde{a})$. This contradicts the fact that the dimension of $\Gamma^{-1}(0)$ is $< k$ ($< k + (3 - l(\mathbf{c}))$) for every $\tilde{a}$ in the neighborhood. □

Proof of Proposition 1.2. If the number of punctures is ≥ 3 the proposition is just Theorem 6.15. The case of fewer punctures can be reduced to that of many punctures as in Section 7.7. □

Corollary 6.16. *For chord generic admissible Legendrian submanifolds in a Baire set of such manifolds, no rigid holomorphic disk with boundary on L decays faster than $e^{-(\theta+\delta)|\tau|}$ close to any of its punctures mapping to a Reeb chord c. Here θ is the smallest complex angle of the Reeb chord c, $\delta > 0$ is arbitrary, and $\tau + it$ are coordinates near the puncture.*

Proof. Such a holomorphic disk would lie in $\mathcal{W}_{2,\epsilon}(\mathbf{c})$, where the component of ϵ corresponding to the puncture mapping to the Reeb chord c is larger than θ. By Proposition 5.16 this change of weight lowers the Fredholm index of $d\Gamma$ by at least 1. Since the Fredholm index of $d\Gamma$ with smaller weight (e.g. 0-weight) is the minimal which allows for existence of disks the lemma follows from Theorem 6.15. □

Proof of Proposition 1.9. The first statement in the proposition follows exactly as above. To see that handle slides appear at distinct times, let $(a_1\mathbf{b}_1; A_1)$ and $(a_2\mathbf{b}_2; A_2)$ be such that

$$\mu(A_1) + |a_1| - |\mathbf{b}_1| = \mu(A_2) + |a_2| - |\mathbf{b}_2| = 0$$

and consider the bundle $\mathcal{W}_{2,\Lambda}(a_1\mathbf{b}_1; A_1)\tilde{\times}\mathcal{W}_{2,\Lambda}(a_2\mathbf{b}_2; A_2)$. Here $\tilde{\times}$ denotes the fiberwise product where, in the fibers, the deformation coordinates (t_1, t_2) are restricted to lie in the diagonal: $t_1 = t_2 = t$. This is a bundle over Λ, and Γ induces a bundle map to the bundle $\mathcal{H}_{1,\Lambda}(D_{m_1}, \mathbb{C}^n) \times \mathcal{H}_{1,\Lambda}(D_{m_2}, \mathbb{C}^n)$, where $\times$ denotes fiberwise product. It is then easy to check that Γ is a Fredholm section of index -1. As in Theorem 6.15 we see that $d\Gamma$ is surjective and that the inverse image of the 0-section intersected with $\mathrm{pr}^{-1}(h)$ is empty for generic h. This shows that the handle slides appear at distinct times.

The statement about all rigid disks being transversely cut out at a handle slide instant can be proved in a similar way: let $(a_1\mathbf{b}_1; A_1)$ be as above and let $(a_3\mathbf{b}_3; A_3)$ be such that

$$\mu(A_3) + |a_3| - |\mathbf{b}_3| = 1.$$

Consider the bundle

$$\mathcal{W}_{2,\Lambda}(a_3\mathbf{b}_3; A_3)\tilde{\times}\mathcal{W}_{2,\Lambda}(a_3\mathbf{b}_3; A_3)\tilde{\times}\mathcal{W}_{2,\Lambda}(a_1\mathbf{b}_1; A_1),$$

and the bundle map Γ defined in the natural way with target

$$\mathcal{H}_{1,\Lambda}(D_{m_3}, \mathbb{C}^n) \times \mathcal{H}_{1,\Lambda}(D_{m_3}, \mathbb{C}^n) \times \mathcal{H}_{1,\Lambda}(D_{m_1}, \mathbb{C}^n).$$

Then the map Γ has Fredholm index 0 and as above we see $d\Gamma$ is surjective. Hence $\Gamma^{-1}(0) \cap \mathrm{pr}^{-1}(h)$ is a transversely cut out 0-manifold for generic h. We show that this implies that if t is such that $\mathcal{M}^t_{A_1}(a_1; \mathbf{b}_1) = \{(v, g)\} \neq \emptyset$ then $\mathcal{M}^t_{A_3}(a_3; \mathbf{b}_3)$ is transversally cut out. Let $(u, f) \in \mathcal{M}^t_{A_3}(a_3; \mathbf{b}_3)$ and assume the differential $d\Gamma^t_{(u,f)}$, which is a Fredholm operator of index 0 is not surjective. Then it has a cokernel of dimension $d > 0$. Furthermore, the image of the tangent space to the fiber under the differential $d\Gamma$ at the point $\Big(((u, f), (u, f), (v, g)), h\Big)$ is contained in a subspace of codimension $\geq 2d - 1$ in the tangent space to the fiber of the target space. This contradicts $\Gamma^{-1}(0) \cap \mathrm{pr}^{-1}(h)$ being transversely cut out. □

6.9. **Transversality in the semi-admissible case.**

Lemma 6.17. *Suppose L is admissible chord-semi-generic and $\Lambda = \mathrm{Ham}_0(L, \delta, s)$, then the bundle maps*

$$(\Gamma, \mathrm{pr})\colon \mathcal{W}'_{2,\epsilon,\Lambda}(\mathbf{c}) \to \mathcal{H}_{1,\epsilon,\Lambda}(D_m, T^{0,1}D_m \otimes \mathbb{C}^n),$$
$$(\Gamma, \mathrm{pr})\colon \widetilde{\mathcal{W}}'_{2,\epsilon,\Lambda}(\mathbf{c}) \to \mathcal{H}_{1,\epsilon,\Lambda}(D_m, T^{0,1}D_m \otimes \mathbb{C}^n)$$

are transverse to the 0*-section.*

Proof. We proceed as in the proof of Lemma 6.12. Let u be an element in the annihilator. The argument of Lemma 6.12 still applies up to the point where we conclude $\alpha|I_+$ equals 0. In the present setup not all Hamiltonian vector fields are allowed (see Definition 6.4). However, the ones that are allowed can be used exactly as in the proof of Lemma 6.12 to conclude the last $(n-1)$ components of u must vanish identically.

Since D_m is conformally equivalent to the unit disk Δ_m with m punctures on the boundary and since integrals as in (6.7) are conformally invariant, we have for any smooth compactly supported v with appropriate boundary conditions

$$0=\int_{\Delta_m}\langle\bar\partial v,u\rangle\,dA=\int_{\Delta_m}\langle v,\partial u\rangle\,dA+\int_{\partial\Delta_m}\langle u,e^{-i\theta}v\rangle\,d\theta. \tag{6.14}$$

As usual the first term in (6.14) vanishes and we find that u is orthogonal to $e^{i\theta}T_{w(e^{i\theta})}\Pi_{\mathbb{C}}(L)$.

Now the boundary of the holomorphic disk must cross the region $X=B(0,2+\epsilon)\setminus B(0,2)$, and the inverse image of this region contains an arc A in the boundary. The intersection between the tangent plane of $T_p\Pi_{\mathbb{C}}(L)$, $p\in X$ and the z_1-line equals 0 and the z_1-line is invariant under multiplication by $e^{i\theta}$. Hence the orthogonal complement of $e^{i\theta}T_{w(e^{i\theta})}\Pi_{\mathbb{C}}(L)$ intersects the z_1-line trivially as well (for $\theta\in A$). We conclude that the first component of u must vanish identically along A and by anti-analytic continuation vanish identically. It follows that u is identically zero. □

In analogy with Corollary 6.14 we get (with c denoting the degenerate Reeb chord of L)

Corollary 6.18. *For a dense open set of* $h\in\mathrm{Ham}_0(L,\delta,s)$, $\Gamma^{-1}(0)\cap\mathrm{pr}^{-1}(h)\subset\mathcal{W}'_{2,\epsilon,\Lambda}(\mathbf{c};A)$ *and* $\Gamma^{-1}(0)\cap\mathrm{pr}^{-1}(h)\subset\widetilde{\mathcal{W}}'_{2,\epsilon,\Lambda}(\mathbf{c};A)$ *are finite dimensional manifolds. If* $\mathbf{c}=ab_1\dots b_m$ *with* $a\neq c$ *then the dimensions are*

$$\mu(A)+\nu(a)-\sum_{r=1}^{m}(\nu(b_j)+\delta(b_j,c))+\max(0,m-2)\ \textit{and}$$

$$\mu(A)+\nu(a)-\sum_{r=1}^{m}(\nu(b_j))+\max(0,m-2),\ \textit{respectively.}$$

If $\mathbf{c}=cb_1\dots b_k$ *then the dimensions are*

$$\mu(B)+\nu(c)-\sum_{r=1}^{m}(\nu(b_j))+\max(0,m-2)\ \textit{and}$$

$$\mu(B)+\nu(c)+1-\sum_{r=1}^{m}(\nu(b_j))+\max(0,m-2),\ \textit{respectively.}$$

The same argument as in the proof of Theorem 6.15 gives

Theorem 6.19. *For a dense open set of* $h\in\mathrm{Ham}_0(L,\delta,s)$, $\Gamma^{-1}(0)\cap\mathrm{pr}^{-1}(h)\cap\mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{c})$ *and* $\Gamma^{-1}(0)\cap\mathrm{pr}^{-1}(h)\cap\widetilde{\mathcal{W}}_{2,\epsilon,\Lambda}(\mathbf{c})$ *are finite dimensional manifolds of dimensions given by the dimension formula in Corollary 6.18, provided this dimension is* ≤ 1 *if* $l(\mathbf{c})\geq 3$ *and* $\leq 1+(3-l(\mathbf{c}))$ *otherwise.*

Remark 6.20. Note that the expected dimension of the set of disks with dimension count in $\tilde{\mathcal{W}}_{2,\epsilon}$ equal to 1 in $\mathcal{W}_{2,\epsilon}$ is equal to $-k$ (or $-k+(3-l(\mathbf{c}))$ if $l(\mathbf{c}\leq 2)$), where k is the number of punctures mapping to the self-tangency Reeb-chord. Therefore for a dense open set in the space of chord semi-generic Legendrian submanifolds this space is empty. Since any disk with exponential decay at the self-tangency point has a neighborhood in $\mathcal{W}_{2,\epsilon}$, we see that generically such disks do not exist, provided their dimension count in $\tilde{\mathcal{W}}_{2,\epsilon}$ is as above.

6.10. **Enhanced transversality.** Let L be a (semi-)admissible submanifold. If $q \in L$ and $\zeta_0 \in \partial D_m$ then define

$$\mathcal{W}_{2,\epsilon}(\mathbf{c}, \zeta_0, p) = \{(w, f) \in \mathcal{W}_{2,\epsilon}(\mathbf{c})\colon (w, f)(\zeta_0) = p\}$$

and in the semi-admissible case also $\widetilde{\mathcal{W}}_{2,\epsilon}(\mathbf{c}, \zeta_0, p)$ in a similar way.

If $\mathrm{ev}_{\zeta_0}\colon \mathcal{W}_{2,\epsilon}(\mathbf{c}) \to L$ denotes the map $\mathrm{ev}_{\zeta_0}(w, f) = (w, f)(\zeta_0)$. Then ev_{ζ_0} is smooth and transverse to p (as is seen by using local coordinates on $\mathcal{W}_{2,\epsilon}(\mathbf{c})$). Moreover, $\mathrm{ev}_{\zeta_0}^{-1}(p) = \mathcal{W}_{2,\epsilon}(\mathbf{c}, p)$ and hence $\mathcal{W}_{2,\epsilon}(\mathbf{c}, p)$ is a closed submanifold of $\mathcal{W}_{2,\epsilon}(\mathbf{c})$ of codimension $\dim(L)$. Note that the tangent space $T_{(w,f)}\mathcal{W}_{2,\epsilon}(\mathbf{c}, p, \zeta_0)$ is the closed subspace of elements (v, γ) in the tangent space $T_{(w,f)}\mathcal{W}_{2,\epsilon}(\mathbf{c})$ which are such that $v\colon D_m \to \mathbb{C}^n$ satisfies $v(\zeta_0) = 0$.

We consider

$$\mathcal{W}_{2,\epsilon}(\mathbf{c}, p) = \bigcup_{\zeta \in \partial D_m} \mathcal{W}_{2,\epsilon}(\mathbf{c}, \zeta, p)$$

as a locally trivial bundle over ∂D_m. Local trivializations are given compositions with suitable diffeomorphisms which move the boundary point ζ a little.

We define perturbation spaces as the closed subspaces $\mathrm{Ham}^p(L, \delta, R) \subset \mathrm{Ham}(L, \delta, R)$ and $\mathrm{Ham}_0^p(L, \delta) \subset \mathrm{Ham}_0(L, \delta)$ of functions h such that $h(p) = 0$ and $Dh(p) = 0$. Thus, $\tilde{\Phi}_h$ fixes p. (Note that if p is the projection of a Reeb chord this is no *additional* restriction.) If Λ denotes one of these perturbation spaces we form the bundles

$$\mathcal{W}_{2,\epsilon,\Lambda}(\mathbf{c}, p) = \bigcup_{L_\lambda, \lambda \in \Lambda} \mathcal{W}_{2,\epsilon}(\mathbf{c}, p),$$

$$\widetilde{\mathcal{W}}_{2,\epsilon,\Lambda}(\mathbf{c}, p) = \bigcup_{L_\lambda, \lambda \in \Lambda} \mathcal{W}_{2,\epsilon}(\mathbf{c}, p)$$

with local coordinates as before.

As before let $'$ denote exclusion of exceptional holomorphic maps.

Lemma 6.21. *Assume that $p \in L$ has a neighborhood U such that $\Pi_{\mathbb{C}}(U)$ is real analytic. Then the bundle maps*

$$(\Gamma, \mathrm{pr})\colon \mathcal{W}'_{2,\epsilon,\Lambda}(\mathbf{c}, p) \to \mathcal{H}_{1,\epsilon,\Lambda}(D_m, T^{0,1^*}D_m \otimes \mathbb{C}^n)$$

$$(\Gamma, \mathrm{pr})\colon \widetilde{\mathcal{W}}'_{2,\epsilon,\Lambda}(\mathbf{c}, p) \to \mathcal{H}_{1,\epsilon,\Lambda}(D_m, T^{0,1^*}D_m \otimes \mathbb{C}^n) \tag{6.15}$$

are transverse to the 0*-section.*

Proof. The proof is the same as the proof of Lemma 6.12 in the admissible case and the same as that of Lemma 6.17 in the semi-admissible case provided the arcs γ and γ' used there do not contain the special point p. On the other hand, if one of these arcs does contain p we may shorten it until it does not. (The key point is that the condition that the Hamiltonian vanishes at a point does not destroy the approximation properties of the elements in the perturbation space for smooth functions supported away from this point). □

Corollary 6.22. *Let $n > 1$. For L in a Baire subset of the space of (semi-)admissible Legendrian n-submanifolds no rigid holomorphic disk passes through the end point of any Reeb chord of L.*

Note, when $n = 1$ this corollary is not true.

Proof. The proof of Theorem 6.15 shows that for a Baire set there are no exceptional holomorphic disks. The Sard-Smale theorem in combination with Lemma 6.21 implies that for a Baire subset of this Baire set the dimension of the space of rigid holomorphic disks with some point mapping to the end point of a specific Reeb chord is given by the Fredholm index of the operator $d\Gamma$ corresponding to Γ in (6.15). Since the source space of this operator is the sum of a copy of $\mathbb{R}$ (from the movement of ζ on the boundary) and a closed codimension $\dim L$ subspace of the source space of $d\Gamma$ in Lemma 6.12 which has minimal index for disks to appear generically, we see the index in the present case is too small. This implies that the subset is generically empty. Taking the intersection of these Baire subsets for the finite collection of Reeb chord endpoints of L we get a Baire subset with the required properties. □

Corollary 6.23. *If L is as in Corollary 6.22 then there are no rigid holomorphic disks with boundary on L which are nowhere injective on the boundary.*

Proof. Let $w\colon D_{m+1} \to \mathbb{C}^n$ represent a holomorphic disk with boundary on L. By Corollary 6.22 we may assume that no point in the boundary of ∂D_m maps to an intersection point of $\Pi_{\mathbb{C}}(L)$.

Assume that w has no injective point on the boundary and let the punctures of D_{m+1} map to the Reeb chords $(c_0, \dots, c_m)$ where the positive puncture maps to c_0. Let C be the holomorphic chain which is the closure of image $w(D_m)$ of w with local multiplicity 1 everywhere. Then

$$\operatorname{Area}(C) < \operatorname{Area}(w) \tag{6.16}$$

since close to the point in C most distant from the origin in $\mathbb{C}^n$, w has multiplicity at least two.

The corners of C is a subset S of $c_0^*, \dots, c_m^*$ and by integrating $\sum_j y_j dx_j$ along the boundary ∂C of C which lies in the exact Lagrangian $\Pi_{\mathbb{C}}(L)$ we find

$$\operatorname{Area}(C) = \mathcal{Z}(c_0) - \sum_{c_j^* \in S, j>0} \mathcal{Z}(c_j), \tag{6.17}$$

where the first term must be present (i.e. C must have a corner at c_0^*) since otherwise the area of C would be negative contradicting the fact that C is holomorphic. On the other hand

$$\operatorname{Area}(w) = \mathcal{Z}(c_0) - \sum_{j>0} \mathcal{Z}(c_j). \tag{6.18}$$

Hence

$$\operatorname{Area}(C) \geq \operatorname{Area}(w), \tag{6.19}$$

which contradicts (6.16). This contradiction finishes the proof. □

6.11. **Transversality in a split problem.** In this section we discuss transversality for disks, with one or two punctures, lying entirely in one complex coordinate plane. Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be an admissible Legendrian submanifold. Let $\Delta \subset \mathbb{R}^2$ denote the standard simplex. Let Δ_1 (Δ_2) be the subsets of $\mathbb{R}^2$ which is bounded by $\partial\Delta$, smoothened at two (one) of its corners. Let $(z_1, \dots, z_n)$ be coordinates on $\mathbb{C}^n$. Let $\pi_i\colon \mathbb{C}^n \to \mathbb{C}$ denote projection to the i-th coordinate and let $\hat{\pi}_i\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ denote projection to the Hermitian complement of the z_i-line. Finally, if $\gamma(t)$, $t \in I \subset \mathbb{R}$ is a one parameter family of lines then we let $\int_\gamma d\theta$ denote the (signed) angle $\gamma(t)$ turns as t ranges over I.

Lemma 6.24. *Let $(u, h) \in \mathcal{W}_2(ab; A)$, $\mu(A) + |a| - |b| = 1$, be a holomorphic disk with boundary on L such that $\hat{\pi}_1 \circ u$ is constant and such that $\pi_1 \circ u = f \circ g$, where $g\colon \Delta_2 \to D_2$ is a diffeomorphism and $f\colon \Delta_2 \to \mathbb{C}$ is an immersion. Furthermore, if t_1, t_2 are coordinates along components of ∂D_2, assume that the paths $\Gamma(t) = d\Pi_{\mathbb{C}}(T_{(u,h)(t_j)}L)$ of Lagrangian subspaces are split: $\Gamma(t_j) = \gamma(t_j) \times \hat{V}_j$, where $\gamma(t) \subset \mathbb{C}$ is a (real line) and $\hat{V}_j \subset \mathbb{C}^{n-1}$, $j = 1, 2$, are transverse Lagrangian subspaces. Then $d\Gamma_{(u,h)}$ is surjective. (In other words, (u, h) is transversely cut out).*

Proof. The Fredholm index of $d\Gamma$ at (u,h) equals 1. If v is the vector field on D_2 which generates the 1-parameter family of conformal automorphisms of D_2 (the vector field ∂_τ in coordinates $\tau + it \in \mathbb{R}\times[0,1]$ on D_2) then $\xi = du\cdot v$ lies in the kernel of $d\Gamma$ and $d\hat{\pi}\cdot\xi = 0$.

Since the boundary conditions are split we may consider them separately. It follows from Section 5.4 that the $\hat{\pi}_1 d\Gamma$ with boundary conditions given by the two transverse Lagrangian subspaces $\hat{V}_1$ and $\hat{V}_2$ has index 0, no kernel and no cokernel.

Let θ_1 and θ_2 be the interior angles at the corners of the immersion f. Since $f(\partial\Delta_2)$ bounds an immersed disk we have

$$\int_{\gamma_1} d\theta + \int_{\gamma_2} d\theta + (\pi - \theta_1) + (\pi - \theta_2) = 2\pi.$$

If $\eta_1 = \pi_1\circ\eta$, where η is in the kernel of $d\Gamma$ then, thinking of D_2 as $\mathbb{R}\times[0,1]$, we find that, asymptotically, for some integers $n_1\ge 0$ and $n_2 \ge 0$

$$\eta_1(\tau+it) = \begin{cases} c_1 e^{-(\theta_1+n_1\pi)(\tau+it)}, & \text{for } \tau\to+\infty,\\ c_2 e^{(\theta_2+n_2\pi)(\tau+it)}, & \text{for } \tau\to-\infty,\end{cases}$$

where c_1 and c_2 are real constants. Cutting D_2 off at $|\tau| = M$ for some sufficiently large M we thus find a solution of the classical Riemann-Hilbert problem with Maslov-class

$$\frac{1}{\pi}\Big(\theta_1+\theta_2-\theta_1-\theta_2-(n_1+n_2)\pi\Big).$$

Since the classical Riemann-Hilbert problem has no solution if the Maslov class is negative and exactly one if it is 0 we see that the solution $\xi = \xi_1$ produced above is unique up to multiplication with real constants. □

Lemma 6.25. *Let $(u,h)\in\mathcal{W}_2(a;A)$, $\mu(A)+|a| = 1$, be a holomorphic disk with boundary on L such that $\hat{\pi}_1\circ u$ is constant and such that $\pi_1\circ u = f\circ g$, where $g\colon \Delta_1\to D_2$ is a diffeomorphism and $f\colon\Delta_1\to\mathbb{C}$ is an immersion. Furthermore, if t is a coordinate along ∂D_1, assume that the path $\Gamma(t) = d\Pi_{\mathbb{C}}(T_{(u,h)(t_j)}L)$ of Lagrangian subspaces is split: $\Gamma(t) = \gamma_1(t_j)\times\gamma_2(t)\times\cdots\times\gamma_n(t)$, where $\gamma_j(t)\subset\mathbb{C}$ is a (real line) such that*

$$\int_{\gamma_j} d\theta < 0, \text{ for } 2\le j\le n.$$

Then $d\Gamma_{(u,h)}$ is surjective.

Proof. The proof is similar to the one just given. Using asymptotics and the classical Riemann-Hilbert problem it follows that the kernel of $d\Gamma$ is spanned by two linearly independent solutions ξ^j, $j = 1,2$, with $\hat{\pi}_1\xi^j = 0$. □

6.12. **Auxiliary tangent-like spaces in the semi-admissible case.** Let L be a chord semi-admissible Legendrian submanifold and assume that L lies in the open subset of such manifolds where the moduli-space of rigid holomorphic disks with corners at $\mathbf{c}$ is 0-dimensional (and compact by Theorem 8.2). Now if (w,f) is a holomorphic disk with boundary on L then by Lemma 6.17 we know that the operator

$$d\Gamma\colon T_{(w,f)}\widetilde{\mathcal{W}}_{2,\epsilon}(\mathbf{c}) \to \mathcal{H}_{1,\epsilon}(D_m, T^*D_m\otimes\mathbb{C}^n) \tag{6.20}$$

is surjective.

