A NOTE ON THE SUPPORT NORM OF A CONTACT STRUCTURE

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Abstract. In this note we observe that the no two of the three invariants defined for contact structures in [3] – that is, the support genus, binding number and support norm – determine the third.

In [3], the second author and B. Ozbagci define three invariants of contact structures on closed, oriented 3-manifolds in terms of supporting open book decompositions. These invariants are the support genus, binding number and support norm. There are obvious relationships between these invariants, but [3] leaves open the question of whether any two of them determine the third. We show in this note that this is not the case.

Recall that an open book decomposition (L, π) of a 3–manifold M consists of an oriented link L in M and a fibration $\pi : (M - L) \to S^1$ of the complement of L whose fibers are Seifert surfaces for L. The fibers $\pi^{-1}(\theta)$ of π are called *pages* of the open book and L is called the *binding*. It is often convenient to record an open book decomposition (L, π) by a pair (Σ, ϕ), where Σ is a compact surface which is homeomorphic to the closure of a page of (L, π), and $\phi : \Sigma \to \Sigma$ is the monodromy of the fibration π. A contact structure ξ on M is said to be supported by the open book decomposition (L, π) if ξ is the kernel of a 1-form α which evaluates positively on tangent vectors to L that agree with the orientation of L, and for which $d\alpha$ restricts to a positive volume form on each page of (L, π).

With this in mind, we may describe the three invariants defined in [3]. The *support genus* of a contact structure ξ on M is defined to be

$$\mathrm{sg}(\xi) = \min\{g(\pi^{-1}(\theta)) \,|\, (L, \pi) \text{ supports } \xi\},$$

where θ is any point in S^1 and $g(\pi^{-1}(\theta))$ is the genus of the page $\pi^{-1}(\theta)$. The *binding number* of ξ is defined to be

$$\mathrm{bn}(\xi) = \min\{|L| \,|\, (L, \pi) \text{ supports } \xi \text{ and } \mathrm{sg}(\xi) = g(\pi^{-1}(\theta))\},$$

where $|L|$ denotes the number of components of L (or, equivalently, the number of boundary components of any page of (L, π)). And the *support norm* of ξ is defined to be

$$\mathrm{sn}(\xi) = \min\{-\chi(\pi^{-1}(\theta)) \,|\, (L, \pi) \text{ supports } \xi\},$$

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where $\chi(\pi^{-1}(\theta))$ denotes the Euler characteristic of any page $\pi^{-1}(\theta)$. It is a simple observation that $\mathrm{sn}(\xi) \geq -1$, with equality if and only if ξ is the standard tight contact structure on S^3.

Since, for any surface Σ, we have the equality

$$-\chi(\Sigma) = 2g(\Sigma) + |\partial\Sigma| - 2,$$

it is immediately clear that

$$\mathrm{sn}(\xi) \leq 2\,\mathrm{sg}(\xi) + \mathrm{bn}(\xi) - 2.$$

Moreover, if the support norm of ξ is achieved by an open book whose pages have genus $g > \mathrm{sg}(\xi)$ and whose binding has m components, then $\mathrm{sn}(\xi) = 2g + m - 2$, which is at least $2\,\mathrm{sg}(\xi) + 1$. The following lemma from [3] summarizes these bounds.

Lemma 1. *For any contact structure ξ on a closed, oriented 3–manifold,*

$$\min\{2\,\mathrm{sg}(\xi) + \mathrm{bn}(\xi) - 2,\, 2\,\mathrm{sg}(\xi) + 1\} \;\leq\; \mathrm{sn}(\xi) \;\leq\; 2\,\mathrm{sg}(\xi) + \mathrm{bn}(\xi) - 2.$$

Thus, for contact structures with $\mathrm{bn}(\xi) \leq 3$, it follows that that $\mathrm{sn}(\xi) = 2\,\mathrm{sg}(\xi) + \mathrm{bn}(\xi) - 2$. Yet, the results in [3] do not resolve whether the upper bound on the support norm in Lemma 1 can ever be a strict inequality. Our main result is that this bound can indeed be a strict inequality; that is, the support genus and binding number do not, in general, wholly determine the support norm.