For any (w,f) with $m+1$ punctures which maps the punctures $p_1,\dots p_k$ to the self tangency Reeb chord of L_h let $\hat{\epsilon}\in[0,\infty)^{m+1-k}\times(-\delta,0)^k$, where $\delta>0$ is small compared to the complex angle of the self tangency Reeb chord and the components of $\hat{\epsilon}$ which are negative correspond to the punctures $p_1,\dots,p_k$. Define the *tangent-like space*

$$T_{(w,f,h)}\mathcal{W}_{2,\hat{\epsilon}}(\mathbf{c})$$

as the linear space of elements (v,γ) where $\gamma \in T_\kappa \mathcal{C}_{m+1}$ and where $v \in \mathcal{H}_{2,\hat\epsilon}(D_{m+1},\mathbb{C}^n)$ satisfies

$$v(\zeta) \in \Pi_{\mathbb{C}}(T_{(w,f)(\zeta)}L) \text{ for all } \zeta \in \partial D_m,$$

$$\int_{\partial D_m} \langle \bar\partial v, u\rangle \, ds = 0 \text{ for all } u \in \mathbb{C}_0^\infty(\partial D_m, \mathbb{C}^n).$$

and consider the linear operator

$$d\hat\Gamma(v,\gamma) = \bar\partial_\kappa v + i \circ dw \circ \gamma. \tag{6.21}$$

The index of this Fredholm operator equals that of the operator in (6.20) and moreover by asymptotics of solutions to these equations (close to the self-tangency Reeb chord we can use the same change of coordinates in the first coordinate as in the non-linear case, see Section 3.6 to determine the behavior of solutions) we find that the kernels are canonically isomorphic. Thus, since the operator in (6.20) is surjective so is the operator in (6.21).

7. Gluing theorems

In this section we prove the gluing theorems used in Sections 1.3 and 1.5. In Section 7.1 we state the theorems. Our general method of gluing curves is the standard one in symplectic geometry. However, some of our specific gluings require a significant amount of analysis. We first "preglue" the pieces of the broken curves together. For the stationary case this is done in Section 7.5, for the handle slide case in Section 7.6, and for the self-tangency case in Sections 7.12 and 7.17. We then apply Picard's Theorem, stated in Section 7.2. Picard's Theorem requires a sequence of uniformly bounded right inverses of the linearized $\bar\partial$ map. We prove the bound for the stationary case in Section 7.8, for the handle slide case in Section 7.9, and for the self-tangency case in Sections 7.15 and 7.21. Picard's Theorem also requires a bound on the non-linear part of the expansion of $\bar\partial$, which we do in Section 7.22. To handle disks with too few boundary punctures, we show in Sections 7.7 through 7.7.2, how by marking boundary points the disks can be thought of as sitting inside a moduli space of disks with many punctures.

Recall the following notation. Bold-face letters will denote ordered collections of Reeb chords. If $\mathbf{c}$ denotes a non-empty ordered collection $(c_1,\dots,c_m)$ of Reeb chords then we say that the *length* of $\mathbf{c}$ is m. We say that the length of the empty ordered collection is 0. Let $\mathbf{c}^1,\dots,\mathbf{c}^r$ be an ordered collection of ordered collections of Reeb chords. Let the length of $\mathbf{c}^j$ be $l(j)$ and let $\mathbf{a}=(a_1,\dots,a_k)$ be an ordered collection of Reeb chords of length $k>0$. Let $S=\{s_1,\dots,s_r\}$ be r distinct integers in $\{1,\dots,k\}$. Define the ordered collection $\mathbf{a}_S(\mathbf{c}^1,\dots,\mathbf{c}^r)$ of Reeb chords of length $k-r+\sum_{j=1}^r l(j)$ as follows. For each index $s_j \in S$ remove a_{s_j} from the ordered collection $\mathbf{a}$ and insert at its place the ordered collection $\mathbf{c}^j$.

Recall that if a is a Reeb chord and $\mathbf{b}$ is a collection of Reeb chords of a Legendrian submanifold, then $\mathcal{M}_A(a;\mathbf{b})$ denotes the moduli space of holomorphic disks with boundary on L, punctures mapping to $(a,\mathbf{b})$, and boundary in L which after adding the chosen capping paths represents the homology class $A \in H_1(L)$. After Theorem 6.15 we know that if the length of $\mathbf{b}$ is at least 2 then $\mathcal{M}_A(a;\mathbf{b})$ is identified with the inverse image of the regular value 0 of the $\bar\partial$-map Γ in Section 4.7. If the length of $\mathbf{b}$ is 0 or 1 then $\mathcal{M}_A(a;\mathbf{b})$ is identified with the quotient of $\Gamma^{-1}(0)$ under the group of conformal reparameterizations of the source of the holomorphic disk.

Similarly if L_λ, $\lambda \in \Lambda$ is a 1-parameter family of chord generic Legendrian submanifolds we write $\mathcal{M}^\Lambda_A(a;\mathbf{b})$ for the parameterized moduli space of rigid holomorphic disks with boundary in L_λ, and punctures at $(a(\lambda),\mathbf{b}(\lambda))$, $\lambda\in\Lambda$. We also write $\mathcal{M}^\lambda_A(a,\mathbf{b})$ to denote the moduli space for a fixed L_λ, $\lambda \in \Lambda$.

Finally if $K \subset \mathbb{C}^n$ and $\delta > 0$ then $B(K,\delta)$ denotes the subset of all points in $\mathbb{C}^n$ of distance less than δ from K.

7.1. **The Gluing Theorems.** In this section we state the various gluing theorems.

7.1.1. *Stationary gluing.* Let L be an admissible Legendrian submanifold. Recall that a holomorphic disk with boundary on L is defined as a pair of functions (u,f), where $u\colon D_m \to \mathbb{C}^n$ and $f\colon \partial D_m \to \mathbb{R}$. Below we will often drop the function f from the notation and speak of the holomorphic disk u. Let $\mathcal{M}_A(a;\mathbf{b})$ and $\mathcal{M}_C(c;\mathbf{d})$ be moduli spaces of rigid holomorphic disks, where $\mathbf{b}$ has length m, $1 \le j \le m$, and $\mathbf{d}$ has length l.

Theorem 7.1. *Assume that the j-th Reeb chord in $\mathbf{b}$ equals c. Then there exists $\delta > 0$, $\rho_0 > 0$ and an embedding*

$$\mathcal{M}_A(a;\mathbf{b}) \times \mathcal{M}_C(c;\mathbf{d}) \times [\rho_0,\infty) \to \mathcal{M}_{A+C}(a;\mathbf{b}_{\{j\}}(\mathbf{d}));$$
$$(u,v,\rho) \mapsto u\,\sharp_\rho v,$$

such that if $u \in \mathcal{M}_A(a;\mathbf{b})$ and $v \in \mathcal{M}_C(c;\mathbf{d})$ and the image of $w \in \mathcal{M}_{A+C}(a;\mathbf{b}_j(\mathbf{d}))$ lies inside $B(u(D_{m+1}) \cup v(D_{l+1});\delta)$ then $w = u\,\sharp_\rho v$ for some $\rho \in [\rho_0,\infty)$.

Proof. The theorem follows from Lemmas 7.7, 7.13, and 7.22 and Proposition 7.6. □

7.1.2. *Handle slide gluing.* Let L_λ, $\lambda \in (-1,1) = \Lambda$ be a 1-parameter family of admissible Legendrian submanifolds such that

$$\mathcal{M}^\Lambda_A(a;\mathbf{b}) = \mathcal{M}^0_A(a;\mathbf{b}) \tag{7.1}$$

is a transversely cut out handle slide disk, represented by a map $u\colon D_{m+1} \to \mathbb{C}^n$ (the length of $\mathbf{b}$ is m) and such that for all $\lambda \in \Lambda$, all moduli-spaces of rigid disks with boundary on L_λ are transversely cut out.

We formulate two gluing theorems in this case. They differ in the following way. In Theorem 7.2 we consider what happens when the positive punctures of rigid disks are glued to negative punctures of the handle slide disk. In Theorem 7.3 we consider what happens when the positive puncture of the handle-slide disk is glued to negative punctures of a rigid disk.

Theorem 7.2. *Assume that $\mathbf{b}$ in (7.1) has positive length and has c in its j-th position. Let $\mathcal{M}^0_C(c;\mathbf{d})$ be a moduli space of rigid holomorphic disks, where the length of $\mathbf{d}$ is k. Then there exist $\delta > 0$, $\rho_0 > 0$, and an embedding*

$$\mathcal{M}^0_A(a;\mathbf{b}) \times \mathcal{M}^0_C(c;\mathbf{d}) \times [\rho_0,\infty) \to \mathcal{M}^\Lambda_{A+C}(a;\mathbf{b}_{\{j\}}(\mathbf{d})),$$
$$(u,v,\rho) \mapsto u\,\sharp_\rho v,$$

such that if $v \in \mathcal{M}^0_C(c;\mathbf{d})$ and the image of $w \in \mathcal{M}^\Lambda_{A+C}(a;\mathbf{b}_{\{j\}}(\mathbf{d}))$ lies inside $B(u(D_{m+1}) \cup v(D_{k+1});\delta)$ then $w = u\,\sharp_\rho v$ for some $\rho \in [\rho_0,\infty)$.

Proof. The theorem follows from Lemmas 7.8, 7.14, and 7.23 and Proposition 7.6. □

Theorem 7.3. *Let $\mathcal{M}^0_C(c;\mathbf{d})$ be a moduli space of rigid holomorphic disks, where $\mathbf{d} = (d_1,\dots,d_l)$. Let $S = \{s_1,\dots,s_r\} \subset \{1,\dots,l\}$ be such that $d_{s_j} = a$ for all $s_j \in S$. Then there exist $\rho_0 > 0$, $\delta > 0$, and an embedding*

$$\mathcal{M}^0_C(c;\mathbf{d}) \times \Pi_S \mathcal{M}^0_A(a;\mathbf{b}) \times [\rho_0,\infty) \to \mathcal{M}^\Lambda_{C+r\cdot A}(c;\mathbf{d}_S(\mathbf{b},\dots,\mathbf{b})),$$
$$(v,\underbrace{u,\dots,u}_{r},\rho) \mapsto v\,\sharp^S_\rho u,$$

such that if $v \in \mathcal{M}^\Lambda_C(c;\mathbf{d})$ and the image of $w \in \mathcal{M}^\Lambda_{C+r\cdot A}(c;\mathbf{d}_S(\mathbf{b},\dots,\mathbf{b}))$ lies inside $B(v(D_{l+1}) \cup u(D_{m+1}) \cup \dots \cup u(D_{m+1}));\delta)$ then $w = v\,\sharp^S_\rho u$ for some $\rho \in [\rho_0,\infty)$.

Proof. The theorem follows from Lemmas 7.9, 7.15, and 7.23 and Proposition 7.6. □

7.1.3. *Self tangency shortening and self tangency gluing.* Let L_λ, $\lambda \in (-1,1) = \Lambda$ be an admissible 1-parameter family of Legendrian submanifolds such that L_0 is semi-admissible with self-tangency Reeb chord a. For simplicity (see Section 2) we assume that all Reeb chords outside a neighborhood of a remain fixed under Λ. We take Λ so that if $\lambda > 0$ then $L_{-\lambda}$ has two new-born Reeb-chords a^+ and a^-, where $\mathcal{Z}(a^+) > \mathcal{Z}(a^-)$. Assume that all moduli spaces of rigid holomorphic disks with boundary on L_λ are transversely cut out for all fixed $\lambda \in \Lambda$, that for all $\lambda \in \Lambda$, there are no disks with negative formal dimension, and that all rigid disks with a puncture at a satisfy the non-decay condition of Lemma 3.6 (see Remark 6.20).

Theorem 7.4. *Let $\Lambda^- = (-1,0)$. Let $\mathcal{M}^0_A(a,\mathbf{b})$ be a moduli space of rigid holomorphic disks where the length of $\mathbf{b}$ is l. Then there exist $\rho_0 > 0$, $\delta > 0$ and a local homeomorphism*

$$\mathcal{M}^0_A(a;\mathbf{b}) \times [\rho_0,\infty) \to \mathcal{M}^{\Lambda^-}_A(a^+;\mathbf{b});$$
$$(u,\rho) \mapsto \sharp_\rho u,$$

such that if $u \in \mathcal{M}^0_A(a;\mathbf{b})$ and the image of $w \in \mathcal{M}^{\Lambda^-}_A(a^+;\mathbf{b})$ lies inside $B(u(D_{l+1});\delta)$ then $w = \sharp_\rho u$ for some $\rho \in [\rho_0,\infty)$.

Let $\mathcal{M}^0_C(c,\mathbf{d})$ be a moduli space of rigid holomorphic disks where the length of $\mathbf{d}$ is m. Let $S \subset \{1,\dots,m\}$ be the subset of positions of $\mathbf{d}$ where the Reeb chord a appears (to avoid trivialities, assume $S \neq \emptyset$). Then there exists $\rho_0 > 0$ and $\delta > 0$ and a local homeomorphism

$$\mathcal{M}^0_C(c,\mathbf{d}) \times [\rho_0,\infty) \to \mathcal{M}^{\Lambda^-}_C(c,\mathbf{d}_S(a^-));$$
$$(u,\rho) \mapsto \sharp_\rho u,$$

such that if $u \in \mathcal{M}^0_C(c;\mathbf{d})$ and the image of $w \in \mathcal{M}^{\Lambda^-}_C(c;\mathbf{d}_S(a^-))$ lies inside $B(u(D_{m+1});\delta)$ then $w = \sharp_\rho u$ for some $\rho \in [\rho_0,\infty)$.

Proof. Consider the first case, the second follows by a similar argument. Applying Proposition 7.6 and Lemmas 7.16, 7.17 and 7.24 we find a homeomorphism $\mathcal{M}^0_A(a;\mathbf{b}) \to \mathcal{M}^{\lambda_-}_A(a^+,\mathbf{b})$ for $\lambda^- < 0$ small enough. The proof of Corollary 6.14 implies that $\mathcal{M}_{\Lambda^-}(a^+,\mathbf{b})$ is a 1-dimensional manifold homeomorphic to $\mathcal{M}_{\lambda_-}(a;\mathbf{b}) \times \Lambda_-$, the theorem follows. □

Theorem 7.5. *Let $\Lambda^+ = (0,1)$ and let $\mathcal{M}^0_{A_1}(a;\mathbf{b}^1),\dots,\mathcal{M}^0_{A_r}(a;\mathbf{b}^r)$ and $\mathcal{M}^0_C(c;\mathbf{d})$ be a moduli spaces of rigid holomorphic disks where the length of $\mathbf{b}^j$ is $l(j)$, and the length of $\mathbf{d}$ is m. Let $S \subset \{1,\dots,m\}$ be the subset of positions of $\mathbf{d}$ where the Reeb chord a appears and assume that S contains r elements. Then there exists $\delta > 0$, $\rho_0 > 0$ and an embedding*

$$\mathcal{M}^0_C(c;\mathbf{d}) \times \Pi_{j=1}^r \mathcal{M}^0_{A_j}(a;\mathbf{b}^j) \times [\rho_0,\infty) \to \mathcal{M}^{\Lambda^+}_{C+\sum_j A_j}(c;\mathbf{d}_S(\mathbf{b}^1,\dots,\mathbf{b}^r));$$
$$(v,u_1,\dots,u_r,\rho) \mapsto v\,\sharp_\rho(u_1,\dots,u_r),$$

such that if $v \in \mathcal{M}^0_C(c;\mathbf{d})$ and $u_j \in \mathcal{M}^0_{A_j}(a;\mathbf{b}^j)$, $j = 1,\dots,r$ and the image of $w \in \mathcal{M}^{\Lambda^+}_{C+\sum_j A_j}(c;\mathbf{d}_S(\mathbf{b}^1,\dots,\mathbf{b}^r))$ lies inside $B(v(D_{m+1}) \cup u_1(D_{l(1)+1}) \cup \dots \cup u_r(D_{l(r)+1}));\delta)$ then $w = v\,\sharp_\rho(u_1,\dots,u_r)$ for some $\rho \in [\rho_0,\infty)$.

Proof. Apply Proposition 7.6 and Lemmas 7.19, 7.21, and 7.25 and reason as above. □

7.2. **Floer's Picard lemma.** The proofs of the theorems stated in the preceding subsections are all based on the following.

Proposition 7.6. *Let $f\colon B_1 \to B_2$ be a smooth map between Banach spaces which satisfies*

$$f(v) = f(0) + df(0)v + N(v),$$

where $df(0)$ is Fredholm and has a right inverse G satisfying

$$\|GN(u) - GN(v)\| \le C(\|u\| + \|v\|)\|u - v\|,$$

for some constant C. Let $B(0,\epsilon)$ denote the ϵ-ball centered at $0 \in B_1$ and assume that

$$\|Gf(0)\| \le \frac{1}{8C}.$$

Then for $\epsilon < \frac{1}{4C}$, the zero-set of $f^{-1}(0) \cap B(0,\epsilon)$ is a smooth submanifold of dimension $\dim(\mathrm{Ker}(df(0)))$ *diffeomorphic to the ϵ-ball in* $\mathrm{Ker}(df(0))$.

Proof. See [12]. □

In our applications of Proposition 7.6 the map f will be the $\bar\partial$-map, see Section 4.7.

7.3. **Notation and cut-off functions.** To simplify notation, we deviate slightly from our standard notation for holomorphic disks. We use the convention that the neighborhood E_{p_0} of the positive puncture p_0 in the source D_m of a holomorphic disk (u, f) will be parameterized by $[1, \infty) \times [0, 1]$ and that neighborhoods of negative punctures E_{p_j}, $j \ge 1$ are parameterized by $(-\infty, -1] \times [0, 1]$.

In the constructions and proofs below we will use certain cut-off functions repeatedly. Here we explain how to construct them. Let $K > 0$, $a < b$, and let $\phi\colon [a, b + K + 1] \to [0, 1]$ be a smooth function which equals 1 on $[a, b]$ and equals 0 in $[b + K, b + K + 1]$. It is easy to see there exists such functions with $|D^k\phi| = \mathcal{O}(K^{-k})$ for $k = 1, 2$. Let $\epsilon > 0$ be small. Let $\psi\colon [0, 1] \to \mathbb{R}$ be a smooth function such that $\psi(0) = \psi(1) = 0$, $\psi'(0) = \psi'(1) = 1$, with $|\psi| \le \epsilon$. We will use cut-off functions $\alpha\colon [a, b + K + 1] \times [0, 1] \to \mathbb{C}$ of the form

$$\alpha(\tau + it) = \phi(\tau) + i\psi(t)\phi'(\tau).$$

Note that $\alpha|\partial([a, b + K + 1] \times [0, 1])$ is real-valued and $\bar\partial\alpha = 0$ on $\partial([a, b + K + 1] \times [0, 1])$. Also, $|D^k\alpha| = \mathcal{O}(K^{-1})$ for $k = 1, 2$.

7.4. **A gluing operation.** Let L be a chord generic Legendrian submanifold. Let $(u, f) \in \mathcal{W}_{2,\epsilon}(a, \mathbf{b})$ where $\mathbf{b} = (b_1, \dots, b_m)$ and let $(v_j, h_j) \in \mathcal{W}_{2,\epsilon}(b_j, \mathbf{c}^j)$, $j \in S \subset \{1, \dots, m\}$. Denote the punctures on D_{m+1} by p_j, $j = 0, \dots, m$ and the positive puncture on $D_{l(j)+1}$ by q_j.

Let g^σ, $\sigma \in [0, 1]$ be a 1-parameter family of metrics as in Section 4.3. Then for $M > 0$ large enough there exists unique functions

$$\xi\colon E_{p_j}[-M] \to T_{b_j^*}\mathbb{C}^n$$
$$\eta_j\colon E_{q_j}[M] \to T_{b_j^*}\mathbb{C}^n,$$

such that

$$\exp^t_{b_j^*}(\xi(\tau + it)) = u(\tau + it),$$
$$\exp^t_{b_j^*}(\eta_j(\tau + it)) = v_j(\tau + it),$$

where $\exp^\sigma$ denotes the exponential map of the metric g^σ. Note that by our special choice of metrics the functions ξ and η are tangent to $\Pi_{\mathbb{C}}(L)$ and holomorphic on the boundary.

For large $\rho > 0$, let $D_r^S(\rho)$, $r = 1 + m + \sum_{j\in S}(l(j) - 1)$ be the disk obtained by gluing to the end of

$$D_{m+1} \setminus \bigcup_{j\in S} E_{p_j}[-\rho]$$

corresponding to p_j a copy of

$$D_{l(j)+1} - E_{q_j}[\rho]$$

by identifying $\rho\times[0,1] \subset E_{p_j}$ with $-\rho\times[0,1] \subset E_{q_j}$, for each $j \in S$. Note that the metrics (and the complex structures κ_1 and $\kappa_2(j)$) on D_{m+1} and $D_{l(j)+1}$ glue together to a unique metric (and complex structure κ_ρ) on $D_r^S(\rho)$. We consider $D_{m+1} \setminus \bigcup_{j\in S} E_{p_j}[-\rho]$ and $D_{l(j)+1} \setminus E_{q_j}[\rho]$ as subsets of $D_r^S(\rho)$.

For $j \in S$, let $\Omega_j \subset D_r^S(\rho)$ denote the subset

$$E_{q_j}[\rho-2,\rho] \cup E_{p_j}[-\rho,-\rho+2] \approx [-2,2]\times[0,1]$$

of $D_r^S(\rho)$. Let $z = \tau + it$ be a complex coordinate on Ω_j and let $\alpha^\pm \colon \Omega_j \to \mathbb{C}$ be cut-off functions which are real valued and holomorphic on the boundary and with $\alpha^+ = 1$ on $[-2,-1]\times[0,1]$, $\alpha^+ = 0$ on $[0,2]\times[0,1]$, $\alpha^- = 1$ on $[1,2]\times[0,1]$, and $\alpha^- = 0$ on $[-2,0]\times[0,1]$. Define the function $\Sigma_\rho^S(u, v_1, \dots, v_r) \colon D_r \to \mathbb{C}^n$ as

$$\Sigma_\rho^S(u, v_1, \dots, v_r)(\zeta) = \begin{cases} v_j(\zeta), & \zeta \in D_{l(j)+1} \setminus E_{q_j}[\rho-2], \\ u(\zeta), & \zeta \in D_{m+1} \setminus \bigcup_{j\in S} E_{p_j}[-\rho+2], \\ \exp^t_{b_j^*}(\alpha^-(z)\xi_j(z) + \alpha^+(z)\eta_j(z)), & z = \tau + it \in \Omega_j. \end{cases}$$

7.5. **Stationary pregluing.** Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be an admissible Legendrian submanifold. Let $u\colon D_{m+1} \to \mathbb{C}^n$ be a holomorphic disk with its j-th negative puncture p mapping to c, $(u,f) \in \mathcal{W}_2(a, \mathbf{b})$, and let $v\colon D_{l+1} \to \mathbb{C}^n$ be a holomorphic disk with the positive puncture q mapping to c, $(v,h) \in \mathcal{W}_2(c, \mathbf{d})$. Define

$$w_\rho = \Sigma_\rho^{\{j\}}(u, v). \tag{7.2}$$

Lemma 7.7. *The function w_ρ satisfies $w_\rho \in \mathcal{W}_2(a, \mathbf{b}_{\{j\}}(\mathbf{d}))$ and*

$$\|\bar\partial w_\rho\|_1 = \mathcal{O}(e^{-\theta\rho}), \tag{7.3}$$

where θ is the smallest complex angle at the Reeb chord c. In particular, $\|\bar\partial w_\rho\|_1 \to 0$ as $\rho \to \infty$.

Proof. The first statement is trivial. Outside Ω_j, w_ρ agrees with u or v which are holomorphic. Thus it is sufficient to consider the restriction of w_ρ to Ω_j. To derive the necessary estimates we Taylor expand the exponential map at c^*. To simplify notation we let $c^* = 0 \in \mathbb{C}^n$ and let $\xi \in \mathbb{R}^{2n}$ be coordinates in $T_0\mathbb{C}^n$ and $x \in \mathbb{R}^{2n}$ coordinates around $0 \in \mathbb{C}^n$. Then

$$\exp_0^t(\xi) = \xi - \Gamma^k_{ij}(t)\xi^i\xi^k\partial_k + \mathcal{O}(|\xi|^3). \tag{7.4}$$

This implies the inverse of the exponential map has Taylor expansion

$$\xi = x + \Gamma^k_{ij}(t)x^ix^k\partial_k + \mathcal{O}(x^3). \tag{7.5}$$

From (7.4) and (7.5) we get

$$\exp_0^t(\alpha^+\xi_j) = \alpha^+u + ((\alpha^+)^2 - \alpha^+)\Gamma^k_{ij}(t)u^iu^j\partial_k + \mathcal{O}(|u|^3) \tag{7.6}$$

and a similar expression for $\alpha^-\eta_j$ in terms of v_j. Lemma 3.6 implies that u and Du are $\mathcal{O}(e^{-\theta\rho})$ in $E_{p_j}[\rho]$, which together with (7.6) implies (7.3). □

7.6. **Handle slide pregluing.** Let L_λ, $\lambda \in (-1,1) = \Lambda$ be a 1-parameter family of chord-generic Legendrian submanifolds such that

$$\mathcal{M}^{\Lambda}_{A}(a;\mathbf{b}) = \mathcal{M}^{0}_{A}(a;\mathbf{b})$$

is a transversally cut out handle slide disk, represented by a map $u\colon D_{m+1} \to \mathbb{C}^n$ with punctures $q, p_1, \dots, p_m$.

We first consider the case studied in Theorem 7.2. Let $v\colon D_{l+1} \to \mathbb{C}^n$ represent an element in $\mathcal{M}^0_C(c,\mathbf{d})$. Let

$$w^0_\rho = \Sigma^{\{j\}}_\rho(u,v),$$

and define

$$w_\rho = w^0_\rho[\rho],$$

see Section 4.5. (We cut off w^0_ρ close to its punctures in order to be able to use local coordinates as in Lemma 4.14 in a neighborhood of w_ρ.)

Lemma 7.8. *The function w_ρ satisfies $(w_\rho, 0) \in \mathcal{W}_{2,\Lambda}(a, \mathbf{b}_j(\mathbf{d}))$ and*

$$\|\bar\partial w_\rho\|_1 = \mathcal{O}(e^{-\theta\rho}), \tag{7.7}$$

where $\theta > 0$ is the smallest complex angle at any Reeb chord of L_0. In particular $\|\bar\partial w_\rho\| \to 0$ as $\rho \to 0$.

Proof. We must consider the gluing regions and the effect of making w^0_ρ constant close to all punctures. The argument in the proof of Lemma 7.7 gives the desired estimate. □

For Theorem 7.3, we need to handle a few more punctures. Let $v\colon D_{l+1} \to \mathbb{C}^n$ be an element in $\mathcal{M}^0_C(c,\mathbf{d})$ where the elements in $S = \{s_1, \dots, s_r\}$ are indices of punctures which map to a. Let

$$w^{0,S}_\rho = \Sigma^S_\rho(v, u, \dots, u)$$

and define

$$w^S_\rho = w^{0,S}_\rho[\rho]$$

Lemma 7.9. *The function w^S_ρ satisfies $(w^S_\rho, 0) \in \mathcal{W}_{2,\Lambda}(c, \mathbf{d}_S(\mathbf{b}))$ and*

$$\|\bar\partial w^S_\rho\|_1 = \mathcal{O}(e^{-\theta\rho}), \tag{7.8}$$

where $\theta > 0$ is the smallest complex angle at any Reeb chord of L_0. In particular $\|\bar\partial w_\rho\| \to 0$ as $\rho \to 0$.

Proof. See the proof of Lemma 7.8. □

7.7. **Marked points.** In order to treat disks with less than three punctures (i.e., disks with conformal reparameterizations) in the same way as disks with more than three punctures we introduce special points which we call *marked points* on the boundary. When a disks with few punctures and marked points are glued to a disk with many punctures there arises a disk with many punctures and marked points and we must study also that situation.

Remark 7.10. Below we will often write simply $\mathcal{W}_{2,\epsilon}$ to denote spaces like $\mathcal{W}_{2,\epsilon}(\mathbf{c})$, dropping the Reeb chords from the notation.

Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a (semi-)admissible Legendrian submanifold and let $u\colon D_m \to \mathbb{C}^n$ represent $(u,f) \in \mathcal{W}_{2,\epsilon}(\kappa)$ where κ is a fixed conformal structure on D_m. Let $U_r \subset \Pi_{\mathbb{C}}(L)$, $r = 1, \dots, k$ be disjoint open subsets where $\Pi_{\mathbb{C}}(L)$ is real analytic and let $q_r \in \partial D_m$ be points such that $u(q_r) \in U_r$ and $du(q_r) \neq 0$. After possibly shrinking U_r we may biholomorphically identify $(\mathbb{C}^n, U_r, u(q_r))$ with $(\mathbb{C}^n, V \subset \mathbb{R}^n, 0)$. Let $(x_1 + iy_1, \dots, x_n + iy_n)$ be coordinates on $\mathbb{C}^n$ and assume these coordinates are chosen so $du(q_r) \cdot v_0 = \partial_1$, where $v_0 \in T_{q_r} D_m$ is a unit vector tangent to the boundary. Let $H_r \subset \mathbb{R}^n$ denote an open neighborhood of 0 in the submanifold $\{x_1 = 0\}$.

Let S denote the cyclically ordered set of points $S = \{p_1, \dots, p_m, q_1, \dots, q_k\}$ where $p_i \in \partial D_m$ are the punctures of D_m. Fix three points $s_1, s_2, s_3 \in \{p_1, p_2, p_3, q_1, \dots, q_k\}$ then the positions of the other points in S parameterizes the conformal structures on Δ_{m+k}. As in Section 4.6, we pick vector fields $\tilde{v}_j$, $j = 1, \dots, m+k-3$ supported around the non-fixed points in S. Given a conformal structure on Δ_{m+k} we endow it with the metric which makes a neighborhood of each puncture p_j look like the strip and denote disks with such metrics $\tilde{D}_{m,k}$.

If (u, f), $u \colon \tilde{D}_{m,k} \to \mathbb{C}^n$ and $f \colon \partial\tilde{D}_{m,k} \to \mathbb{R}$ are maps and $\tilde{\kappa}$ is a conformal structure on $\tilde{D}_{m+k}$ then forgetting the marked points $q_1, \dots, q_k$ we may view the maps as defined on D_m and the conformal structure $\tilde{\kappa}$ gives a conformal structure κ on D_m. Note though that the standard metrics on D_m corresponding to κ may be different from the metric corresponding to $\tilde{\kappa}$ (this happens when one of the punctures q_j is very close to one of the punctures p_r). However, the metrics differ only on a compact set and thus using this forgetful map we define for a fixed conformal structure $\tilde{\kappa}$ on $\tilde{D}_{m,k}$ the space

$$\mathcal{W}^S_{2,\epsilon}(\tilde{\kappa}) \subset \mathcal{W}_{2,\epsilon}(\kappa)$$

as the subset of elements represented by maps $w \colon D_m \to \mathbb{C}^n$ such that $w(q_r) \in H_r$ for $r = 1, \dots, k$. Using local coordinates on $\mathcal{W}_{2,\epsilon}(\kappa)$ around (u, f) we see that for some ball B around (u, f), $\mathcal{W}^S_{2,\epsilon}(\tilde{\kappa}) \cap B$ is a codimension k submanifold with tangent space at (w, g) the closed subset of $T_{(w,g)}\mathcal{W}_{2,\epsilon}$ consisting of $v \colon D_m \to \mathbb{C}^n$ with $\langle v(q_r), \partial_1 \rangle = 0$. We call $\tilde{D}_{m,k}$ a disk with m punctures and k marked points.

The diffeomorphisms $\tilde{\phi}^{\sigma_j}_j$, $\sigma_j \in \mathbb{R}$ generated by to $\tilde{v}_j$ gives local coordinates $\sigma = (\sigma_1, \dots, \sigma_{m+k-3}) \in \mathbb{R}^{m+k-3}$ on the space of conformal structures on $\tilde{D}_{m,k}$ and the structure of a locally trivial bundle to the space

$$\mathcal{W}^S_{2,\epsilon} = \bigcup_{\tilde{\kappa} \in \mathcal{C}_{m+k}} \mathcal{W}^S_{2,\epsilon}(\tilde{\kappa}).$$

The $\bar{\partial}$-map is defined in the natural way on this space and we denote it $\tilde{\Gamma}$.

7.7.1. *Marked points on disks with few punctures.* Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a (semi-)admissible submanifold, let $m \le 2$ and consider a holomorphic disk (u, f) with boundary on L, represented by a map $u \colon D_m \to \mathbb{C}^n$. We shall put $3 - m$ marked points on D_m.

Pick $U_r \subset \Pi_{\mathbb{C}}(L)$, $1 \le r \le 3 - m$ as disjoint open subsets in which $\Pi_{\mathbb{C}}(L)$ is real analytic and let $q_r \in \partial D_m$ be points such that $u(q_r) \in U_r$ and $du(q_r) \neq 0$. Such points exists by Lemma 6.7. As in Section 7.7 we then consider the q_r as marked points and as there we use the notation H_r for the submanifold into which q_r is mapped.

Then the class in the moduli space of holomorphic disks of every holomorphic disk (w, g) which is sufficiently close to (u, f) in $\mathcal{W}_{2,\epsilon}$ has a unique representative $(\hat{w}, \hat{g}) \in \mathcal{W}^S_{2,\epsilon}$. Namely, any such (w, g) must intersect H_r in a point q'_r close to q_r, $1 \le r \le 3 - m$. If ψ denotes the unique conformal reparameterization of D_m which takes q_r to q'_r, $1 \le r \le 3 - m$ then $\hat{w}(\zeta) = w(\psi(\zeta))$. Moreover, if

$$d\Gamma_{(u,f)} \colon T_{(u,f)}\mathcal{W}_{2,\epsilon} \to \mathcal{H}_{1,\epsilon}[0](D_m, T^{*0,1}D_m \otimes \mathbb{C}^n), \tag{7.9}$$

has index k (note $k \ge 3 - m$ since the space of conformal reparameterizations of D_m is $(3 - m)$-dimensional) then the restriction of $d\Gamma_{(u,f)}$ to $T_{(u,f)}\mathcal{W}^S_{2,\epsilon}$ has index $k - (3 - m)$. In particular if $d\Gamma_{(u,f)}$ is surjective so is its restriction.

We conclude from the above that to study the moduli space of holomorphic disks in a neighborhood of a given holomorphic disk we may (and will) use a neighborhood of that disk in $\mathcal{W}^S_{2,\epsilon}$ and $\tilde{\Gamma}$ rather then a neighborhood in the bigger space $\mathcal{W}_{2,\epsilon}$ and Γ.

7.7.2. *Marked points on disks with many punctures.* Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be as above, let $m \geq 3$ and consider a holomorphic disk (u, f) with boundary on L, represented by a map $u\colon D_m \to \mathbb{C}^n$. We shall put k marked points on D_m.

Pick $U_r \subset \Pi_{\mathbb{C}}(L)$, $1 \leq r \leq k$ as disjoint open subsets in which $\Pi_{\mathbb{C}}(L)$ is real analytic and let $q_r \in \partial D_m$ be points such that $u(q_r) \in U_r$ and $du(q_r) \neq 0$. As in Section 7.7 we then consider the q_r as marked points and as there we use the notation H_r for the submanifold into which q_r is mapped.

Note that (u, f) lies in $\mathcal{W}_{2,\epsilon}$ as well as in $\mathcal{W}^S_{2,\epsilon}$. We define a map

$$\Omega\colon U \subset \mathcal{W}^S_{2,\epsilon} \to \mathcal{W}_{2,\epsilon}; \quad \Omega((w, g, \tilde{\phi}^s)) = (\hat{w}, \hat{g}, \tilde{\phi}^t)$$

where U is a neighborhood of $((u, f), \tilde{\kappa})$ in as follows.

Consider the local coordinates $\omega \in \mathbb{R}^{m+k-3}$ on $\mathcal{C}_{m+k}$ around $\tilde{\kappa}$ and the product structure

$$\mathbb{R}^{m+k-3} = \mathbb{R}^{m-3} \times \mathbb{R}^j \times \mathbb{R}^{k-j},$$

where $\mathbb{R}^{m-3}$ is identified with the diffeomorphisms

$$\phi^\tau = \tilde{\phi}^{\tau_1}_{p_4} \circ \cdots \circ \tilde{\phi}^{\tau_{m-3}}_{p_m}, \quad \tau = (\tau_1, \dots, \tau_{m-3}) \in \mathbb{R}^{m-3},$$

where j is the number of elements in $\{s_1, s_2, s_3\} \setminus \{p_1, p_2, p_3\}$, and where $\mathbb{R}^{k-j}$ is identified with the diffeomorphisms

$$\phi^\sigma = \tilde{\phi}^{\sigma_1}_{\hat{s}_1} \circ \cdots \circ \tilde{\phi}^{\sigma_{k-j}}_{\hat{s}_{k-j}}, \quad \sigma = (\sigma_1, \dots, \sigma_{k-j}) \in \mathbb{R}^{k-j}$$

where $\{\hat{s}_1, \dots, \hat{s}_{k-j}\} = S \setminus (\{p_4, \dots, p_m\} \cup \{s_1, s_2, s_3\})$.

For $\tilde{\theta}$ near $\tilde{\kappa}$, let $\{s'_1, \dots, s'_{m+k-3}\}$ denote the corresponding positions of punctures and marked points in $\partial\Delta$ and let $\psi\colon \Delta \to \Delta$ be the unique conformal reparameterization such that $\psi(p_j) = p'_j$ for $j = 1, 2, 3$ and note that we may view ψ as a map from D_m to $\tilde{D}_{m,k}$. Let $s''_l = \psi^{-1}(s'_l)$ for $3 \leq l \leq k + m - 3$ and $s_l \neq p_i$, $i = 1, 2, 3$ let $(\tau, \sigma) \in \mathbb{R}^{m+k-3-j}$ be the unique element such that $\phi^\tau \circ \phi^\sigma(s_l) = s''_l$. Define

$$\Omega(w, \tilde{\theta}) = (w \circ \psi \circ \phi^\sigma, (\phi^\tau)^{-1}),$$

where $(\phi^\tau)^{-1}$ is interpreted as a conformal structure on D_m in a neighborhood of κ in local coordinates given by ϕ^τ, $\tau \in \mathbb{R}^{m-3}$ and where we drop the boundary function from the notation since it is uniquely determined by the $\mathbb{C}^n$-function component of $\Omega(w, \tilde{\theta})$ and g.

Lemma 7.11. *The map Ω maps $U \cap \tilde{\Gamma}^{-1}(0)$ into $\Gamma^{-1}(0)$. Moreover, Ω is a C^1-diffeomorphism on a neighborhood of (u, f).*

Proof. Assume that $(w, g) \in \tilde{\Gamma}^{-1}(0)$. Then w is holomorphic in the conformal structure $\tilde{\theta}$. Since ψ is a conformal equivalence and since the conformal structure $\tilde{\theta}$ is obtained from $\tilde{\kappa}$ by action of the inverses of $\phi^\tau \circ \phi^\sigma$ this implies

$$0 = dw \circ d\psi + i \circ (dw \circ d\psi) \circ (d\phi^\sigma) \circ (d\phi^\tau) \circ j_\kappa \circ (d\phi^\tau)^{-1} \circ (d\phi^\sigma)^{-1}.$$

Thus

$$0 = \Big(dw \circ d\psi \circ d\phi^\sigma + i \circ (dw \circ d\psi \circ d\phi^\sigma) \circ (d\phi^\tau) \circ j_\kappa \circ (d\phi^\tau)^{-1}\Big) \circ (d\phi^\sigma)^{-1},$$

and $w \circ \psi \circ \phi^\sigma$ is holomorphic in the conformal structure $d\phi^\tau j_\kappa (d\phi^\tau)^{-1}$ as required.

For the last statement we use the inverse function theorem. It is clear that the map Ω is C^1 and that the differential of Ω at (u, f) is a Fredholm operator. In fact, on the complement of all conformal variations on $\tilde{D}_{m,k}$ not supported around any of $p_3, \dots, p_m$ it is just an inclusion into a subspace of codimension k, which consists of elements v which vanish at $q_1, \dots, q_k$. Since $du(q_r) \neq 0$ for all r it follows easily that the image of the remaining k directions in $T_{(u,f)}\mathcal{W}^S_{2,\epsilon}$ spans the complement of this subspace. □

It is a consequence of Lemma 7.11 that if (u, f) is a holomorphic disk with boundary on L and more than 3 punctures then we may view a neighborhood of (u, f) in the moduli space of such disks either as a submanifold in $\mathcal{W}^S_{2,\epsilon}$ or in $\mathcal{W}_{2,\epsilon}$ in a neighborhood of (u, f).

Remark 7.12. Below we extend the use of the notion $\mathcal{W}_{2,\epsilon}$ to include also spaces $\mathcal{W}^S_{2,\epsilon}$, when this is convenient. The point being that after Sections 7.7.1 and 7.7.2 we may always assume the number of marked points and punctures is ≥ 3, so that the moduli space of holomorphic disks (locally) may be viewed as a submanifold of $\mathcal{W}_{2,\epsilon}$.

7.8. **Uniform invertibility of the differential in the stationary case.** Let

$$\Gamma\colon \mathcal{W}_2 \to \mathcal{H}_1[0](D_m, T^{*0,1}D_m \otimes \mathbb{C}^n)$$

be the $\bar{\partial}$-map defined in Section 4.7 (see Remark 7.10 for notation). Let $u\colon D_{m+1} \to \mathbb{C}^n$ and $v\colon D_{l+1} \to \mathbb{C}^n$ be as in Section 7.1.1 and consider the differential $d\Gamma_\rho$ at (w_ρ, κ_ρ), where w_ρ is as in Lemma 7.7 and κ_ρ is the natural metric (complex structure) on $D_r(\rho)$, $r = m + l$. After Sections 7.7.1 and 7.7.2 we know that after adding $3 - m$ or $3 - l$ marked points on holomorphic disks with ≤ 2 punctures, we may assume that $m \geq 2$ and $l \geq 2$ below.

Lemma 7.13. *There exist constants C and ρ_0 such that if $\rho > \rho_0$ then there are continuous right inverses*

$$G_\rho\colon \mathcal{H}_1[0](T^{*0,1}D_r(\rho) \otimes \mathbb{C}^n) \to T_{(w_\rho,\kappa_\rho)}\mathcal{W}_2$$

of $d\Gamma_\rho$ with

$$\|G_\rho(\xi)\| \leq C\|\xi\|_1.$$

Proof. The kernels

$$\ker(d\Gamma_{(u,\kappa_1)}) \subset T_u\mathcal{W}_2 \oplus T_{\kappa_1}\mathcal{C}_{m+1},$$
$$\ker(d\Gamma_{(v,\kappa_2)}) \subset T_v\mathcal{W}_2 \oplus T_{\kappa_2}\mathcal{C}_{l+1}$$

are both 0-dimensional. As in Section 7.7, we view elements $\gamma_1 \in T_{\kappa_1}\mathcal{C}_{m+1}$ ($\gamma_2 \in T_{\kappa_2}\mathcal{C}_{l+1}$) as linear combinations of sections of $\mathrm{End}(TD_{m+1})$ ($\mathrm{End}(TD_{l+1})$) supported in compact annular regions close to all punctures and marked points, except at three. Since these annular regions are disjoint from the regions affected by the gluing of D_{m+1} and D_{l+1}, we get an embedding

$$T_{\kappa_1}\mathcal{C}_{m+1} \oplus T_{\kappa_2}\mathcal{C}_{l+1} \to T_{\kappa_\rho}\mathcal{C}_{m+r},$$

In fact, using this embedding,

$$T_{\kappa_\rho}\mathcal{C}_r = T_{\kappa_1}\mathcal{C}_{m+1} \oplus T_{\kappa_2}\mathcal{C}_{l+1} \oplus \mathbb{R},$$

where the last summand can be taken to be generated by a section γ_0 of $\mathrm{End}(TD_r(\rho))$ supported in an annular region around a puncture (marked point) in $D_r(\rho)$ where there was no conformal variation before the gluing. Then γ_0 spans a subspace of dimension 1 in $T_{(w_\rho,\kappa_\rho)}\mathcal{W}_2$. Let a be a coordinate along this 1-dimensional subspace. We prove that for $(\xi, \gamma) \in W_\rho = \{a = 0\}$ we have the estimate

$$\|(\xi, \gamma)\| \leq C\|d\Gamma_\rho(\xi, \gamma)\|_{1,\epsilon}, \qquad (7.10)$$

for all sufficiently large ρ. Since the Fredholm-index of $d\Gamma_\rho$ equals 1, this shows $d\Gamma_\rho$ are surjective and with uniformly bounded inverses G_ρ as claimed and thus finishes the proof.

Assume (7.10) is not true. Then there exists a sequence of elements $(\xi_N, \gamma_N) \in W_{\rho(N)}$, $\rho(N) \to \infty$ as $N \to \infty$ with

$$\|(\xi_N, \gamma_N)\| = 1, \tag{7.11}$$

$$\|\bar\partial_{\kappa_{\rho(N)}}\xi_N + i \circ dw_{\rho(N)} \circ \gamma_N\|_1 \to 0. \tag{7.12}$$

As in Section 7.4, we glue a negative puncture at p to a positive one at q. Note that on the strip

$$\Theta_\rho = (E_p[-1] \setminus E_p[-\rho]) \cup (E_q[1] \setminus E_q[\rho]) \approx [-\rho, \rho] \times [0,1] \subset D_r(\rho) \tag{7.13}$$

the conformal structure κ_ρ is the standard one and therefore $\bar\partial_{\kappa_\rho}$ is just the standard $\bar\partial$ operator. Also, since γ_N does not have support in Θ_ρ the second term in (7.12) equals 0 when restricted to Θ_ρ.

Let $\alpha_\rho \colon \Theta_\rho \to \mathbb{C}$ be cut-off functions which are real and holomorphic on the boundary, equal 1 on $[-2,2] \times [0,1]$, equal 0 outside $[-\frac{1}{2}\rho, \frac{1}{2}\rho] \times [0,1]$, and satisfy $|D^k\alpha_\rho| = \mathcal{O}(\rho^{-1})$, $k = 1,2$.

Then $\alpha_{\rho(N)}\xi_N$ is a sequence of functions on $\mathbb{R} \times [0,1]$ which satisfy boundary conditions converging to two transverse Lagrangian subspaces. Just as we prove in Lemma 5.9 that the (continuous) index is preserved under small perturbations, we conclude that the (upper semi-continuous) dimension of the kernel stay zero for large enough $\rho(N)$; thus, there exists a constant C such that

$$\|\xi_N|[-2,2] \times [0,1]\|_2 \le \|\alpha\xi_N\|_2 \le C\Big(\|\alpha_{\rho(N)}(\bar\partial\xi_N)\|_1 + \|(\bar\partial\alpha_{\rho(N)})\xi_N\|_1\Big). \tag{7.14}$$

As $N \to \infty$ both terms on the right hand side in (7.14) approaches 0. Hence,

$$\|\xi_N|[-2,2] \times [0,1]\|_2 \to 0, \text{ as } N \to \infty. \tag{7.15}$$

Pick cut-off functions β_N^+ and β_N^- on $D_r(\rho)$ which are real valued and holomorphic on the boundary and have the following properties. On $D_{m+1} \setminus E_p[-\rho+1]$, $\beta_N^+ = 1$ and on $D_{l+1} \setminus E_q[\rho]$, $\beta_N^+ = 0$. On $D_{l+1} \setminus E_q[\rho-1]$, $\beta_N^- = 1$ and on $D_{m+1} \setminus E_p[-\rho]$, $\beta_N^- = 0$. Let $(\xi_N, \gamma_N)^\pm = (\beta_N^\pm\xi_N, \beta_N^\pm\gamma_N)$. Using the invertibility of $d\Gamma_+ = d\Gamma_{(u,\kappa_1)}$ and $d\Gamma_- = d\Gamma_{(v,\kappa_2)}$ we find a constant C such that

$$\begin{aligned}\|(\xi_N, \Gamma_N)^\pm\| &\le C\|d\Gamma_\pm(\xi_N, \gamma)^\pm\|_1 \\ &\le C\Big(\|\beta_N^\pm d\Gamma_\rho(\xi_N, \gamma_N)\|_1 + \|(\bar\partial\beta_N^\pm)\xi_N\|_1\Big).\end{aligned} \tag{7.16}$$

The first term in the last line of (7.16) tends to 0 as $N \to \infty$ by (7.12), the second term tends to 0 by (7.15). Hence, the left hand side of (7.16) tends to 0 as $N \to \infty$. Thus, (7.15) and (7.16) contradict (7.11) and we conclude (7.10) holds. □

7.9. **Uniform invertibility in the handle slide case.** Two handle-slide gluing theorems (Theorem 7.2 and Theorem 7.3) were formulated. Here the handle-slide analogs of Lemma 7.13 corresponding to these two theorems will be proved. The lemma corresponding to Theorem 7.2 is proved first, using a straightforward extension of the proof of Lemma 7.13. Second the more difficult lemma corresponding to Theorem 7.3 is dealt with. Compare with the Appendix in [31].

Let L_λ be as in Section 7.1.2 and let $u \colon D_{m+1} \to \mathbb{C}^n$, be the handle slide disk and let p denote one of its negative punctures mapping to the Reeb chord b. Also, let $v \colon D_{l+1} \to \mathbb{C}^n$ be a rigid disk with positive puncture q mapping to b. Consider the differential $d\Gamma_\rho$ at $(w_\rho, \kappa_\rho, 0)$, where w_ρ is as in Lemma 7.8, κ_ρ is the natural metric (complex structure) on $D_r(\rho)$, $r = m + l$, and $0 \in \Lambda \approx (-1,1)$. Again we assume $m \ge 2$ and $l \ge 2$.

Lemma 7.14. *There exist constants C and ρ_0 such that if $\rho > \rho_0$ then there is a continuous right inverse G_ρ of $d\Gamma_\rho$*

$$G_\rho \colon \mathcal{H}_1[0](T^{*0,1}D_r(\rho)\otimes\mathbb{C}^n) \to T_{(w_\rho,\kappa_\rho,0)}\mathcal{W}_{2,\Lambda}$$

such that

$$\|G_\rho(\xi)\| \le C\|\xi\|_1.$$

Proof. The kernels

$$\ker(d\Gamma_{(u,\kappa_1,0)}) \subset T_{(u,0)}\mathcal{W}_{2,\Lambda}\oplus T_{\kappa_1}\mathcal{C}_{m+1},$$
$$\ker(d\Gamma_{(v,\kappa_2)}) \subset T_{(v,\kappa_2)}\mathcal{W}_2\oplus T_{\kappa_2}\mathcal{C}_{l+1}$$

are 0-dimensional (note that Λ is not involved in the second line). As in the proof of Lemma 7.13, we consider the embedding

$$T_{\kappa_1}\mathcal{C}_{m+1}\oplus T_{\kappa_2}\mathcal{C}_{l+1} \to T_{\kappa_\rho}\mathcal{C}_{m+l},$$

and use this inclusion to get the isomorphism

$$T_{\kappa_\rho}\mathcal{C}_{m+l} = T_{\kappa_1}\mathcal{C}_{m+1}\oplus T_{\kappa_2}\mathcal{C}_{l+1}\oplus\mathbb{R},$$

where the last summand can be taken to be generated by a section of $\mathrm{End}(TD_r(\rho))$ supported around a puncture (marked point) where no conformal variation was supported before the gluing. Let γ_0 denote a tangent vector in this direction. Then γ_0 spans a subspace of dimension 1 in $T_{(w_\rho,\kappa_\rho,0)}\mathcal{W}_{2,\epsilon,\Lambda}$. Let a be a coordinate along this 1-dimensional subspace. We prove that for $(\xi,\gamma,\lambda)\in W_\rho=\{a=0\}$ we have the estimate

$$\|(\xi,\gamma,\lambda)\| \le C\|d\Gamma_\rho(\xi,\gamma,\lambda)\|_1, \tag{7.17}$$

for all sufficiently large ρ. Since the Fredholm-index of $d\Gamma_\rho$ equals 1, this finishes the proof.

Assume (7.17) is not true. Then there exists a sequence of elements $(\xi_N,\gamma_N,\lambda_N)\in W_{\rho(N)}$, such that, $\rho(N)\to\infty$ as $N\to\infty$ and

$$\|(\xi_N,\gamma_N,\lambda_N)\| = 1, \tag{7.18}$$

$$\|\bar\partial_{\kappa_{\rho(N)}}\xi_N + i\circ dw_{\rho(N)}\circ\gamma_N + \lambda_N\bar\partial_{\kappa_{\rho(N)}}Y(w_{\rho(N)})\|_1 \to 0. \tag{7.19}$$

Let $\alpha_\rho\colon D_r(\rho)\to\mathbb{C}$ be smooth functions which equal 1 on $(D_{m+1}\setminus E_p[-\frac{1}{2}\rho])$, equal 0 on the complement of $D_{m+1}\setminus E_p[-\rho+3]$, are holomorphic and real valued on the boundary, and satisfy $|D^k\alpha_\rho| = \mathcal{O}(\rho^{-1})$, $k=1,2$. Consider the support of α_ρ as a subset of D_{m+1}. Then $(\xi_N^+,\gamma_N^+) = (\alpha_{\rho(N)}\xi_N,\alpha_{\rho(N)}\gamma_N)$ is a sequence of elements in $T_{(u,\kappa_1,0)}\mathcal{W}_{2,\epsilon,\Lambda}\oplus T_{\kappa_1}\mathcal{C}_{m+1}$ The invertibility of $d\Gamma_{(u,\kappa_1,0)}$ implies there exists a constant C such that

$$\|(\xi_N^+,\gamma_N^+,\lambda_N)\| \le C\|\bar\partial_{\kappa_1}\xi_N^+ + i\circ du\circ\gamma_N^+ + \lambda_N\bar\partial Y(u)\|_1. \tag{7.20}$$

On the other hand, noting that $\bar\partial_{\kappa_1}$ is the standard $\bar\partial$-operator in a neighborhood of q and that $\bar\partial_{\kappa_\rho}$ is the standard $\bar\partial$-operator in the gluing region we find that

$$\begin{aligned}\|\bar\partial_{\kappa_1}\xi_N^+ + i\circ du\circ\gamma_N^+ + \lambda\bar\partial Y(u)\|_1 \le{}& C\Big(\|\alpha_\rho(\bar\partial_{\kappa_\rho}\xi_N + i\circ dw\circ\gamma_N + \lambda\bar\partial Y(w_\rho))\|_1\\ &+ \|(\bar\partial\alpha_\rho)\xi_N\|_1 + \|(1-\alpha_\rho)\bar\partial Y(u)\|_1\Big).\end{aligned}$$

The first term in the right hand side goes to zero as $\rho\to\infty$ by (7.19). The second term goes to 0 since $|\bar\partial\alpha_\rho| = \mathcal{O}(\rho^{-1})$. Finally, the last term goes to 0 since $\bar\partial Y(u)\in\mathcal{H}_1(D_{m+1},T^{*0,1}D_{m+1}\otimes\mathbb{C}^n)$. Hence, the right hand side goes to 0 as $\rho=\rho(N)\to\infty$. We conclude from (7.20) that $\|(\xi_N^+,\gamma_N^+,\lambda_N)\|\to 0$ as $\rho\to\infty$. In particular, it follows that $\lambda_N\to 0$ as $\rho\to\infty$ and once this has been established we can repeat the argument from the proof of Lemma 7.13 to conclude that (7.17) holds true. □

We now turn to the second case. Let L_λ be as in Section 7.1.2 and let again $u\colon D_{m+1} \to \mathbb{C}^n$, be the handle slide disk and let q denote its positive puncture mapping to the Reeb chord a. Also, let $v\colon D_{l+1} \to \mathbb{C}^n$ be a rigid disk with a set S of negative punctures $S = \{p_1, \dots, p_k\}$ mapping to a. Again we assume $m \geq 2$ and $l \geq 2$.

Consider the differential $d\Gamma^S_\rho$ at $(w^S_\rho, \kappa^S_\rho, 0)$, where w^S_ρ is as in Lemma 7.9, κ^S_ρ is the natural metric (complex structure) on $D^S_r(\rho)$, $r = l + km$, and $0 \in \Lambda \approx (-1, 1)$. We let D^j_{m+1}, $j = 1, \dots, k$ denote k distinct copies of D_{m+1}.

Lemma 7.15. *There exist constants C and ρ_0 such that if $\rho > \rho_0$ then there is a continuous right inverse G^S_ρ of $d\Gamma^S_\rho$*

$$G^S_\rho\colon \mathcal{H}_1[0](T^{*0,1}D_r(\rho) \otimes \mathbb{C}^n) \to T_{(w^S_\rho, \kappa^S_\rho, 0)}\mathcal{W}_{2,\Lambda}$$

such that

$$\|G^S_\rho(\xi)\| \leq C\|\xi\|_1.$$

Proof. The kernels

$$\ker(d\Gamma_{(v,\kappa_2,0)}) \subset T_{(v,0)}\mathcal{W}_{2,\Lambda} \oplus T_{\kappa_2}\mathcal{C}_{l+1},$$
$$\ker(d\Gamma_{(u,\kappa_1,0)}) \subset T_{(u,0)}\mathcal{W}_{2,\Lambda} \oplus T_{\kappa_1}\mathcal{C}_{m+1}$$

are 1-dimensional, respectively 0-dimensional. Note that $\ker(d\Gamma_{(v,\kappa_2,0)})$ is *not* contained in the subspace $\{\lambda = 0\}$. We consider first the special case $S = \{1\}$. As in previous proofs we write,

$$T_{\kappa_\rho}\mathcal{C}_{l+m} = T_{\kappa_2}\mathcal{C}_{l+1} \oplus T_{\kappa_1}\mathcal{C}_{m+1} \oplus \mathbb{R},$$

where the generators of $\mathbb{R}$ is a section γ_0, of $\mathrm{End}(TD_{l+m}(\rho))$ which is supported around one of the previously fixed punctures of D_{m+1}. As in the proof of Lemma 7.14, we prove the the estimate

$$\|(\xi, \gamma, \lambda)\| \leq C\|d\Gamma^{\{1\}}_\rho(\xi, \gamma, \lambda)\|_1. \tag{7.21}$$

on the complement of the new conformal direction. This finishes the proof in the special case when $k = 1$.

In order to move to higher k, we prove that the kernel of $d\Gamma^{\{1\}}_\rho$ is *not* contained in $\{\lambda = 0\}$ for ρ large enough. Assume this is not the case. Then there exists a sequence $(\xi_N, \gamma_N, 0)$ such that

$$\|(\xi_N, \gamma_N, 0)\| = 1, \quad d\Gamma^{\{1\}}_\rho(\xi_N, \gamma_N, 0) = 0. \tag{7.22}$$

Let α be a smooth function, real valued and holomorphic on the boundary of $D^{\{1\}}_r$, equal to 1 on $(D_{l+1} \setminus E_{p_1}[-\frac{1}{2}\rho]) \cup (D_{m+1} \setminus E_q[\frac{1}{2}\rho])$ and equal to 0 on the gluing region Ω. Using the uniform invertibility of $d\Gamma_{(v,\kappa_2,0)}$ and of $d\Gamma_{(u,\kappa_1,0)}$ on the complement of $\{\lambda = 0\}$ we conclude that $\|\alpha(\xi_N, \gamma_N)\| \to 0$. Finally, using the elliptic estimate on the strip we also find that the 2-norm of ξ_N restricted to $(E_{p_1}[-\frac{1}{2}\rho] \setminus E_{p_1}[-\rho]) \cup (E_q[\frac{1}{2}\rho] \setminus E_q[\rho])$ goes to zero. This contradicts (7.22) and we find the kernel has λ-component.

Now consider the case $k > 1$. We assume by induction that the desired invertibility of $d\Gamma^S_\rho$ is proved for S with $|S| < k$ and that the kernel of $d\Gamma^S_\rho$ has λ component. Letting T be the union of S and the new puncture, we view w^T_ρ as obtained by gluing w^S_ρ and u. Repeating the above argument we prove the estimate on the complement of the new-born conformal structure and also the fact that the kernel of $d\Gamma^T_\rho$ has λ component. Since k is finite this finishes the proof. □

7.10. **Self-tangencies, coordinates and genericity assumptions.** Let $z = x + iy = (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n)$ be coordinates on $\mathbb{C}^n$. Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a semi-admissible Legendrian submanifold with self-tangency double point at 0. We assume that the self-tangency point is standard, see Section 2.

Theorems 6.19 and 8.2 imply that the moduli-space of rigid holomorphic disks with boundary on L is a 0-dimensional compact manifold. Moreover, because of the enhanced transversality discussion in Section 6.10, we may assume that there exists $r_0 > 0$ such that for all $0 < r < r_0$, if $u\colon D_m \to \mathbb{C}^n$ is a rigid holomorphic disk with boundary on L then $\partial D_m \cap u^{-1}(B(0,r))$ is a disjoint union of subintervals of $\partial E_{p_j}[\pm M]$, for some $M > 0$ and some punctures p_j on ∂D_m mapping to 0.

By Lemma 3.6, if $u\colon D_m \to \mathbb{C}^n$ is a rigid holomorphic disk with q^+ a positive (q^- a negative) puncture mapping to 0 then there exists $c \in \mathbb{R}$ such that for $\zeta = \tau + it \in E_{q^\pm}[\pm M]$

$$u(\zeta) = \left(-2(\zeta + c)^{-1}, 0, \dots, 0\right) + \mathcal{O}(e^{-\theta|\tau|}),$$

for some $\theta > 0$. For simplicity, we assume below that coordinates on $E_{q^\pm}[M]$ are chosen in such a way that $c = 0$ above.

7.11. **Perturbations for self-tangency shortening.** For $0 < a < 1$, with a very close to 1 and $R > 0$ with $R^{-1} \ll r_0$, let $b_R\colon [0,\infty) \to \mathbb{R}$ be a smooth non-increasing function with support in $[0, R^{-1})$ and with the following properties

$$\begin{aligned} &b_R(r) = (R + R^a)^{-2} \text{ for } r \in \left[0, (R + \tfrac{1}{2}R^a)^{-1}\right],\\ &|Db_R(r)| = \mathcal{O}(R^{-a}),\\ &|D^2 b_R(r)| = \mathcal{O}(R^{2-2a}),\\ &|D^3 b_R(r)| = \mathcal{O}(R^{4-3a}),\\ &|D^4 b_R(r)| = \mathcal{O}(R^{6-4a}). \end{aligned} \tag{7.23}$$

The existence of such a function is easily established using the fact that the length of the interval where Db_R is supported equals

$$R^{-1} - \left(R + \frac{1}{2}R^a\right)^{-1} = \frac{1}{2}R^{a-2} + \mathcal{O}(R^{2(a-2)}).$$

Let

$$h_R(z) = -x_1(z) b_R(|z|). \tag{7.24}$$

Let L^1 and L^2 be the two branches of the local Lagrangian projection near the self-tangency, see Section 2 or Figure 2. For $s > 0$, let Ψ_R^s denote the time s Hamiltonian flow of h_R and let $L_R(s)$ denote the Legendrian submanifold which results when Ψ_R^s is lifted to a contact flow on $\mathbb{C}^n \times \mathbb{R}$ (see Section 2) which is used to move L^2. Let $L_R^2(s) = \Psi_R^s(L^2)$. Let $g(R, s, \sigma)$ be a 3-parameter family of metrics on $\mathbb{C}^n$ such that L^1 is totally geodesic for $g(R, s, 0)$, $L_R^2(s)$ is totally geodesic for $g(R, s, 1)$ and such that $g(R, s, 0)$ and $g(R, s, 1)$ have properties as the metrics constructed in Section 4.3.

Note that $L_R^2(1) \cap L^1$ consists of exactly two points with coordinates $(\pm(R + R^a)^{-1} + \mathcal{O}(R^{-3}), 0, \dots, 0)$.

We will use Ψ_R^s to deform holomorphic disks below. It will be important for us to know they remain almost holomorphic in a rather strong sense, for which we need to derive some estimates on the flow Ψ_R^s and its derivatives. Let X_R denote the Hamiltonian vector field of

h_R. Then if D denotes derivative with respect to the variables in $\mathbb{C}^n$ and $\cdot$ denotes contraction of tensors

$$\frac{d}{ds}\Psi_R^s = X_R; \quad \Psi_\mu^0 = \mathrm{id}, \tag{7.25}$$

$$\frac{d}{ds}D\Psi_R^s = DX_R \cdot D\Psi_R^s; \quad D\Psi_\mu^0 = \mathrm{id}, \tag{7.26}$$

$$\frac{d}{ds}D^2\Psi_R^s = D^2X_R \cdot D\Psi_R^s \cdot D\Psi_R^s + DX_R \cdot D^2\Psi_R^s; \quad D^2\Psi_R^0 = 0, \tag{7.27}$$

$$\begin{aligned}\frac{d}{ds}D^3\Psi_R^s =& D^3X_R \cdot D\Psi_R^s \cdot D\Psi_R^s \cdot D\Psi_R^s + 2D^2X_R \cdot D^2\Psi_R^s \cdot D\Psi_R^s \\ &+ D^2X_R \cdot D\Psi_R^s \cdot D^2\Psi_R^s + DX_R \cdot D^3\Psi_R^s; \quad D^3\Psi_R^0 = 0.\end{aligned} \tag{7.28}$$

Since $X_R = i \cdot Dh_R$ and $x_1(z) = \mathcal{O}(R^{-1})$ for $|z|$ in the support of b_R, (7.23) implies

$$|X_R| = \mathcal{O}(R^{-(1+a)}), \tag{7.29}$$

$$|DX_R| = \mathcal{O}(R^{(1-2a)}), \tag{7.30}$$

$$|D^2X_R| = \mathcal{O}(R^{(3-3a)}), \tag{7.31}$$

$$|D^2X_R| = \mathcal{O}(R^{(5-4a)}). \tag{7.32}$$

If $0 \le s \le 1$ then

F0 (7.25) and (7.29) imply $|\Psi_R^s - \mathrm{id}\,| = \mathcal{O}(R^{-(1+a)})$.

F1 (7.26) and (7.30) first give $|D\Psi_R^s| = \mathcal{O}(1)$. This together with (7.30) imply $|D\Psi_R^s - \mathrm{id}\,| = \mathcal{O}(R^{1-2a})$.

F2 (7.27), (7.30), (7.31), **F1**, and Duhamel's principle imply $|D^2\Psi_R^s| = \mathcal{O}(R^{3-3a})$.

F3 In a similar way as in **F2** we derive $|D^3\Psi_R^s| = \mathcal{O}(R^{5-4a})$.

Let $u\colon \mathbb{R}\times[0,1] \to \mathbb{C}^n$ be a holomorphic function and and let $\omega\colon [0,1] \to [0,1]$ be a smooth non-decreasing surjective approximation of the identity which is constant in a δ-neighborhood of the ends of the interval. Consider the function $u_R(\tau + it) = \Psi_R^{\omega(t)}(u(\tau+it))$. We want estimates for u_R, $\bar\partial u_R$ and $D\bar\partial u_R$ and $\partial_\tau D\bar\partial u_R$.

F0 implies

$$u_R = u + \mathcal{O}(R^{-(1+a)}). \tag{7.33}$$

For the estimates on $\bar\partial u_R$ and its derivatives we note

$$\bar\partial u_R = D\Psi_R^{\omega(t)}\frac{\partial u}{\partial \tau} + i\Big(D\Psi_R^{\omega(t)}\frac{\partial u}{\partial t} + \frac{d\omega}{dt}X_R(u)\Big) \tag{7.34}$$

By (7.29), (7.30), **F1**, and the holomorphicity of u,

$$|\bar\partial u_R| = \mathcal{O}(R^{1-2a})|Du| + \mathcal{O}(R^{-(1+a)}). \tag{7.35}$$

Taking derivatives of (7.34) with respect to τ and t we find (using **F0-3** and (7.29)-(7.32))

$$\begin{aligned}|D\bar\partial u_R| =& \mathcal{O}(R^{1-2a})|D^2u| + \mathcal{O}(R^{3-3a})|Du|^2 \\ &+ \mathcal{O}(R^{1-2a})|Du| + \mathcal{O}(R^{-(1+a)})\end{aligned} \tag{7.36}$$

$$\begin{aligned}|\partial_\tau D\bar\partial u_R| =& \mathcal{O}(R^{1-2a})|D^3u| + \mathcal{O}(R^{3-3a})|Du||D^2u| + \mathcal{O}(R^{5-4a})|Du|^3 \\ &+ \mathcal{O}(R^{1-2a})|D^2u| + \mathcal{O}(R^{3-3a})|Du|^2 + \mathcal{O}(R^{-(1+a)})|Du|.\end{aligned} \tag{7.37}$$

Finally, let $\theta\colon [0,1] \to \mathbb{R}$ be a smooth function supported in a $\frac{1}{2}\delta$-neighborhood of the endpoints of the interval with $\theta'(0) = \theta'(1) = 1$. Define

$$\hat u_R(\tau+it) = u_R(\tau+it) + i\theta(t)\bar\partial u_R(\tau+it). \tag{7.38}$$

Then $u_R = \hat{u}_R$ on the boundary and $\hat{u}_R$ is holomorphic on the boundary. Also for some constant C

$$|\hat{u}_R| \leq C(|u_R| + |\bar{\partial} u_R|), \tag{7.39}$$

$$|\bar{\partial}\hat{u}_R| \leq C(|\bar{\partial} u_R| + |D\bar{\partial} u_R|), \tag{7.40}$$

$$|D\bar{\partial}\hat{u}_R| \leq C(|\bar{\partial} u_R| + |D\bar{\partial} u_R| + |\partial_\tau D\bar{\partial} u_R|). \tag{7.41}$$

7.12. **Self-tangency preshortening.** Let $u\colon D_{m+1} \to \mathbb{C}^n$ be a rigid holomorphic disk with boundary on L and with negative punctures $p_1, \dots, p_k$ mapping to 0. (The case of one positive puncture mapping to 0 is completely analogous to the case of one negative puncture so for simplicity we consider only the case of negative punctures.)

For large $\rho > 0$ let $R = R(\rho)$ be such that the intersection points of L^1 and $L^2_R(1)$ are $a^\pm = (\pm(\rho + \rho^a)^{-1}, 0, \dots, 0)$. Then $R(\rho) = \rho + \mathcal{O}(\rho^{-1})$. Define

$$u_\rho(\zeta) = \begin{cases} u(\zeta) & \text{for } \zeta \in D_{m+1} \setminus \left(\bigcup_{j=1}^k E_{p_j}[-\frac{1}{2}\rho]\right), \\ \hat{u}_{R(\rho)}(\tau + it) & \text{for } \zeta = \tau + it \in E_{p_j}[-\frac{1}{2}\rho]. \end{cases}$$

Then there exist unique functions

$$\xi_R(j)\colon E_{p_j}[-\rho] \to T_{a^-}\mathbb{C}^n$$

such that

$$\exp^{R,t}(\xi_R(j)(\zeta)) = u_\rho(\zeta), \quad \zeta \in E_{p_j}[-\rho],$$

where $\exp^{R,t}$ denotes the exponential map in the metric $g(R, \omega(t), t)$ at a^-.

Let $\alpha_\rho\colon (-\infty, -\rho] \times [0,1] \to \mathbb{C}$ be a smooth cut-off function, real valued and holomorphic on the boundary and such that $\alpha_\rho(\tau + it) = 1$ for τ in a small neighborhood of $-\rho$, $\alpha_\rho(\tau+it) = 0$ for $\tau \leq -\rho - \frac{1}{2}\rho^a$, and $|D^k\alpha_\rho| = \mathcal{O}(\rho^{-a})$, $k = 1, 2$. Define $w_\rho\colon D_{m+1} \to \mathbb{C}^n$ as

$$w_\rho(\zeta) = \begin{cases} u_\rho(\zeta) & \text{for } \zeta \in D_{m+1} \setminus \left(\bigcup_j E_{p_j}[-\rho]\right), \\ \exp^{R,t}(\alpha_\rho(\zeta)\xi_R(j)(\zeta)) & \text{for } \zeta = \tau + it \in E_{p_j}[-\rho], \quad j = 1, \dots, k, \\ a_- & \text{for } \zeta \in \bigcup_j E_{p_j}[-\rho - \frac{1}{2}\rho^a]. \end{cases} \tag{7.42}$$

7.13. **Weight functions for shortened disks.** Let $u\colon D_{m+1} \to \mathbb{C}^n$ be a rigid holomorphic disk with boundary on L. Let $\epsilon > 0$ be small and let $e_\rho\colon D_{m+1} \to \mathbb{R}$ be a function which equals $e^{-\epsilon|\tau|}$ for $\tau + it \in E_{p_j} \setminus E_{p_j}[-\rho]$ and is constantly equal to $e^{-\epsilon\rho}$ for $\tau + it \in E_{p_j}[-\rho]$. Define $\mathcal{W}_{2,-\epsilon,\rho}$ just as in Section 4.8 but replacing the weight function e_ϵ with the new weight function e_ρ. The corresponding weighted norms will be denoted $\|\cdot\|_{2,-\epsilon,\rho}$. We also write $\mathcal{H}_{1,-\epsilon,\rho}[0](D_{m+1}, T^{*0,1} \otimes \mathbb{C}^n)$ to denote the subspace of elements in the Sobolev space with the weight function e_ρ which vanishes on the boundary.

7.14. **Estimates for self-tangency preshortened disks.**

Lemma 7.16. *The function w_ρ in (7.42) lies in $\mathcal{W}_{2,-\epsilon,\rho}$ (see Remark 7.10 for notation) and there exists a constant C such that*

$$\|\bar{\partial} w_\rho\|_{1,-\epsilon,\rho} \leq Ce^{-\epsilon\rho}\rho^{-1-\frac{1}{2}a}.$$

Proof. The first statement is obvious. Consider the second. In $D_{m+1} \setminus \left(E_{p_j}[-\rho]\right)$, w_ρ equals u which is holomorphic. It thus remains to check $E_{p_j}[-\rho] \approx (-\infty, -\rho] \times [0, 1]$.

Taylor expansion of $\exp^{R,t}$ gives

$$\exp^{R,t}\xi = \xi - \Gamma^k_{ij}(R, t)\xi^i\xi^j\partial_k + \mathcal{O}(|\xi|^3). \tag{7.43}$$

The Taylor expansion of the inverse then gives

$$\xi_R = \hat{u}_R + \Gamma^k_{ij}(R,t)\hat{u}^i_R\hat{u}^j_R\partial_k + \mathcal{O}(|\hat{u}_R|^3). \tag{7.44}$$

Thus in $(-\infty,-\rho]\times[0,1]$ we have

$$w_\rho = \alpha_\rho\hat{u}_R + (\alpha_\rho - \alpha_\rho^2)\Gamma^k_{ij}(R,t)\hat{u}^i_R\hat{u}^j_R\partial_k + \mathcal{O}(|\hat{u}_R|^3). \tag{7.45}$$

Now $R = \rho + \mathcal{O}(\rho^{-1})$ from Section 7.12, $|D^k\alpha_\rho| = \mathcal{O}(\rho^{-a})$ for all cut-off functions, and by Lemma 3.6 $|D^k u| = \mathcal{O}(\rho^{-(1+k)})$, in $(-\infty,-\rho]\times[0,1]$; thus, applying (7.39) through (7.41) to (7.45) we get

$$|\bar\partial w_\rho| + |D\bar\partial w_\rho| = \mathcal{O}(\rho^{-(1+a)}).$$

Noting that $\bar\partial w_\rho$ is supported on an interval of length $\frac{1}{2}\rho^a$, so multiplying with the weight function we find

$$\|\bar\partial w_\rho\|_{1,-\epsilon,\rho} \le Ce^{-\epsilon\rho}\rho^{-1-\frac{1}{2}a}.$$

□

7.15. **Controlled invertibility for self-tangency shortening.** Let $d\Gamma_\rho$ denote the differential of the map

$$\Gamma_\rho\colon \mathcal{W}_{2,-\epsilon,\rho} \to \mathcal{H}_{1,-\epsilon,\rho}[0](D_{m+1}, T^{*0,1}\otimes\mathbb{C}^n).$$

Referring to Sections 7.7.1 and 7.7.2, we assume that $m \ge 2$ and $l(j) \ge 2$ for each j.

Lemma 7.17. *There exist constants C and ρ_0 such that if $\rho > \rho_0$ then there is a continuous right inverse G_ρ of $d\Gamma_\rho$*

$$G_\rho\colon \mathcal{H}_{1,-\epsilon,\rho}[0](T^{*0,1}D_r(\rho)\otimes\mathbb{C}^n) \to T_{(w_\rho,\kappa_\rho,0)}\mathcal{W}_{2,-\epsilon,\rho}$$

such that for any $\delta > 0$

$$\|G_\rho(\xi)\| \le C\rho^{1+\delta}\|\xi\|_{1,-\epsilon,\rho}. \tag{7.46}$$

Proof. The kernel

$$\ker(d\Gamma_u) \subset T_u\mathcal{W}_{2,-\epsilon} \oplus T_\kappa\mathcal{C}_{m+1}$$

has dimension 0. By the invertibility of $d\Gamma_u$ we conclude there is a constant C such that for $\xi \in T_u\mathcal{W}_{\epsilon,2,\rho}$ we have

$$\|\xi\| \le C\|d\Gamma_{u,\rho}\xi\|_{1,-\epsilon}. \tag{7.47}$$

Assume that (7.46) is not true. Then there exists a sequence $\xi_N \in T_{w_\rho}\mathcal{W}_{2,-\epsilon,\rho(N)}$ with $\rho(N)\to\infty$ as $N\to\infty$ such that

$$\|\xi_N\| = 1, \tag{7.48}$$

$$\|d\Gamma_\rho\xi_N\|_{1,-\epsilon,\rho(N)} \le C\rho^{-1-\frac{\delta}{2}}. \tag{7.49}$$

Let $\alpha\colon D_{m+1}\to\mathbb{C}$ be a smooth function which equals 0 on $E_{p_j}[-\rho-\frac{1}{4}\rho^a]$ and equals 1 on $D_{m+1}\setminus\left(\bigcup E_{p_j}[-\rho-10]\right)$, which is real valued and holomorphic on the boundary and with $|D^k\alpha| = \mathcal{O}(\rho^{-a})$, $k=1,2$. Then (7.47) implies

$$\|\alpha\xi_N\| \le C(\|(\bar\partial\alpha)\xi_N\|_{1,-\epsilon} + \|\alpha d\Gamma_{u,\rho}\xi_N\|_{1,-\epsilon}) = \mathcal{O}(\rho^{-a}) \tag{7.50}$$

Finally, we let $\hat\phi\colon(-\infty,-\rho+\rho^a]\to\mathbb{C}$ be the function which equals $\theta(\rho)-\theta(\tau)$, where $\theta(\tau)$ denotes the angle between the tangent line of $L^2_\rho(1)$ intersected with the z_1-plane (the plane of the first coordinate in $\mathbb{C}^n$) at $u(\tau+i)$ and the real line in that plane. From Lemma 3.6 we calculate that $|D^k\hat\phi| = \mathcal{O}(\rho^{a-2})$, $0\le k\le 2$. Using the same procedure as for cut-off functions we extend it to a function $\phi\colon(-\infty,-\rho+\rho^a)\times[0,1]$ which is holomorphic on the boundary, which equals $\hat\phi$ on $(-\infty,-\rho+\rho^a)\times\{1\}$ and which equals 0 on $(-\infty,-\rho+\rho^a)\times\{0\}$ and with the same derivative estimates. Let $\mathbf{M} = \mathrm{Diag}(\phi,1,\dots,1)$.

Let α be a cut-off function which is 0 in $D_{m+1} \setminus E_{p_j}[-\rho+\rho^a]$ and 1 on $E_{p_j}[-\rho]$. Having frozen the angle away from 0, we can use Lemmas 5.8 and 5.9 (assuming that ϵ is smaller than the smallest component of the complex angle) to get

$$\|e^{-\epsilon\rho}\alpha\mathbf{M}\xi_N\| \le C\rho(\|e^{-\epsilon\rho}(\bar\partial\alpha\mathbf{M})\xi_N\| + \|e^{-\epsilon\rho}\alpha\mathbf{M}d\Gamma_\rho\xi_N\|) \tag{7.51}$$

The first term on the right hand side inside the brackets is $\mathcal{O}(\rho^{a-2}) + \mathcal{O}(\rho^{-2a})$ the second term is $\mathcal{O}(\rho^{-1-\delta})$. Hence as $\rho \to \infty$ the right hand side goes to 0. This together with (7.50) contradicts (7.48) and we conclude the lemma holds. □

7.16. **Perturbations for self-tangency gluing.** For $R > 0$ with $R^{-1} << r_0$, let $a_R\colon [0,\infty) \to \mathbb{R}$ be a smooth non-increasing function with support in $[0, \frac{1}{2}R^{-1})$ and with the following properties

$$\begin{aligned} a_R(r) &= R^{-1} \text{ for } r \in [0, R^{-2}],\\ |Da_R(r)| &= \mathcal{O}(1),\\ |D^2a_R(r)| &= \mathcal{O}(R). \end{aligned} \tag{7.52}$$

The existence of such functions is easily established. Let $h_R\colon \mathbb{C}^n \to \mathbb{C}^n$ be given by

$$h_R(z) = x_1(z)a_R(|z_1|). \tag{7.53}$$

For $s > 0$, let Φ_R^s denote the time s Hamiltonian flow of h_R and let $L_R(s)$ denote the Legendrian submanifold which results when Φ_R^s is lifted to a local contact flow on $\mathbb{C}^n \times \mathbb{R}$ which is used to move L^2. (Note that Φ_R^s fixes the last $n-1$ coordinates and has small support in the z_1-direction and so its lift can be extended to the identity outside $L^2(s)$.) Let $L_R^2(s) = \Phi_R^s(L^2)$. We pick a_R so that $L_R^2(s) \cap L^1 = \emptyset$, for $0 < s \le (KR)^{-1}$ for some fixed $K > 4$.

As in Section 7.11 we derive the estimates

$$|\Phi_R^s - \mathrm{id}\,| \le \mathcal{O}(R^{-2}), \tag{7.54}$$

$$|D\Phi_R^s - \mathrm{id}\,| = \mathcal{O}(R^{-1}), \tag{7.55}$$

$$|D^2\Phi_R^s| = \mathcal{O}(1). \tag{7.56}$$

For convenient notation we write $\gamma_R^2(s)$ for the curve in which $L_R^2(s)$ intersects the z_1-line in a neighborhood of 0.

7.17. **Self-tangency pregluing.** Let $u\colon D_m \to \mathbb{C}^n$ be a rigid holomorphic disk with boundary on L and with negative punctures $p_1,\dots,p_k$ (as above we write $S = \{p_1,\dots,p_k\}$) mapping to 0. Let $v_j\colon D_{l(j)+1} \to \mathbb{C}_n$ be rigid holomorphic disks with positive punctures q_j mapping to 0.

For $0 < \rho < \infty$ let $R = \rho$, $s = (K\rho)^{-1}$ and let L_ρ be the Legendrian submanifold which results when Φ_R^s is applied. Consider the region Ξ_ρ in the z_1-line bounded by the curves $\gamma_R^2(s)$, $\gamma_R^1(s)$, $u^1(\rho+it)$, $0 \le t \le 1$, and $v(j)^1(\rho+it)$, $0 \le t \le 1$. By the Riemann mapping theorem there exists a holomorphic map from a rectangle $\phi_\rho\colon [-A(\rho), A(\rho)] \times [0,1] \to \mathbb{C}$ which parameterizes this region in such a way that $[-A(\rho), A(\rho)] \times \{j-1\}$ maps to $\gamma_j(s)$, $j = 1,2$. Moreover, since Ξ_ρ is symmetric with respect to reflections in the $\operatorname{Im} z_1 = y_1$-axis we have $\phi_\rho(0 + i[0,1]) \subset \{\operatorname{Re} z_1 = x_1 = 0\}$.

Lemma 7.18. *The shape of the rectangle depends on ρ. More precisely, there exists constants $0 < K_1 < K_2 < \infty$ such that $K_1\rho \le A(\rho) \le K_2\rho$ for all ρ.*

Proof. Identify the z_1-line with $\mathbb{C}$. Consider the region Θ_ρ bounded by the circles of radius 1 and $1 + 4\rho^{-2}$ both centered at $i \in \mathbb{C}$, and the lines through i which intersects the x_1-axis in the points $\pm 2(\rho)^{-1}$. Mark the straight line segments of its boundary. The conformal modulus of this region is easily seen to be $\rho + \mathcal{O}(\rho^{-1})$.

On the other hand, using (7.54) and (7.55) one constructs a $(K + \mathcal{O}(\rho^{-1}))$-quasi conformal map from Θ_ρ to Ξ_ρ, for some $K > 0$ independent of ρ. This implies the conformal modulus m_ρ of Θ_ρ satisfies

$$(K + \mathcal{O}(\rho^{-1}))^{-1}(\rho + \mathcal{O}(\rho^{-1})) \leq m_\rho \leq (K + \mathcal{O}(\rho^{-1}))(\rho + \mathcal{O}(\rho^{-1})) \tag{7.57}$$

and the lemma follows. □

Let u^1 and v_j^1 denote the z_1-components of the maps u and v_j. Since Φ_R^s fixes γ_2 outside $|x_1| \leq (2\rho)^{-1}$ we note that

$$u^1 \text{ maps the region } E_{p_j}[-\rho] \setminus E_{p_j}[-2\rho] \text{ into } \Theta_\rho \setminus \phi_\rho(0 \times [0,1]). \tag{7.58}$$

and that

$$v_j^1 \text{ maps the region } E_{p_j}[\rho] \setminus E_{p_j}[2\rho] \text{ into } \Theta_\rho \setminus \phi_\rho(0 \times [0,1]). \tag{7.59}$$

Fix $0 < a < \frac{1}{4}$. Using u^1, v_j^1, the conformal map ϕ_ρ and their inverses, we construct a complex 1-dimensional manifold $D_r(\rho)$ by gluing $\Omega_j(\rho) = [-A(\rho), A(\rho)] \times [0,1]$ to $D_{m+1} \setminus \left(\bigcup_j E_{p_j}[-(1+a)\rho]\right)$ and $D_{l(j)+1} E_q[(1+a)\rho]$. Note that, by construction, $D_r(\rho)$ comes equipped with a holomorphic function

$$w_\rho^1 \colon D_r(\rho) \to \mathbb{C}, \tag{7.60}$$

which equals u^1 on $D_{m+1} \setminus \bigcup_j E_{p_j}[-(1+a)\rho]$, which equals v_j^1 on $D_{l(j)+1} \setminus E_{q_j}[(1+a)\rho]$, and which equals ϕ_ρ on Ω_j, for all j.

We next exploit the product structure of $\Pi_{\mathbb{C}} L$ in a neighborhood of 0. If u' and v_j' denotes the remaining components of u and v_j so that $u = (u^1, u')$ and $v_j = (v_j^1, v_j')$ then in some neighborhood of the punctures q_j and p_j, v_j' and u_j' are holomorphic functions with boundary on the two transverse Lagrangian manifolds P_1 and P_2, see Section 2. As in Section 4.3 we find a 1-parameter family $g(\sigma)$ of metrics on $\mathbb{C}^{n-1} \approx \{z_1 = 0\}$. Then, for M sufficiently large, there exist unique vector valued functions ξ_j' and η_j' such that

$$\exp_0^t \xi_j'(\tau + it) = u'(\tau + it), \quad \tau + it \in E_{p_j}[-M], \tag{7.61}$$

$$\exp_0^t \eta_j'(\tau + it) = v_j'(\tau + it), \quad \tau + it \in E_{q_j}[M]. \tag{7.62}$$

Now pick a cut-off function α^+ which equals 1 on $D_{m+1} \setminus \bigcup_j E_{p_j}[-\rho+5]$ and 0 on $E_{p_j}[-\rho+3]$. Pick similar cut-off functions α^- on $D_{l(j)+1}$. Define $w_\rho' \colon D_r(\rho) \to \mathbb{C}^{n-1}$ by

$$w_\rho'(\zeta) = \begin{cases} u'(\zeta), & \zeta \in D_{m+1} \setminus \bigcup E_{p_j}[-\rho+5], \\ v_j'(\zeta), & \zeta \in D_{l(j)+1} \setminus E_{q_j}[\rho-5], \\ \exp_0^t(\alpha^+(\zeta)\xi_j(\zeta)), & \zeta \in E_{p_j}[-\rho+5] \setminus E_{p_j}[-\rho], \\ \exp_0^t(\alpha^-(\zeta)\eta_j(\zeta)), & \zeta \in E_{q_j}[\rho-5] \setminus E_{p_j}[\rho], \\ 0, & \zeta \in \Omega_j. \end{cases} \tag{7.63}$$

Finally, combining (7.60) and (7.63), we define

$$w_\rho = (w_\rho^1, w_\rho') \tag{7.64}$$

7.18. **Weight functions and Sobolev norms for self tangency gluing.** Consider $D_r(\rho)$ from the previous section, $\epsilon > 0$, and a smooth function $f\colon D_r(\rho) \to \mathbb{C}^n$. Let

- f^+ denote the restriction of f to
$$\operatorname{int}\Big(D_{m+1} \setminus \bigcup_j E_{p_j}[-(1+a)\rho]\Big),$$
which we consider as a subset of D_{m+1}.
- f^- denote the restriction of f
$$\bigcup_j \operatorname{int}\Big(D_{l(j)+1} \setminus E_{q_j}[(1+a)\rho]\Big),$$
which we consider as subset of the disjoint union $\bigcup_j D_{l(j)+1}$
- f^0 denote the restriction of f to the disjoint union $\bigcup_j \operatorname{int}(\Omega_j(\rho))$.

For $\epsilon > 0$, let e^-_ϵ denote the weight function on $\bigcup_j D_{l(j)+1}$ which equals 1 on $D_{l(j)+1} \setminus E_{q_j}$ and equals $e^{\delta|\tau|}$ in E_{q_j}, each j. Let $\|\cdot\|_{k,\epsilon,-}$ denote the Sobolev norm with weight e^-_δ. Let e^0_ϵ denote the weight function on Ω_j which equals $e^{\epsilon(A(\rho)+\rho+\tau)}$ and $\|\cdot\|_{k,\epsilon,0}$ denote the Sobolev norm with this weight. Finally, let e^+_ϵ be the function on D_{m+1} which equals $e^{2\epsilon(A(\rho)+\rho)}$ on $D_{m+1}\setminus\bigcup_j E_{p_j}]$ and equals $e^{2\epsilon(A(\rho)+\rho)-\epsilon|\tau|}$ in E_{p_j}. Let $\|\cdot\|_{k,\epsilon,+}$ denote the corresponding norm.

Define

$$\|f\|_{k,\epsilon,\rho} = \|f_+\|_{k,\epsilon,+} + \|f_0\|_{k,\epsilon,0} + \|f_-\|_{k,\epsilon,-}. \tag{7.65}$$

Using this norm we define as in the shortening case the spaces $\mathcal{W}_{2,\epsilon,\rho}$ and $\mathcal{H}_{1,\epsilon,\rho}[0](D_{m+1}, T^{*0,1}\otimes \mathbb{C}^n)$. The $\bar\partial$-map $\Gamma\colon \mathcal{W}_{2,\epsilon,\rho} \to \mathcal{H}_{1,\epsilon,\rho}[0](T^{*0,1}D_m \otimes \mathbb{C}^n)$ is defined in the natural way.

7.19. **Estimates for self tangency glued disks.**

Lemma 7.19. *The function w_ρ in (7.64) lies in $\mathcal{W}_{2,\epsilon,\rho}$ and there exists a constant C such that*
$$\|\bar\partial w_\rho\|_{1,\epsilon,\rho} \le Ce^{(-\theta+2K_2\epsilon)\rho},$$
where $\theta \gg \epsilon$ is the smallest non-zero complex angle at the self tangency point 0 and where K_2 is as in Lemma 7.18.

Proof. Note that the first coordinate of w_ρ is holomorphic and that the support of $\bar\partial w_\rho$ is disjoint from Ω_j. Using the asymptotics of u' and v'_j the proof of Lemma 7.7 applies to give the desired estimate once we note that the weight function is $\mathcal{O}(e^{K_2\epsilon\rho})$ by Lemma 7.18. □

7.20. **Estimates for real boundary conditions.** In order to prove the counterpart of Lemma 7.17 in the self tangency gluing case we study an auxiliary non-compact counterpart of the gluing region.

Let $\Omega(\rho) = [-A(\rho), A(\rho)] \times [0,1]$ and let M_ρ be the complex manifold which results when $(-\infty, -(1-a)\rho] \times [0,1]$ and $[(1-a)\rho, \infty) \times [0,1]$ are glued to $\Omega(\rho)$ with the holomorphic gluing maps $u^1 \circ (\phi_\rho)^{-1}$ and $v^1_j \circ (\phi_\rho)^{-1}$, respectively. (That is, the maps which were used to construct $D_r(\rho)$.) We consider Sobolev norms on M_ρ similar to those used above.

For $\epsilon > 0$, let

- $e^0_\epsilon\colon : [-A(\rho), A(\rho)] \times [0,1] \to \mathbb{R}$ be the function $e^0_\epsilon(\tau + it) = e^{\epsilon\tau}$,
- $e^-_\epsilon\colon (-\infty, -(1-a)\rho] \times [0,1] \to \mathbb{R}$ be the function $e^-_\epsilon(\tau+it) = e^{\epsilon(\rho - A(\rho)+\tau)}$
- $e^+_\epsilon\colon [(1-a)\rho, \infty) \times [0,1] \to \mathbb{R}$ be the function $e^+_\epsilon(\tau+it) = e^{\epsilon(-\rho+A(\rho)+\tau)}$

If $f\colon M_\rho \to \mathbb{C}$ is function we let as above f^-, f^0, f^+ denote the restrictions of f to the interiors of the pieces from which M_ρ was constructed and define the Sobolev norm

$$\|f\|_{k,\rho,\epsilon} = \|f^-\|_{k,\epsilon} + \|f^0\|_{k,\epsilon} + \|f^+\|_{k,\epsilon}. \tag{7.66}$$

Lemma 7.20. *There are constants C and ρ_0 if $\rho > 0$ and if $f\colon M_\rho \to \mathbb{C}$ is function which is real valued and holomorphic on the boundary and has $\|f\|_{k,\rho,\epsilon} \le \infty$ then*

$$\|f\|_{k,\rho,\epsilon} \le C\|\bar\partial f\|_{k-1,\rho,\epsilon}, \tag{7.67}$$

for $k = 1, 2$.

Proof. To prove the lemma we first study the gluing functions. Let $\psi\colon [-\rho, -(1-a)\rho) \times [0,1] \to [-\mathcal{A}(\rho), 0] \times [0,1]$ be the function $u^1 \circ (\phi_\rho)^{-1}$. Note that ψ is holomorphic and that by (7.58) has a holomorphic extension (still denoted ψ) to $[-\rho, 0) \times [0,1]$.

To simplify notation we change coordinates and think of the source $[-\rho, 0) \times [0,1]$ as $[0,\rho) \times [0,1]$ and of the target $[-A(\rho), 0] \times [0,1]$ as $[0, A(\rho)] \times [0,1]$. Thus

$$\psi\colon [0,\rho] \times [0,1] \to [0, A(\rho)] \times [0,1] \tag{7.68}$$

is a holomorphic map. Consider the complex derivative $\frac{\partial\psi}{\partial z}$. This is again a holomorphic function which is real on the boundary of $[0,\rho) \times [0,1]$. In analogy with Lemma 5.2 we conclude that

$$\frac{\partial\psi}{\partial z} = \sum_{n\in Z} c_n' e^{n\pi z}, \tag{7.69}$$

for some real constants c_n'. Integrating this and using $\psi(0) = 0$, we find

$$\psi(z) = c_0 z + \sum c_n e^{n\pi z}, \tag{7.70}$$

for some real constants c_n. Then

$$i = \psi(i) = c_0 i + \sum c_n e^{n\pi i} \tag{7.71}$$

and we conclude $c_0 = 1$. Moreover, if ψ^d denotes the double of ψ (which has the same Fourier expansion) then since $\psi^d(it)$ is purely imaginary for $0 \le t \le 2$, we find that $c_n = -c_n$ for all $n \ne 0$. Thus

$$\psi(z) = z + \sum_n c_n(e^{n\pi z} - e^{-n\pi z}). \tag{7.72}$$

The area of the image of ψ^d is $\mathcal{O}(\rho)$ by Lemma 7.18. Since this area equals the L^2-norm of the derivative of ψ^d we conclude that

$$2\int_0^\rho 1^2\, d\tau + \sum_{n\in Z}\int_0^\rho n^2\pi^2|c_n|^2 e^{2n\pi\tau}\, d\tau = \mathcal{O}(\rho). \tag{7.73}$$

Integrating we find there exists a constant K and $0 < \delta \ll 1$ such that

$$|c_n| \le K\rho(n)^{-\frac{1}{2}} e^{-n\pi\rho} \le K e^{-n(\pi-\delta)\rho}, \tag{7.74}$$

for each $n \ne 0$. Thus, in the gluing region $[0, a\rho)$ we find

$$|\psi(z) - z| \le K\sum_{n>0} e^{-n(\pi-\delta-a)\rho} \le K' e^{-(\pi-2(\delta+a))\rho} = K' e^{-\eta\rho}, \tag{7.75}$$

where $\eta > 0$. Similarly one shows $|D\psi - \mathrm{id}\,| \le K e^{-\frac{1}{2}\eta\rho}$ and $|D^2\psi| \le K e^{-\frac{1}{2}\eta\rho}$.

Assume (7.67) is not true then there exists a sequence f_j of functions on $M_{\rho(j)}$, $\rho(j) \to \infty$ as $j \to \infty$, with

$$\|f_j\|_{2,\rho,\epsilon} = 1, \tag{7.76}$$

$$\|\bar\partial f_j\|_{1,\rho,\epsilon} \to 0, \quad \text{as } j \to \infty. \tag{7.77}$$

Let $\gamma\colon (-\infty, -(1-a)\rho] \times [0,1]$ be a cut-off function which equals 1 on $(-\infty, -(1-\frac{1}{4}a)\rho] \times [0,1]$ which equals 0 on $[-(1-\frac{1}{2}a)\rho, -(1-a)\rho)$, has $|D^k\gamma| = \mathcal{O}(\rho^{-1})$, $k = 1, 2$, and is real valued and holomorphic on the boundary. Then by uniform invertibility of the $\bar\partial$-operator on the strip with constant weight ϵ we find

$$\|\gamma f\|_{2,\epsilon} \le C(\|(\bar\partial\gamma)f\|_{1,\epsilon} + \|\gamma\bar\partial f\|_{1,\epsilon}). \tag{7.78}$$

Here both terms on the right hand side goes to 0 as $\rho\to\infty$. In a similar way we conclude

$$\|\beta f\|_{2,\epsilon} \to 0, \tag{7.79}$$

for β a cut-off function on $[(1-a)\rho,\infty)$.

Now let α be a cut-off function on $[-A(\rho),A(\rho)]\times[0,1]$ which equals 1 on $[-A(\rho)+2,A(\rho)-2]\times[0,1]$ and equals 0 outside $[-A(\rho)+1,A(\rho)-1]\times[0,1]$. We find

$$\|\alpha f\|_{2,\epsilon} \le C(\|(\bar\partial\alpha)f\|_{1,\epsilon} + \|\alpha\bar\partial f\|_{1,\epsilon}). \tag{7.80}$$

Here the second term on the right hand side goes to 0 as $\rho\to\infty$ by (7.77). The first goes to 0 as well since $\|\gamma f\|\to 0$ and $\|\beta f\|\to 0$ and since the transition functions are very close to the identity for ρ large.

In conclusion we find $\|f\|_{2,\rho,\epsilon}\to 0$, contradicting (7.76), and (7.67) holds. □

7.21. **Uniform invertibility for self tangency gluing.** Let $d\Gamma_\rho$ denote the differential of the map

$$\Gamma\colon \mathcal{W}_{2,\epsilon,\rho} \to \mathcal{H}_{1,\epsilon,\rho}[0](D_{m+1}, T^{*0,1}\otimes\mathbb{C}^n),$$

at w_ρ. Referring to Sections 7.7.1 and 7.7.2, we assume that $m\ge 2$ and $l(j)\ge 2$ for each j.

Lemma 7.21. *There exist constants C and ρ_0 such that if $\rho>\rho_0$ and then there is a continuous right inverse G_ρ of $d\Gamma_\rho$*

$$G_\rho\colon \mathcal{H}_{1,\epsilon,\rho}[0](T^{*0,1}D_r(\rho)\otimes\mathbb{C}^n) \to T_{(w_\rho,\kappa_\rho)}\mathcal{W}_{2,\epsilon,\rho}$$

such that

$$\|G_\rho(\xi)\| \le C\|\xi\|_{1,\epsilon,\rho}.$$

Proof. Recall $0<\epsilon\ll\theta$, where $\theta>0$ is the smallest non-zero complex angel at the self-tangency point. Assume we glue k disks $v_1,\dots,v_k$ to u. The kernels

$$\begin{aligned} d\Gamma_{(u,\kappa_1)} &\subset T_{(u,\kappa_1)}\mathcal{W}_{2,-\epsilon},\\ d\Gamma_{(v_j,\kappa_2(j))} &\subset T_{(v_j,\kappa_2(j))}\widetilde{\mathcal{W}}_{2,\epsilon}, \end{aligned} \tag{7.81}$$

are both of dimension 0 and $d\Gamma_{(u,\kappa_1)}$ and $d\Gamma_{(v_j,\kappa_2(j))}$ are invertible.

As usual we consider the embedding

$$T_{\kappa_1}\mathcal{C}_{m+1} \oplus \bigoplus_{j=1}^{k} T_{\kappa_2(j)}\mathcal{C}_{l(j)+1} \to T_{\kappa_\rho}\mathcal{C}_r, \tag{7.82}$$

which identifies the left-hand side with a subspace of codimension k in $T_{\kappa_\rho}\mathcal{C}_r$. Let W_ρ denote the complement of this subspace in $T_{(w_\rho,\kappa_\rho)}\mathcal{W}_{2,\epsilon,\rho}$. We show that there exists a constant C such that for ρ large enough and $(\xi,\gamma)\in W_\rho$

$$\|(\xi,\gamma)\| \le C\|d\Gamma_\rho(\xi,\gamma)\|. \tag{7.83}$$

Assume (7.83) is not true then there exists a sequence $(\xi_N,\gamma_N)\in W_{\rho(N)}$, where $\rho(N)\to\infty$ as $N\to\infty$ with

$$\|(\xi_N,\gamma_N)\| = 1, \tag{7.84}$$

$$\|d\Gamma_{\rho(N)}(\xi_N,\gamma_N)\| \to 0, \quad \text{as } N\to\infty. \tag{7.85}$$

Let $\beta^0_\rho\colon D_r(\rho)\to\mathbb{C}$ be a cut-off function which equals 1 on $D_{m+1}\setminus(\bigcup_j E_{p_j}[-\frac{1}{2}\rho])$, equals 0 outside $D_{m+1}\setminus(\bigcup_j E_{p_j}[-\frac{3}{4}\rho])$, with $|D^k\beta^0_\rho|=\mathcal{O}(\rho^{-1})$, $k=1,2$. By the uniform invertibility of $d\Gamma_{(u,\kappa_1)}$ we find

$$\|\beta^0_{\rho(j)}(\xi_N,\gamma_N)\|_{2,\epsilon,\rho}\le C\|d\Gamma_{(u,\kappa_1)}\beta^0_{\rho(N)}(\xi_N,\Gamma_N)\|_{1,\epsilon,\rho}\le \tag{7.86}$$

$$C\Big(\|(\bar\partial\beta^0_{\rho(N)})\xi_N\|_{1,\epsilon,\rho}+\|\beta^0_{\rho(N)}d\Gamma_\rho(\xi_N,\gamma_N)\|_{1,\epsilon,\rho}\Big). \tag{7.87}$$

Both terms on the right hand side goes to 0 as $N\to\infty$. Hence

$$\|\beta^0_{\rho(N)}(\xi_N,\gamma_N)\|_{2,\epsilon,\rho}\to 0,\qquad\text{as } N\to\infty. \tag{7.88}$$

Similarly, with $\beta^j_\rho\colon D_r(\rho)\to\mathbb{C}$ a cut-off function which equals 1 on $D_{l(j)+1}\setminus E_{q_j}[\frac{1}{2}\rho]$, equals 0 outside $D_{l(j)+1}\setminus E_{q_j}[\frac{3}{4}\rho]$, with $|D^k\beta^0_\rho|=\mathcal{O}(\rho^{-1})$, $k=1,2$, we find, by the uniform invertibility of $d\Gamma_{(v_j,\kappa_2(j))}$ that

$$\|\beta^j_{\rho(N)}(\xi_N,\gamma_N)\|_{2,\epsilon,\rho}\to 0,\qquad\text{as } N\to\infty\text{ for all } j. \tag{7.89}$$

For $1\le j\le k$ we consider the region

$$\Theta_j(\rho)= \tag{7.90}$$

$$\Big(E_{p_j}\setminus E_{p_j}[-(1+a)\rho]\Big)\cup_{\big((\phi_\rho)^{-1}\circ u^1\big)}\Omega_j\cup_{\big((\phi_\rho)^{-1}\circ v^1_j\big)}\Big(E_{q_j}\setminus E_{q_j}[(1+a)\rho]\Big). \tag{7.91}$$

Note that there is a natural inclusion $\Theta_j(\rho)\subset M_\rho$, where M_ρ is as in Lemma 7.20. Also note that the boundary conditions of the linearized equation over $\Omega_j(\rho)$ splits into a 1-dimensional problem corresponding to the first coordinate and an $(n-1)$-dimensional problem with boundary conditions converging to two transverse Lagrangian subspaces in the remaining $(n-1)$ coordinate directions.

Let α^+_ρ be a cut-off function on $\Theta_j(\rho)$ which equals 1 on

$$E_{p_j}[-\frac{1}{4}\rho]\setminus E_{p_j}[-(1+\frac{1}{2}a)\rho], \tag{7.92}$$

equals 0 outside

$$E_{p_j}[-\frac{1}{8}\rho]\setminus E_{p_j}[-(1+\frac{2}{3}a)\rho], \tag{7.93}$$

with $|D^k\alpha^+|=\mathcal{O}(\rho^{-1})$, $k=1,2$, and which is real valued and holomorphic on the boundary. Note that over the region where α^+ is supported the boundary conditions of w_ρ agrees with those of u. Thus the angle between the line giving the boundary conditions of w_ρ and the real line is $\mathcal{O}(\rho^{-1})$ and it is easy to construct a unitary diagonal matrix function $\mathbf{M}$ on the support α^+ with $|D^k\mathbf{M}|=\mathcal{O}(\rho^{-1})$, $k=1,2$ with the property that $\mathbf{M}\xi_N$ has the boundary conditions of w_ρ in the last $(n-1)$ coordinates and has real boundary conditions in the first coordinate. Thus Lemma 7.20 implies that

$$\|\alpha^+\xi_N\|_{2,\rho,\epsilon}\le C\|\mathbf{M}\alpha^+\xi_N\|\le C\Big(\|(\bar\partial\alpha^+\mathbf{M})\xi\|_{1,\epsilon,\rho}+\|\mathbf{M}\bar\partial\xi_N\|_{1,\epsilon,\rho}\Big). \tag{7.94}$$

Here both terms in the right hand side goes to 0 as $N\to\infty$.

In exactly the same way we show that

$$\|\alpha^-\xi_N\|\to 0\qquad\text{as }\rho\to\infty, \tag{7.95}$$

for a cut-off function α^- with support on the other end of Θ_ρ.

Let α^0 be a cut-off function which equals 1 on $[-A(\rho)+2,A(\rho)-2]\times[0,1]$ and equals 0 outside $[-A(\rho)+1,A(\rho)-1]\times[0,1]$ and with the usual properties. Then the function

$$(\tau+it)\mapsto (d\Phi^{ts(N)}_{R(N)}(\phi_\rho(\tau+it)))^{-1}\cdot\alpha^0(\tau+it)\xi_N(\tau+it) \tag{7.96}$$

has the boundary conditions of w_ρ in the last $(n-1)$ coordinates (two transverse Lagrangian subspaces in this region) and has real boundary conditions in the first coordinate.

Lemma 7.20 implies

$$\begin{aligned}&\|\alpha^0\xi_N\|_{1,\rho,\epsilon} \le C\|(d\Phi_R^{ts(N)}(N))^{-1}\cdot\alpha^0\xi_N\|_{1,\rho,\epsilon} \le \\ &C\Big(\|\bar\partial(\alpha^0 d\Phi_R^{ts(N)}(N))^{-1})\cdot\xi_N\|_{0,\rho,\epsilon} + \|(\alpha^0 d\Phi_R^{ts(N)}(N))^{-1})\cdot\bar\partial\xi_N\|_{0,\rho,\epsilon}\Big).\end{aligned} \tag{7.97}$$

Using (7.55) and (7.56) in combination with (7.94) and (7.95) we find that the first term in (7.97) goes to 0 as $N\to 0$. By (7.85), so does the second. Hence

$$\|\alpha^0\xi_N\|_{1,\rho,\epsilon}\to 0. \tag{7.98}$$

Applying the same argument to $\partial_\tau\xi_N$ and $i\partial_t\xi_N$ we conclude that

$$\|\alpha^0\xi_N\|_{2,\rho,\epsilon}\to 0. \tag{7.99}$$

Now (7.89), (7.88), (7.94), (7.95), and (7.99) contradict (7.84) and we find that (7.83) holds.

To finish the proof we let $\mu_j = \bar\partial\frac{\partial\phi^{C_j}}{\partial C_j}$, see Section 6.9. Then μ_j anti-commutes with j_{κ_ρ} and we consider the μ_j as newborn conformal variations spanning the complement of W_ρ in $T_{(w_\rho,\kappa_\rho)}\mathcal{W}_{2,\epsilon,\rho}$.

The images of μ_j, $j=1,\dots,k$ under $d\Gamma_\rho$ are clearly linearly independent since they have mutually disjoint supports. We show that their images stays a uniformly bounded distance away from the subspace $d\Gamma_\rho(W_\rho)$. Assume not, then there exists a sequence of elements (ξ_ρ,γ_ρ) in W_ρ with

$$\|d\Gamma_\rho\big((\xi_\rho,\gamma_\rho)-\mu_j\big)\|_{1,\epsilon,\rho}\to 0 \quad \text{as } \rho\to\infty. \tag{7.100}$$

Since $\|d\Gamma_\rho\mu_j\|_{1,\epsilon,\rho} = \mathcal{O}(1)$ we conclude from (7.83) that $\|(\xi_\rho,\gamma_\rho)\|_{2,\epsilon,\rho} = \mathcal{O}(1)$. Then, with the cut-off function β_ρ^j from above and notation as in Section 6.9 we find

$$\|d\Gamma_{(v_j,\kappa_2(j))}(\beta_\rho^j(\xi_\rho,\gamma_\rho)-\hat C_j)\|_{1,\epsilon} = \|d\Gamma_\rho(\beta_\rho^j(\xi_\rho,\gamma_\rho)-\mu_j)\|_{1,\epsilon,\rho}\le \tag{7.101}$$

$$\|\beta_\rho^j(d\Gamma_\rho(\xi_\rho,\gamma_\rho)-\mu_j)\|_{1,\epsilon,\rho} + \|(\bar\partial\beta_\rho^j)((\xi_\rho,\gamma_\rho)-\mu_j)\|_{1,\epsilon,\rho}. \tag{7.102}$$

The right hand side of the above equation goes to 0 as $\rho\to\infty$. Hence so does the left hand side. This however contradicts the invertibility of $d\Gamma_{(v_j,\kappa_2(j))}$ and we conclude $d\Gamma_\rho(W_\rho)$ stays a bounded distance away from $d\Gamma_\rho(\mu_j)$. Thus, defining $G_\rho(d\Gamma_\rho\mu_j)=\mu_j$ finishes the proof. □

7.22. **Estimates on the non-linear term.** In Section 4.7, we linearized the map Γ using local coordinates B around $(w,f)\in\mathcal{W}_{2,\epsilon}$. To apply Floer's Picard lemma, we must study also higher order variations of Γ.

For $(w,f,0)\in\mathcal{W}_{2,\epsilon,\Lambda}$, $w\colon D_m\to\mathbb{C}^n$ and conformal structure κ on D_m, we take as in Section 4.5 local coordinates $(v,\kappa)\in B_{2,\epsilon}\times\mathbb{R}^{m-3}\times\Lambda$ on $\mathcal{W}_{2,\epsilon,\lambda}$ around (w,f) and write (in these coordinates)

$$\Gamma(v,\lambda,\gamma) = \bar\partial_\kappa v + i\circ dw\circ\gamma + \lambda\cdot\bar\partial_\kappa Y_0^\sigma + N(v,\lambda,\gamma).$$

We refer to $N(v,\lambda,\gamma)$ as the *non-linear term.* We consider Λ to have dimension 0 in the stationary and the self tangency cases and have dimension 1 in the handle slide case. We first consider stationary gluing

Lemma 7.22. *There exists a constant C such that the non-linear term $N(v,\gamma)$ of Γ in a neighborhood w_ρ, where w_ρ is as in Section 7.5 satisfies*

$$\|N(u,\beta)-N(v,\kappa)\|_1 \le C\Big(\|u\|_2+|\beta|+\|v\|_2+|\gamma|\Big)\Big(\|u-v\|_2+|\beta-\gamma|\Big) \tag{7.103}$$

Proof. With notation as in Section 4.5 we have

$$\Gamma(v,\gamma) = \bar{\partial}_{\kappa+\gamma}\Big(\exp^{\sigma(\zeta)}_{w_\rho(\zeta)} v(\zeta)\Big).$$

We prove (7.103) first in the special case $\gamma = \beta = 0$. We perform our calculation in coordinates $x+iy$ on $D_r(\rho)$, which agree with the standard coordinates on the ends and in the gluing region on $D_r(\rho)$. On the remaining parts of the disk the metric of these coordinates differs from the usual metric by a conformal factor but since the remaining part is compact the estimates are unaffected by this change of metric. In these coordinates we write $\bar{\partial}_\kappa = \partial_x + i\partial_y$. Now, as in Lemma 4.12 we find

$$\partial_x \exp^{\sigma}_{w_\rho} v = J[w_\rho, v, \partial_x w_\rho, \partial_x v, \sigma](1) + \partial_\sigma(\exp^\sigma_{w_\rho} v)\cdot\partial_x\sigma,$$

where $J[x,\xi,x',\xi',\sigma]$ denotes the Jacobi field in the metric $g(\sigma)$ along the geodesic $\exp^\sigma_x t\xi$ with initial conditions $J(0) = x'$, $J'(0) = \xi'$. Of course a similar equation holds for $\partial_y \exp^\sigma_{w_\rho} v$.

Let $G\colon (\mathbb{C}^n)^4 \times [0,1]\times\mathbb{R} \to \mathbb{C}^n$ be the function

$$G(x,\xi,x',\xi',\sigma,\sigma') = J[x,\xi,x',\xi',\sigma](1) - x' - \xi' + \partial_\sigma \exp^\sigma_x \xi\cdot\sigma'$$

(unrelated to the earlier right inverses G_ρ) then with $w_\rho = w$,

$$N(v) = G(w,v,\partial_x w,\partial_x v,\sigma,\partial_x\sigma) + iG(w,v,\partial_y w,\partial_y v,\sigma,\partial_y\sigma).$$

Moreover, the function G is smooth with uniformly bounded derivatives, it is linear in x',ξ',σ', and satisfies

$$\begin{aligned} &G(x,0,x',\xi',\sigma,\sigma') = 0,\\ &D_2G(x,0,x',\xi',\sigma,\sigma') = 0, \end{aligned} \tag{7.104}$$

where the last equation follows from Taylor expansion of the exponential map and the Jacobi field.

We estimate the 1-norm of

$$G(w,u,\partial_x w,\partial_x u,\sigma,\partial_x\sigma) - G(w,v,\partial_x w,\partial_x v,\sigma,\partial_x\sigma).$$

For the 0-norm, we write

$$\begin{aligned} &\Big|G(w,u,\partial_x w,\partial_x u,\sigma,\partial_x\sigma) - G(w,v,\partial_x w,\partial_x v,\sigma,\partial_x\sigma)\Big| \le\\ &\Big|G(w,u,\partial_x w,\partial_x u,\sigma,\partial_x\sigma) - G(w,u,\partial_x w,\partial_x v,\sigma,\partial_x\sigma)\Big|\\ &+\Big|G(w,u,\partial_x w,\partial_x v,\sigma,\partial_x\sigma) - G(w,v,\partial_x w,\partial_x v,\sigma,\partial_x\sigma)\Big| \le\\ &C\Big(|u||Du-Dv| + |Dv||u-v|\Big) \le\\ &C\Big((|u|+|v|)|Du-Dv| + (|Du|+|Dv|)|u-v|\Big), \end{aligned} \tag{7.105}$$

where we use (7.104). Noting that the $\|\cdot\|_2$-norm controls the sup-norm we square and integrate (7.105) to conclude

$$\|N(u)-N(v)\|_0 \le C\Big(\|u\|_2 + \|v_2\|\Big)\|u-v\|_2. \tag{7.106}$$

For the 1-norm we find an L^2-estimate of

$$
\begin{aligned}
&\Big|D_1G(w,u,\partial_x w,\partial_x u,\sigma,\partial_x\sigma) - D_1G(w,v,\partial_x w,\partial_x v,\sigma,\partial_x\sigma)\Big||Dw|\\
&+\Big|D_2G(w,u,\partial_x w,\partial_x u,\sigma,\partial_x\sigma)\cdot Du - D_2G(w,v,\partial_x w,\partial_x v,\sigma,\partial_x\sigma)\cdot Dv\Big|\\
&+\Big|D_3G(w,u,\partial_x w,\partial_x u,\sigma,\partial_x\sigma)\cdot D\partial_x w - D_3G(w,v,\partial_x w,\partial_x v,\sigma,\partial_x\sigma)\cdot D\partial_x w\Big|\\
&+\Big|D_4G(w,u,\partial_x w,\partial_x u,\sigma,\partial_x\sigma)\cdot D\partial_x u - D_4G(w,v,\partial_x w,\partial_x v,\sigma,\partial_x\sigma)\cdot D\partial_x v\Big|\\
&+\Big|D_5G(w,u,\partial_x w,\partial_x u,\sigma,\partial_x\sigma) - D_5G(w,v,\partial_x w,\partial_x v,\sigma,\partial_x\sigma)\Big||D\sigma|\\
&+\Big|D_6G(w,u,\partial_x w,\partial_x u,\sigma,\partial_x\sigma)\cdot D\partial_x \sigma - D_6G(w,v,\partial_x w,\partial_x v,\sigma,\partial_x\sigma)\cdot D\partial_x\sigma\Big|. \qquad (7.107)
\end{aligned}
$$

Using (7.104) the first and fifth terms in (7.107) are estimated by

$$C(|u|+|v|)(|u-v|+|Du-Dv|)(|Dw|+|DF|), \qquad (7.108)$$

where F is the extension of $f\colon \partial D_r(\rho)\to\mathbb{R}$ as in Section 4.5. The second term in (7.107) is estimated by

$$C\Big((|u|+|v|)|Du-Dv| + (|Du|+|Dv|)|Du-Dv| + |u-v|(|Du|+|Dv|)\Big). \qquad (7.109)$$

For the remaining terms we use the linearity of G in x',ξ',σ' to write them as

$$
\begin{aligned}
&\Big|D_3G(w,u,D\partial_x w,\partial_x u,\sigma,\partial_x\sigma) - D_3G(w,v,D\partial_x w,\partial_x v,\sigma,\partial_x\sigma)\Big|\\
&+\Big|D_4G(w,u,\partial_x w,D\partial_x u,\sigma,\partial_x\sigma) - D_4G(w,v,\partial_x w,D\partial_x v,\sigma,\partial_x\sigma)\Big|\\
&+\Big|D_6G(w,u,\partial_x w,\partial_x u,\sigma,D\partial_x\sigma) - D_6G(w,v,\partial_x w,\partial_x v,\sigma,D\partial_x\sigma)\Big|.
\end{aligned}
$$

Thus the third and sixth terms in (7.107) are estimated as in (7.105) by

$$C\Big((|u|+|v|)|Du-Dv| + (|Dv|+|Du|)|u-v|\Big). \qquad (7.110)$$

Finally, the fourth term in (7.107) is estimated by

$$C\Big((|u|+|v|)|D^2u-D^2v| + (|D^2u|+|D^2v|)|u-v|\Big) \qquad (7.111)$$

To estimate the L^2-norms of these expressions we use the sup-norm bound of $|u|$ and $|v|$ and the fact that the $\|\cdot\|_1$-norm controls the L^p-norm for all $p>2$. For example

$$
\begin{aligned}
&\int_{D_m}(|Du|+|Dv|)^2|Du-Dv|^2\,dA \le\\
&\Big(\int_{D_m}(|Du|+|Dv|)^4\Big)^{\frac12}\Big(\int_{D_m}(|Du-Dv|)^4\Big)^{\frac12}\le\\
&(\|u\|_2+\|v\|_2)^2(\|u-v\|_2)^2. \qquad (7.112)
\end{aligned}
$$

We conclude that we have the estimate

$$\|N(u)-N(v)\|_1 \le C\Big(\|u\|_2+\|v\|_2\Big)\|u-v\|_2,$$

as desired.

Finally in the case when also the conformal structures changes we note that if j_κ in coordinates $x+iy$ is represented by the matrix

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

then $j_{\kappa+\gamma}$ is represented by the matrix

$$\begin{pmatrix} \phi_\gamma & -1 \\ 1+\phi_\gamma^2 & -\phi_\gamma \end{pmatrix},$$

where $\phi_\gamma\colon D_m \to \mathbb{R}$ is a compactly supported function. The extra term which enters in the non-linear term is then

$$(\phi_\beta^2 G(w,u,\partial_y w,\partial_y u,\sigma,\partial_y\sigma) - \phi_\gamma^2 G(w,v,\partial_y w,\partial_y v,\sigma,\partial_y\sigma)),$$

which is easily estimated using the techniques above. □

For handle slide gluing the corresponding lemma reads.

Lemma 7.23. *There exists a constant C such that the non-linear term $N(v,\lambda,\gamma)$ of Γ in a neighborhood of w_ρ, where w_ρ is as in Section 7.6 satisfies*

$$\|N(u,\beta,\lambda)-N(v,\kappa,\mu)\|_1 \le C\Big(\|u\|_2+|\beta|+|\lambda|+\|v\|_2+|\gamma|+|\mu|\Big)\Big(\|u-v\|_2+|\beta-\gamma|+|\lambda-\mu|\Big)$$

Proof. The proof is similar to the proof of Lemma 7.22. The main difference arises since the local coordinate map also depends on $\lambda \in \Lambda$. Replacing the function G in the proof of Lemma 7.22 with the function (notation as in Section 4.5)

$$H(x,\xi,x',\xi',\sigma,\sigma',\lambda) = \\ J[x,\xi,x',\xi',\sigma,\lambda](1) - x' - \xi' - \lambda\cdot DY_0^\sigma(x)\cdot x' + \partial_\sigma(\exp^{\lambda,\sigma}_{\psi^\sigma_\lambda(x)} A^\lambda_\sigma\xi)\cdot\sigma',$$

where $J[x,\xi,x',\xi',\sigma,\lambda]$ is the Jacobi field along the geodesic $\exp^{\lambda,\sigma}_{\psi^\sigma_\lambda(x)} A^\sigma_\lambda\xi$ with initial conditions $J(0) = d\psi^\sigma_\lambda\cdot x' + \partial_\sigma\psi^\sigma_\lambda\cdot\sigma'$, $J'(0) = A^\sigma_\lambda\xi' + dA^\sigma_\lambda\cdot x'\cdot\xi + \partial_\sigma A^\sigma_\lambda\cdot\sigma'\cdot\xi$ and repeating the argument given there proves the lemma. □

In the self-tangency shortening case the estimate is somewhat changed since we work in Sobolev spaces with negative exponential weights in the gluing region. Here we have

Lemma 7.24. *There exists a constant C such that the non-linear term $N(v,\gamma,\lambda)$ of Γ in a neighborhood of w_ρ, where w_ρ is as in Section 7.12 satisfies*

$$\|N(u,\beta)-N(v,\kappa)\|_{1,-\epsilon,\rho} \le Ce^{\epsilon\rho}\Big(\|u\|_{2,-\epsilon,\rho}+|\beta|+\|v\|_{2,-\epsilon,\rho}+|\gamma|\Big) \\ \times\Big(\|u-v\|_{2,-\epsilon,\rho}+|\beta-\gamma|\Big)$$

Proof. The proof is exactly the same as the proof of Lemma 7.22. We must only take into account in what way the weights affect the estimates. Starting with (7.106), we see that the norm $\|\cdot\|_{2,\rho,-\epsilon}$ does not control the sup-norm uniformly in ρ. But it does control $e^{-\epsilon\rho}$ times the sup-norm. Thus we conclude instead of (7.106)

$$\|N(u)-N(v)\| \le Ce^{\epsilon\rho}(\|u\|_{2,-\epsilon,\rho}+\|v\|_{2,-\epsilon,\rho})\|u-v\|_{2,-\epsilon,\rho}. \tag{7.113}$$

Similarly, we loose this factor in the other estimates where we use the sup-norm. Let e_ρ denote the weight function from Section 7.13. When we use the L^4-estimate we have instead of (7.112) the following

$$\begin{aligned}&\int_{D_m}(|Du|+|Dv|)^2|Du-Dv|^2e_\rho^2\,dA\le\\ e^{2\epsilon\rho}&\int_{D_m}(|Du|+|Dv|)^2|Du-Dv|^2e_\rho^4\,dA\le\\ e^{2\epsilon\rho}&\Big(\int_{D_m}(|Du|+|Dv|)^4e_\rho^4\Big)^{\frac{1}{2}}\Big(\int_{D_m}(|Du-Dv|)^4e_\rho^4\Big)^{\frac{1}{2}}\le\\ Ce^{2\epsilon\rho}&(\|u\|_{2,-\epsilon,\rho}+\|v\|_{2,-\epsilon,\rho})^2(\|u-v\|_{2,-\epsilon,\rho})^2.\end{aligned}$$

We conclude finally

$$\|N(u)-N(v)\|_{1,-\epsilon,\rho}\le Ce^{\epsilon\rho}(\|u\|_{2,-\epsilon,\rho}+\|v\|_{2,-\epsilon,\rho})\|u-v\|_{2,-\epsilon,\rho}.$$

The same argument as in Lemma 7.22 then takes care of the conformal structures and the lemma follows. □

Finally, we consider self-tangency gluing, where we have a large weight function which does not interfere (destructively) with the sup-norm and the L^4 estimates.

Lemma 7.25. *There exists a constant C such that the non-linear term $N(v,\gamma)$ of Γ in a neighborhood of w_ρ, where w_ρ is as Section 7.17 satisfies*

$$\begin{aligned}\|N(u,\beta)-N(v,\kappa)\|_1\le &C\Big(\|u\|_{2,\epsilon,\rho}+|\beta|+\|v\|_{2,\epsilon,\rho}+|\gamma|\Big)\\ &\times\Big(\|u-v\|_{2,\epsilon,\rho}+|\beta-\gamma|\Big)\end{aligned}$$

Proof. See the proof of Lemma 7.22 □

8. Gromov compactness

In this section we prove a version of the Gromov compactness theorem. In Section 8.2, we discuss the compactification of the space of conformal structures which is done is [17]. In Section 8.3, we translate the notions of convergence and (limiting) broken curves from [22] to our setting. There are two notions of convergence we must prove: a strong local convergence and a weak global convergence. In Sections 8.5 and 8.6, we adopt Floer's original approach, [13], to prove the strong local convergence. In proving local convergence, we in fact prove that our holomorphic disks, away from the punctures, are smooth up to and including the boundaries, see Remark 8.6. To prove the weak global convergence in Section 8.7, we analyze where the area (or energy) of a sequence of disks accumulates, and construct an appropriate sequence of reparameterizations of the domain to recover this area.

We note that although our holomorphic curves map to a non-compact space, $\mathbb{C}^n$, the set of curves we consider lives in a compact subset. This follows because $\mathbb{C}^n$ is a symplectic manifold with "finite geometry at infinity": a holomorphic curve with a non-compact image must contain infinite area. And the area of any disk we consider is bounded above by the action of the chords mapped to at its corners. Thus, we can prove the Gromov compactness theorem in this non-compact set-up. For a review of finite geometry at infinity (also known as "tame"), see [1] Chapter 5, as well as [7, 18, 31].

8.1. **Notation and conventions for this section.** Unlike in the other sections, we need to consider Sobolev spaces with derivatives in L^p for $p\neq 2$. We define in the obvious way the spaces $W^{p,\mathrm{loc}}_k(\Delta_m,\mathbb{C}^n)$ to indicate $\mathbb{C}^n$-valued functions on Δ_m whose first k derivatives are locally L^p-integrable. For this section only, we denote the corresponding norm by $\|\cdot\|_{k,p}$.

In order to define broken curves in the next subsection, we will need to extend the disk continuously to the boundary punctures. Of course the extra Legendrian boundary condition, h, does not extend continuously. For this reason, we will only extend u to $\bar{\Delta}_m$, the closure of Δ_m; thus, $u\colon \bar{\Delta}_m \to \mathbb{C}^n$. Note that $\|u\|_{p,k}$ might still blow up at these punctures. We sometimes only consider u and $u|\partial\bar{\Delta}_m$ in which case we write $u : (\bar{\Delta}_m, \partial\bar{\Delta}_m) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(L))$. For $X \subset \bar{\Delta}_m$, let $\|u\|_{k,p:X} = \|u|X\|_{k,p}$, and $\|u\|_{k,p:\epsilon}$ denote the norm restricted to some disk (or half-disk) of radius ϵ.

Because we sometimes change the number of boundary punctures, we will denote by D the unit disk in $\mathbb{C}^n$.

8.2. **Compactification of space of conformal structures.** Recall $\mathcal{C}_m$ is the space of conformal structures (modulo conformal reparameterizations) on the unit disk in $\mathbb{C}$ with m boundary punctures.

When $m \geq 3$, we define a *stable cusp disk representative* with m *marked boundary points*, $(\Sigma; p_1, \dots, p_m)$, to be a connected, simply-connected union of unit disks in $\mathbb{C}$ where pairs of disks may overlap at isolated boundary points (which we call *double points* of Σ) and each disk in Σ has at least 3 points, called *marked points*, which correspond either to double points or the original boundary marked points. When $m = 1$ or 2, the stable cusp disk representative shall be a single disk. Two stable cusp disk representatives are *equivalent* if there exist a conformal reparameterization of the disks taking one set of marked points to the other. We define a *stable (cusp) disk* with m marked points to be an equivalence class of stable disk representatives with m marked points.

In Section 10 of [17], Fukaya and Oh prove that $\bar{\mathcal{C}}_m$, the compactification of $\mathcal{C}_m$, is the space of stable disks with m marked points.

8.3. **The statement.** A *broken curve* $(u,h) = ((u^1,h^1), \dots (u^N,h^N))$ is a connected union of holomorphic disks, (u^j,h^j), (recall u^j is extended to $\bar{\Delta}_{m_j}$) where each u^j has exactly one positive puncture and except for one disk, say u^1, the positive puncture of u^j agrees with the negative puncture of some other $u^{j'}$. One may easily check that a broken curve can be parameterized by a single smooth $v : (D_m, \partial D) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(L))$, such that v^{-1} is finite except at points where two punctures were identified, here v^{-1} is an arc in Δ_m.

Definition 8.1. A sequence of holomorphic disks (u_α, h_α) *converges* to a broken curve $(u,h) = ((u^1,h^1), \dots, (u^N,h^N))$ if the following holds

(1) (*Strong local convergence*) For every $j \leq N$, there exists a sequence $\phi^j_\alpha : D \to D$ of linear fractional transformations and a finite set $X^j \subset D$ such that $u_\alpha \circ \phi^j_\alpha$ converges to u^j uniformly with all derivatives on compact subsets of $D \setminus X^j$
(2) (*Weak global convergence*) There exists a sequence of orientation-preserving diffeomorphisms $f_\alpha : D \to D$ such that $u_\alpha \circ f_\alpha$ converges in the C^0-topology to a parameterization of u.

Henceforth, to simplify notation when passing to a subsequence, we will not change the indexing.

Theorem 8.2. *Assume $(u_\alpha, h_\alpha) \in \mathcal{M}(a; b_1, \dots, b_m)$ is a sequence of holomorphic disks with L_α Legendrian boundary condition. Let $\kappa_\alpha \in \mathcal{C}_{m+1}$ denote the conformal structure on the domain of u_α. Assume L_α converges to an embedded Legendrian L in the C^∞-topology. Then there exists a subsequence $(u_\alpha, h_\alpha, \kappa_\alpha)$ such that κ_α converges to $\kappa \in \bar{\mathcal{C}}_{m+1}$ and (u_α, h_α) converges to a broken curve (u,h) whose domain is a stable disk representative of κ.*

Note that using the strong local convergence property *a posteriori*, this compactness result proves that all derivatives of a holomorphic disk (u,h) are locally integrable away from the finite set of points. In particular, such disks are smooth at the boundary away from these points. See Remark 8.6.

We also remark that, the appropriately modified, Theorem 8.2 holds if the disks have more than one positive puncture.

8.4. **Area of a disk.** For holomorphic $u : (D, \partial D) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(L))$, recall that $\text{Area}(u) = \int u^*\omega$, where $\omega = \sum_i dx_i \wedge dy_i$, denotes its (signed) area.

Lemma 8.3. *Consider an admissible Legendrian isotopy parameterized by $\lambda \in \Lambda$. We assume $\Lambda \subset \mathbb{R}$ is compact. Denote by L_λ the moving Legendrian submanifold. There exists a positive upper semi-continuous function $\hbar : \Lambda \to \mathbb{R}^+$ such that for any non-constant holomorphic map $u : (D, \partial D) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(L_\lambda))$, $\text{Area}(u) \geq \hbar(\lambda)$.*

Proof. We need the following statement from Proposition 4.3.1 (ii) of Sikorav in [1]: There are constants r_1, k (depending only on $\mathbb{C}^n$) such that if $r \in (0, r_1]$ and $u \colon \Sigma \to B(x, r)$ is a holomorphic map of a Riemann surface containing x in its image and with $u(\partial\Sigma) \subset \partial B(x, r)$ then $\text{Area}(\Sigma) \geq kr^2$.

Since u is non-constant, Stokes Theorem implies u must have boundary punctures. Choose $r > 0$, an upper semi-continuous function of λ, such that:

- for all Reeb chords c, $\Pi_{\mathbb{C}}(L_\lambda) \cap B(c^*, r)$ is real analytic and diffeomorphic either to $\mathbb{R}^n \times \{0\} \cup \{0\} \times \mathbb{R}^n$ or the local picture of the singular moment in a standard self-tangency move (see Definition 2.3).
- for all distinct Reeb chords c_1, c_2, $B(c_1^*, r) \cap B(c_2^*, r) = \emptyset$ and
- $r < r_1$.

Let θ_λ be the smallest angle among all the complex angles associated to all the tranverse double points of $\Pi_{\mathbb{C}}(L_\lambda)$. Now set

$$\hbar(\lambda) = \min\left\{\min_{c \in \mathcal{C}(L_\lambda)} \mathcal{Z}(c), \frac{kr^2\cos^2\theta_\lambda}{8}\right\} > 0.$$

Suppose u maps all n of its punctures to the same double point c^*, then by (1.4)

$$\text{Area}(u) \geq \mathcal{Z}(c) \geq \hbar.$$

(Note the number of positive punctures of u must be larger than the number of negative ones since u is not constant.)

Otherwise, assume u maps boundary punctures to at least two distinct double points c_1^*, c_2^* where c_1^* is a non-degenerate double point. Then $c_2^* \notin \bar{B}(c_1^*, r)$ implies that there exists a point $x \in u(D) \cap \Pi_{\mathbb{C}}(L) \cap \partial B(c_1^*, \frac{r}{2})$. Moreover, $B(x, \frac{r\cos\theta_\lambda}{2}) \subset B(x, r)$ intersects $\Pi_{\mathbb{C}}(L)$ in only one sheet. Using the real-analyticity of the boundary, we double $u(D) \cap B(x, \frac{r\cos\theta_\lambda}{2})$ and apply the proposition of Sikorav to conclude

$$\text{Area}(u) \geq \text{Area}(u(D) \cap B(x, \frac{r\cos\theta_\lambda}{2})) \geq \frac{kr^2\cos^2\theta_\lambda}{8} \geq \hbar.$$

□

We introduce one more area-related notion, again borrowed from [22]. Given a sequence of holomorphic maps u_α we say $z \in D$ is a *point mass* of $\{u_\alpha\}$ with *mass* m if there exists a sequence $z_\alpha \in D$ converging to $z \in D$ such that

$$\lim_{\epsilon \to 0} \lim_{\alpha \to \infty} \text{Area}\,(u\,|B_\epsilon(z_\alpha) \cap D\,) = m.$$

8.5. **Strong local convergence I: bootstrapping.** In this subsection we prove the following "bootstrap" elliptic estimate: if we know a holomorphic curve lies locally in W_k^p with $p > 2, k \geq 1$, then the $\|\cdot\|_{p',k}$-(local)-norm controls the $\|\cdot\|_{p',k+1}$-(local)-norm for $p' \in [2,p)$.

This proof first appeared as Lemma 2.3 in [13] and later corrected as Proposition 3.1 [24]. Floer and Oh both prove the $k = 1$ case and state the general case. Although there are no new techniques here, we present for the reader the general case in more detail.

Let $A \subset \mathbb{C}$ denote the open disk or half-disk with boundary on the real line. Let $W_k^p(A, \mathbb{C}^n)$ denote the closure, under the $\|\cdot\|_{k,p}$-norm, of the set of all smooth compactly supported functions from A to $\mathbb{C}^n$.

Lemma 8.4. *For every $l > k$, $l - 2/q > k - 2/p$, there exists a constant C such that if $\xi \in W_k^p(A, \mathbb{C}^n)$ is compactly supported, $\xi|\partial A \subset \mathbb{R}^n$, and $\bar\partial\xi \in W_{l-1}^q(A, \mathbb{C}^n)$ then*

$$\|\xi\|_{l,q} \leq C\|\bar\partial\xi\|_{l-1,q}. \tag{8.1}$$

This is stated as Lemma 2.2 of [13] and Lemma 3.2 of [24]. Floer attributes this result to Theorem 20.1.2 of [20]. However, we were unable to deduce Lemma 8.4 for $k > 1$ from Hörmander's theorem. Alternatively, one can use the Seeley extension theorem (see [23], section 1.4 for example) to extend the map to the full disk (in the case of the half disk) and then use the well-known full disk version of Lemma 8.4.

We can now state and prove this subsection's main theorem.

Theorem 8.5. *Fix $k \geq 1$ and (not necessarily small) $\delta_{k-1} > \delta_k \geq 0$. For any compact $K \subset A$, there exists a "constant" $C_1 = C_1(\|u\|_{k,2+\delta_{k-1}})$ depending continuously on $\|u\|_{k,2+\delta_{k-1}}$ such that for all holomorphic maps $u \in W_k^{2+\delta_{k-1}}(A, \mathbb{C}^n)$ with $u(\partial A) \subset \Pi_{\mathbb{C}}(L)$, we have*

$$\|u\|_{k+1,2+\delta_k:K} \leq C_1\|u\|_{k,2+\delta_k:A}. \tag{8.2}$$

Moreover, if u_α is a sequence of holomorphic maps in $W_k^{2+\delta_{k-1}}(A, \mathbb{C}^n)$ such that $u_\alpha(\partial A) \subset \Pi_{\mathbb{C}}(L)$ and $\|u_\alpha\|_{k,2+\delta_{k-1}}$ is uniformly bounded, then there exists a subsequence u_α converging in $W_k^{2+\delta_k}(K, \mathbb{C}^n)$ to some holomorphic map $u : K \to \mathbb{C}^n$.

Remark 8.6. Note how we can use the Sobolev embedding theorem to conclude that all derivatives of the curve lie in L^2 locally, assuming we have a finite local $\|\cdot\|_{1,2+\delta_0}$ norm to begin with. In particular, a holomorphic disk (h, u) with boundary punctures becomes smooth at the boundary away from the punctures. We did not have to assume this smoothness *a priori.*

Proof. We shall only prove the first statement. The second one easily follows from the first and the Sobolev embedding theorem.

Our goal is to prove (8.2) in some small ϵ ball in K. The claim will then follow from the compactness of K.

Because $W_k^{2+\delta_{k-1}}$ compactly sits in C^0 for $k \geq 1$, we can choose small ϵ (continuously in $\|u\|_{k,2+\delta_{k-1}}$) and ϵ' such that given $z_0 \in K$, $u(B(z_0,\epsilon)) \subset B(u(z_0,\epsilon'))$. We fix ϵ and ϵ' at the end of the proof. Assume ϵ is small enough such that $B(z_0,\epsilon)$ does not contain any boundary punctures of u.

Choose a diffeomorphism ϕ of $\mathbb{C}^n$ so that $\phi(\Pi_{\mathbb{C}}(L)) \cap B_{\epsilon'}(\phi \circ u(z_0))$ corresponds to a piece of $\mathbb{R}^n \subset \mathbb{C}^n$ if $z_0 \in \partial A$. Assume $z_0 = 0 \in \partial A$ as we will not consider the easier interior estimate. Locally near 0 define $v = \phi \circ u$; thus, v has $\mathbb{R}^n$ boundary conditions and $\bar\partial_{\phi^* i} v = 0$ where $\bar\partial_{\phi^* i}$ uses the pull-back (almost) complex structure.

Choose a compactly supported smooth function $\gamma : A \to \mathbb{R}$ such that $\gamma(z) = 1$ for $|z| \leq \frac{1}{2}$ and set $\gamma_\epsilon(z) = \gamma(z/\epsilon)$. Note that we can choose γ such that the C^k-norm $\|\gamma_\epsilon\|_{C^k}$ is of order $\frac{1}{\epsilon^k}$. By Lemma 8.4, there exists C_2 such that for any ϵ,

$$(8.3)\qquad \begin{aligned}\|\gamma_\epsilon v\|_{k+1,2+\delta_k} &\leq C_2\|\bar\partial(\gamma_\epsilon v)\|_{k,2+\delta_k}\\ &\leq C_2(\|(\bar\partial\gamma_\epsilon)v\|_{k,2+\delta_k}+\|\gamma_\epsilon\bar\partial v\|_{k,2+\delta_k})\\ &\leq C_2\left(\|\gamma_\epsilon\|_{C^k}\|v\|_{k,2+\delta_k:\epsilon}+\left\|\gamma_\epsilon(\phi^*i-i)\frac{\partial v}{\partial y}\right\|_{k,2+\delta_k}\right)\end{aligned}$$

where $x+iy$ is the complex coordinate in A.

To simplify notation, let $J(v)=\phi^*i-i$. We can assume that all derivatives of $J(\cdot)$ are uniformly bounded on $B(\phi\circ u(z_0),\epsilon')$. We consider the last term of (8.3):

$$(8.4)\qquad \begin{aligned}\left\|\gamma_\epsilon J(v)\frac{\partial v}{\partial y}\right\|_{k,2+\delta_k} &\leq C_3\sum_{\{i,j\ \mid\ i+j=k\}}\left\|D^i(J(v))D^j\left(\gamma_\epsilon\frac{\partial v}{\partial y}\right)\right\|_{0,2+\delta_k}\\ &\leq C_4C_3\|J(v)\|_{0,\infty:\epsilon}\left\|D^k\left(\gamma_\epsilon\frac{\partial v}{\partial y}\right)\right\|_{0,2+\delta_k}+\\ &\quad C_3\sum_{\{i,j\ \mid\ i+j=k,j<k\}}\|D^i(J(v))\|_{0,2+\delta_{k-1}:\epsilon}\left\|D^j\left(\gamma_\epsilon\frac{\partial v}{\partial y}\right)\right\|_{0,p:\epsilon}\end{aligned}$$

where $\frac{1}{p}=\frac{1}{2+\delta_k}-\frac{1}{2+\delta_{k-1}}$. (Here we use $\delta_{k-1}>\delta_k$.)

Choose ϵ small enough such that $|J(v)|\leq\frac{1}{3C_4C_3C_2}$; thus, combining (8.3) and (8.4), we get

$$(8.5)\qquad \begin{aligned}\frac{2}{3}\|\gamma_\epsilon v\|_{k+1,2+\delta_k} &\leq C_2\|\gamma_\epsilon\|_{C^k}\|v\|_{k,2+\delta_k:\epsilon}+\\ &\quad C_2C_3\sum_{\{i,j\ \mid\ i+j=k,j<k\}}\|D^i(J(v))\|_{0,2+\delta_{k-1}:\epsilon}\left\|D^j\left(\gamma_\epsilon\frac{\partial v}{\partial y}\right)\right\|_{0,p}.\end{aligned}$$

Since, $i\cdot(2+\delta_{k-1})>2$ for $1\leq i\leq k$, we get (see [22] Proposition B.1.7)

$$(8.6)\qquad \left\|D^i(J(v))\right\|_{0,2+\delta_{k-1}:\epsilon}\leq\|J(v)\|_{i,2+\delta_{k-1}:\epsilon}\leq C_5\left(\|J\|_{C^i}+1\right)\|v\|_{i,2+\delta_{k-1}:\epsilon}\leq C_6,$$

where $C_6=C_6\left(\|v\|_{k,2+\delta_{k-1}:\epsilon}\right)$.

Fix any $s>p$. Then for any $\kappa>0$, if we set

$$\mu=\frac{\frac{1}{2+\delta_k}-\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}}$$

we get

$$(8.7)\qquad \begin{aligned}\left\|D^j\left(\gamma_\epsilon\frac{\partial v}{\partial y}\right)\right\|_{0,p} &\leq \kappa\left\|D^j\left(\gamma_\epsilon\frac{\partial v}{\partial y}\right)\right\|_{0,s}+\kappa^{-\mu}\left\|D^j\left(\gamma_\epsilon\frac{\partial v}{\partial y}\right)\right\|_{0,2+\delta_k}\\ &\leq \kappa C_7\left\|D^j\left(\gamma_\epsilon\frac{\partial v}{\partial y}\right)\right\|_{1,2+\delta_k}+\kappa^{-\mu}\left\|D^j\left(\gamma_\epsilon\frac{\partial v}{\partial y}\right)\right\|_{0,2+\delta_k}\\ &\leq \kappa C_8\left\|\gamma_\epsilon v\right\|_{j+2,2+\delta_k}+\kappa^{-\mu}\left\|D^j\left(\gamma_\epsilon\frac{\partial v}{\partial y}\right)\right\|_{0,2+\delta_k}.\end{aligned}$$

The first inequality uses an interpolation result and the second the embedding theorem.

Choose κ such that

$$(8.8)\qquad C_2C_3C_6\kappa C_8\leq\frac{1}{3}.$$

Using (8.6) to (8.8) we bound the first term of the sum on the left hand side of (8.5):

$$
\begin{aligned}
(8.9)\qquad & C_2C_3\left\|D^1J(v)\right\|_{0,2+\delta_{k-1}:\epsilon}\left\|D^{k-1}\left(\gamma_\epsilon\frac{\partial v}{\partial y}\right)\right\|_{0,p} \\
&\leq \frac{1}{3}\|\gamma_\epsilon v\|_{k+1,2+\delta_k}+C_2C_3C_6\cdot\kappa^{-\mu}\|v\|_{k,2+\delta_k:\epsilon}
\end{aligned}
$$

Combining (8.5) with (8.9), there exists $C_9 = C_9\left(\|v\|_{k,2+\delta_{k-1}:\epsilon}\right)$ and $C_{10} = C_{10}\left(\|v\|_{k,2+\delta_{k-1}:\epsilon}\right)$ such that

$$
\begin{aligned}
\frac{1}{3}\|\gamma_\epsilon v\|_{k+1,2+\delta_k} &\leq (C_2\|\gamma_\epsilon\|_{C^k}+C_9)\|v\|_{k,2+\delta_k:\epsilon}+ \\
& C_2C_3\sum_{\{i,j\ |\ i+j=k,j<k-1\}}C_6\left\|D^j\left(\gamma_\epsilon\frac{\partial v}{\partial y}\right)\right\|_{0,p} \\
&\leq C_{10}\|v\|_{k,2+\delta_k:\epsilon}.
\end{aligned}
$$

This last line follows since $\|v\|_{k,2+\delta_k:\epsilon}$ controls $\|v\|_{j+1,p:\epsilon}$ for $j<k-1$.

□

8.6. **Strong local convergence II: uniformly bounding higher Sobolev norms.** In order to apply Theorem 8.5, we need a uniform bound on the $\|\cdot\|_{k,2+\delta}$-norm where $\delta > 0$ might be large. Our holomorphic disk only come with a bound on the $\|\cdot\|_{1,2}$-norm in terms of the action. In this subsection we indicate how the latter norm controls the former.

Theorem 8.7. *Consider the sequence of holomorphic disks $(u_\alpha, h_\alpha) \in \mathcal{M}(a; b_1, \dots, b_m)$. There exists a finite number of points $z_1, \dots, z_l \in \partial\Delta_m$ and a "constant" $C_{11} = C_{11}(K, p, k)$ such that for any positive integer k, for any $p \in \mathbb{R}$ with $k > \frac{2}{p}$, and for any compact set $K \subset \Delta_m \setminus \{z_1, \dots, z_l\}$,*

$$\|D^k u_\alpha\|_{0,p:K} \leq C_{11}.$$

This result in the special case when $m = 2$ and $k = 1$ was proved as Theorem 2 in [13] and then reproved as Proposition 3.3 in [24]. Thus, the first part of this proof uses some ideas of [13]. For the cross-referencing inclined reader, we borrow the notation from [24].

Proof. Let $a = \mathrm{Area}(u_\alpha)$ which is independent of α. Let

$$
\begin{aligned}
\eta_\alpha(K) &= \inf\{\eta>0:\ \text{there exists } z\in K \text{ such that} \\
&\qquad \|D^k u_\alpha\|_{0,p:B(z,\eta)} \geq \eta^{\frac{2}{p}-k}\}.
\end{aligned}
$$

Assume $\eta_\alpha(K) \to 0$ for some subsequence, otherwise our theorem holds. Choose $z_\alpha \in K$ such that

$$\left\|D^k u_\alpha\right\|^p_{0,p:B(z_\alpha,\eta_\alpha)} \geq \frac{1}{2}\eta_\alpha^{2-pk}.$$

Define $r_\alpha = \eta_\alpha^{-1}\mathrm{dist}(z_\alpha, \partial\Delta_m)$. There are two cases to consider. We must derive a contradiction for both.

Case 1: $r_\alpha \to \infty$.

Pass to a subsequence and assume z_α converges to some $z_0 \in \bar{\Delta}_m$.

Let $A \subset \mathbb{R}^2$ be a compact subset and use variables (s,t) on $\mathbb{R}^2$. Let $f : A \to \mathbb{R}^{2n}$. We remark that when rescaling variables $(s,t) \mapsto (\beta s, \beta t)$, the L^p-norm changes like:

$$(8.10)\qquad \left\|D^k f\right\|_p \to \left((\beta^{-k})^p(\beta^2)\right)^{\frac{1}{p}}\left\|D^k f\right\|_p = \beta^{-k+\frac{2}{p}}\left\|D^k f\right\|_p.$$

This remark allows us to use the Floer technique for our more general case: (k,p), $k > \frac{2}{p}$ versus $(1,p)$, $p > 2$.

Since $r_\alpha \to \infty$, for every $R > 0$ and large enough α we have holomorphic maps

$$v_\alpha : \mathbb{C} \supset B(0,R) \to \mathbb{C}^n, \quad v_\alpha(z) = u_\alpha(\eta_\alpha(z - z_\alpha)),$$

satisfying

$$\|Dv_\alpha\|_{0,2} \leq a \tag{8.11}$$

$$\left\|D^k v_\alpha\right\|_{0,p:B(0,1)} \geq \frac{1}{2} \tag{8.12}$$

$$\left\|D^k v_\alpha\right\|_{0,p:B(z,1)} \leq 1, \text{ for all } z \in B(0, R-1). \tag{8.13}$$

Equations (8.11) to (8.13) follow from (8.10) and the exponent in the definition of η_α. We use (8.13) and the last statement of Theorem 8.5 to find a subsequence, v_α converging in $W^p_k(B(0,R), \mathbb{C}^n)$ to some holomorphic map $w_R : B(0,R) \to \mathbb{C}^n$. (Technically, the convergence is in $W^{p'}_k$ for some $p' < p$; however, since we still have $p' > 2$ we will ignore this.) Repeating this procedure for all positive integers R and choosing converging subsequences, we obtain a holomorphic map $w : \mathbb{C} \to \mathbb{C}^n$ satisfying

$$\|Dw\|_{0,2} \leq a \tag{8.14}$$

$$\left\|D^k w\right\|_{0,p:B(0,1)} \geq \frac{1}{2}. \tag{8.15}$$

By (8.15), w is non-constant; hence, $\|Dw\|_{0,2} = a' > 0$.

To derive a contradiction, consider the sequence of annuli

$$\mathbb{C} \supset A_\rho := \{re^{i\theta} \ : \ \rho \leq r < \rho + 1\}.$$

(8.14) implies that $\|Dw\|_{0,2:A_\rho} \to 0$. Thus, for some circle $C_{\rho'} = \{\rho' e^{i\theta}\}$ where $\rho' \in (\rho, \rho+1)$, we have $\|Dw\|_{0,2:C_{\rho'}} \to 0$. By Sobolev's Theorem, this implies $\|w\|_{C^0:C'_\rho} \to 0$.

Thus, for sufficiently large ρ, $w|C_{\rho'}$ spans a small disk $D_{\rho'} \subset \mathbb{C}^n$ whose (absolute) area is bounded by $\frac{a'}{3}$. Let $B_{\rho'} \subset \mathbb{C}$ be the disk spanned by $C_{\rho'}$. Choose ρ large enough such that $\|Dw\|_{0,2;B_{\rho'}} > \frac{2a'}{3}$. Let $\tilde{w} : S^2 \to \mathbb{C}^n$ be the (not necessarily holomorphic) map $w|B_{\rho'}$ capped off with $D_{\rho'}$. Then, with $\omega = \sum_i dx_i \wedge dy_i$ we find, since $\pi_2(\mathbb{C}^n) = 0$,

$$0 = \int_{S^2} \tilde{w}^*(\omega) \geq \|Dw\|_{0,2:B_{\rho'}} - \int_{D_{\rho'}} \omega \geq \frac{a'}{3} > 0$$

Case 2: $r_\alpha \to r < \infty$.

In this case, $z_\alpha \to z_0 \in \partial\Delta_m$. We proceed as before to construct a limiting holomorphic map w, where this time

$$w : \mathbb{C}_r = \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) \geq -r\} \to \mathbb{C}^n$$

with $w(\partial\mathbb{C}_r) \subset \Pi_{\mathbb{C}}(L)$ and satisfying (8.14) and (8.15). Here $B_1(0)$ is the unit ball in $\mathbb{C}_r$.

At this point our proof deviates from [13] and [24] because of multiple boundary punctures.

Repeat the second part of the discussion of Case 1, where $A_\rho \subset \mathbb{C}_r$ is a "partial" annulus and $C_{\rho'} \subset A_\rho$ a "partial" circle. Instead of constructing a smooth (but not necessarily holomorphic) map $\tilde{w}$ as before, we use the convergence $\|w\|_{C^0:C'_\rho} \to 0$ to conclude that (after precomposing with a linear fractional transformation from the unit disk $D \subset \mathbb{C}$ to $\mathbb{C}_r$ which takes -1 to ∞)

$$w : (D \setminus \{-1\}, \partial D \setminus \{-1\}) \to (\mathbb{C}^n, \Pi_{\mathbb{C}}(L))$$

can be continuously extended to -1.

By Lemma 8.3, $\|w\|_{0,2} \geq \hbar$.

From the construction of w and v_α, it is easy to see that for large enough α,

$$\text{Area}\left(u_\alpha \left| \bar{\Delta}_m \setminus B(z_\alpha, \eta_\alpha)\right.\right) \leq a - \frac{\hbar}{2}.$$

Note that z_0 is an example of a point mass for the sequence u_α. We will relabel it z_1 to coincide with the statement of the theorem.

Now suppose there is a subsequence $z_\alpha \to z_2 \neq z_1$ such that

$$\left\| D^k u_\alpha \right\|_{0,p:B(z_\alpha,\eta_\alpha)} \geq \frac{1}{2}\eta_\alpha^{2-pk}.$$

Repeating Case 1, we again conclude that $z_2 \in \partial\Delta_m$. Furthermore,

$$\text{Area}\left(u_\alpha \left| \bar{\Delta}_m \setminus (B(z_\alpha, \eta_\alpha) \cup B(z_{\alpha,\eta_\alpha}))\right.\right) \leq a - 2\frac{\hbar}{2}.$$

Assume α is large enough such that $B(z_\alpha, \eta_\alpha) \cup B(z_{\alpha,\eta_\alpha}) = \emptyset$. Since a is finite, this can only happen a finite number of times: $z_1, \dots z_l$. So as long as $K \subset \Delta_m \setminus \{z_1, \dots z_l\}$, the bound of the theorem holds.

□

8.7. **Recovering the bubbles.** The goal of this subsection is to construct a (not necessarily conformal) reparameterization of $\bar{\Delta}_m$ which recovers all disks which bubble off. This reparameterization implies the second convergence in Definition 8.1.

Consider a sequence (u_α, h_α) which converges strongly on any compact $K \subset \Delta_m \setminus \{z_1, \dots, z_l\}$. By the proof of Theorem 8.7, we can assume that z_1 is a point mass with mass $m_1 > 0$.

Let $\mathbb{C}_+ \subset \mathbb{C}$ denote the upper-half plane. Let $B_r = \{z \in \mathbb{C}_+ : \|z\| < r\}$ and $C_r = \partial B_r$. Define the conformal map

$$\psi_\alpha : \mathbb{C}_+ \to \bar{\Delta}_m, \quad \psi_\alpha(z) = \frac{-z + iR_\alpha^2}{z + iR_\alpha^2} \cdot z_1$$

where $R_\alpha \in \mathbb{R}$ is such that

$$\text{Area}\left(u_\alpha \left| \psi_\alpha\left(B_{R_\alpha}\right)\right.\right) = m_1.$$

Pass to a subsequence and assume $\alpha < \alpha'$ implies $R_\alpha < R_{\alpha'}$, which can be done since by the definition of point mass, $\lim_{\alpha\to\infty} R_\alpha = \infty$. Note that

$$\lim_{\alpha\to\infty} \psi_\alpha\left(B_{R_\alpha}\right) = \lim_{\alpha\to\infty} \psi_\alpha\left(B_{R_\alpha^{3/2}}\right) = z_1. \tag{8.16}$$

Assume α is large enough so that $\psi_\alpha\left(B_{R_\alpha^{3/2}}\right)$ contains no other point masses of the sequence u_α. However, $\psi_\alpha\left(B_{R_\alpha^{3/2}}\right)$ might contain boundary punctures.

After passing to a subsequence, we can use Theorems 8.5 and 8.7 to assume that u_α converges to some u on any compact set in $\Delta_m \setminus \left(\{z_2, \dots, z_l\} \cup \psi_\alpha\left(B_{R_\alpha^{3/2}}\right)\right)$.

The definition of R_α and (8.16) imply

$$\lim_{\alpha\to\infty} \text{Area}\left(u_\alpha \left| \psi_\alpha\left(B_{R_\alpha^{3/2}} \setminus B_{R_\alpha}\right)\right.\right) = 0. \tag{8.17}$$

Use (8.17) and argue as in the previous subsection to find some half circle $C_{R'_\alpha} \subset \mathbb{C}_+$, with $R'_\alpha \in (R_\alpha^{3/2} - 1, R_\alpha^{3/2}]$ such that

$$\|u_\alpha \circ \psi_\alpha\|_{C^0:C_{R'_\alpha}} \to 0.$$

Define the center of mass of $u_\alpha \circ \psi_\alpha$ to be

$$z_\alpha = x_\alpha + iy_\alpha = \frac{1}{m_1}\int_{B_{R_\alpha}} |D(u_\alpha \circ \psi_\alpha)|^2 (x + iy)\, dx \wedge dy \in B_{R_\alpha},$$

where $x+iy$ are coordinates on $\mathbb{C}_+$. Define the conformal map ϕ_α which sends i to z_α:

$$\phi_\alpha : \mathbb{C}_+ \to \mathbb{C}_+, \quad \phi_\alpha(z) = y_\alpha z + x_\alpha.$$

Note that although $\phi_\alpha^{-1}(C_{R_\alpha})$ might remain bounded, $\phi_\alpha^{-1}\left(C_{R'_\alpha}\right)$ converges to ∞ because

$$\left|\phi_\alpha^{-1}\left(R'_\alpha e^{i\theta}\right)\right| \geq \frac{(R_\alpha^{3/2}-1)}{|y_\alpha|} \max\{|\cos\theta|, |\sin\theta|\}$$

and $y_\alpha < R_\alpha$.

Define the conformal map

$$\Psi : D \to \mathbb{C}_+, \quad \Psi(z) = \frac{z-1}{iz+i}$$

where $D \subset \mathbb{C}$ is the unit disk. Note that $\Psi^{-1}\phi_\alpha^{-1}\left(C_{R'_\alpha}\right) \to -1$ and that $u_\alpha \circ \psi_\alpha \circ \phi_\alpha \circ \Psi$ all have center of mass at $0 \in D$. (Recall that the center of mass uses the Euclidean metric on $\mathbb{C}_+$, not on D.)

Since

$$\|u_\alpha \circ \psi_\alpha \circ \phi_\alpha \circ \Psi\|_{C^0:\Psi^{-1}\circ\phi_\alpha^{-1}\left(C_{R'_\alpha}\right)} \to 0,$$

pass to a subsequence as before and conclude that $u_\alpha \circ \psi_\alpha \circ \phi_\alpha \circ \Psi$ converges to some holomorphic w on compact sets outside of some boundary point masses and punctures, as well as -1 (since $u_\alpha \circ \psi_\alpha \circ \phi_\alpha \circ \Psi$ is not defined at -1).

As before, w can be continuously extended to -1. We claim that under this reparameterization, -1 is not a point mass of $u_\alpha \circ \psi_\alpha \circ \phi_\alpha \circ \Psi$. Otherwise, in the $\mathbb{C}_+$ set-up, as some mass escaped to ∞, the center of mass would have to go to ∞ as well, contradicting the fact that it is fixed at $i \in \mathbb{C}_+$.

Because u_α converges to u outside of $\psi_\alpha\left(B_{R_\alpha^{3/2}}\right)$, and because no area is "unaccounted" for by (8.17), we can continuously extend u to z_1 so that $u(z_1) = w(-1)$. Considering how u and w were obtained from u_α it is easy to see that the sign of the punctures (z_1 for u and -1 for w) will be opposite. Thus since each of u and w must have a positive puncture each will have exactly one. Repeat the above argument at all the other point masses z_j. Then repeat for any new point masses in the sequences defining the holomorphic disks w_j associated to z_j. Continuing until all point masses have been dealt with we see no holomorphic curves were overlooked in the reparameterization.

8.8. **Proof of Theorem 8.2.** Let $\Pi_{\mathbb{C}}(L)$ denote the limiting Lagrangian boundary condition. Let $\hbar = \hbar\left(\Pi_{\mathbb{C}}(L)\right)$ be the minimal area of non-constant maps defined in Section 8.4. Use the discussion in Section 8.2 to pass to a subsequence whose conformal structures converge to a stable disk.

We wish to apply Theorem 8.5 to derive strong local convergence. To achieve the required uniform bound on $\|u_\alpha\|_{k,2+\delta_{k-1}:K}$ for some compact set $K \subset \Delta_m$ which lies away from point masses, we apply Theorem 8.7 k times to bound $\|u_\alpha\|_{i,2+\delta_{k-1}:K}$ for $i = 1, \ldots, k$. The reparameterizations ϕ_α^j in Definition 8.1 come from the discussion in Section 8.7.

The weak global convergence follows readily from Section 8.7. □

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