For the rest of this note, Σ will denote the genus one surface with one boundary component. Let $\phi_{\mathbf{n},m}$ be the diffeomorphism of Σ given by

$$\phi_{\mathbf{n},m} = D_\delta^m \cdot D_x D_y^{-n_1} \cdots D_x D_y^{-n_k},$$

where x, y and δ are the curves pictured in Figure 1, and $\mathbf{n} = (n_1, \ldots, n_k)$ is a k-tuple of non-negative integers for which some $n_i \neq 0$. Let $\xi_{\mathbf{n},m}$ denote the contact structure supported by the open book $(\Sigma, \phi_{\mathbf{n},m})$, and let $M_{\mathbf{n},m}$ denote the 3-manifold with this open book decomposition.

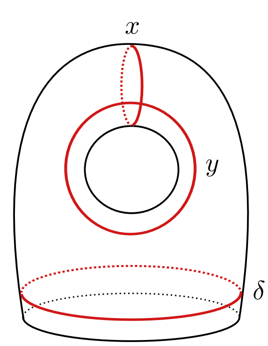

FIGURE 1. The surface Σ and the curves x, y and δ.

Theorem 2. *For $m \leq 0$, the contact structure $\xi_{\mathbf{n},m}$ satisfies*

$$\mathrm{sg}(\xi_{\mathbf{n},m}) = 0.$$

For any fixed tuple $\mathbf{n}$, there is a finite subset $E_{\mathbf{n}}$ of the integers such that

$$\mathrm{bn}(\xi_{\mathbf{n},m}) > 3 \text{ and } \mathrm{sn}(\xi_{\mathbf{n},m}) = 1$$

for all $m \leq 0$ which are not in $E_{\mathbf{n}}$. In particular,

$$\mathrm{sn}(\xi_{\mathbf{n},m}) < 2\,\mathrm{sg}(\xi_{\mathbf{n},m}) + \mathrm{bn}(\xi_{\mathbf{n},m}) - 2$$

for all $m \leq 0$ which are not in $E_{\mathbf{n}}$.

In contrast, the support genus $\mathrm{sg}(\xi_{\mathbf{n},m}) = 1$ when $m > 0$, [1]. Therefore, for $m > 0$, $\mathrm{bn}(\xi_{\mathbf{n},m}) = 1$ and $\mathrm{sn}(\xi_{\mathbf{n},m}) = 1$, and, hence,

$$\mathrm{sn}(\xi_{\mathbf{n},m}) = 2\,\mathrm{sg}(\xi_{\mathbf{n},m}) + \mathrm{bn}(\xi_{\mathbf{n},m}) - 2.$$

That is, the upper bound in Lemma 1 is achieved for $\xi = \xi_{\mathbf{n},m}$ when $m > 0$.

Proof of Theorem 2. One can easily see that, for $m \leq 0$, the open book $(\Sigma, \phi_{\mathbf{n},m})$ is not *right-veering*; therefore, for such m, the contact structure $\xi_{\mathbf{n},m}$ is overtwisted [4] and $\xi_{\mathbf{n},m}$ is supported by an open book with planar pages [2].

Observe that if a contact structure ξ on M is supported by an open book with planar pages and the binding number of ξ is three or less, then M must be a Seifert fibered space. More precisely, if the binding number of ξ is two then M is a lens space, and if the binding number is three then M is a small Seifert fibered space. One can see this by drawing a surgery picture corresponding to the open book supporting ξ which realizes the binding number.

It is well known that the diffeomorphism $\phi_{\mathbf{n},0} = D_x D_y^{-n_1} \cdots D_x D_y^{-n_k}$ is pseudo-Anosov (for instance, $\phi_{(1),0}$ is the monodromy of the figure eight knot in S^3). Therefore, the binding of the open book given by $(\Sigma, \phi_{\mathbf{n},0})$ is a hyperbolic knot, and the manifold $M_{\mathbf{n},m}$ is obtained from $M_{\mathbf{n},0}$ via $-\frac{1}{m}$ surgery on this knot. Thurston's Dehn Surgery Theorem then implies that there is some finite subset $E_{\mathbf{n}}$ of the integers for which $M_{\mathbf{n},m}$ is hyperbolic for all m not in $E_{\mathbf{n}}$ [6]. In particular, $M_{\mathbf{n},m}$ is not a Seifert fibered space, save, perhaps, for some of the m in the exceptional set $E_{\mathbf{n}}$. Hence, the binding number of $\xi_{\mathbf{n},m}$ must be greater than three for all $m \leq 0$ which are not in $E_{\mathbf{n}}$.

We are left to check that the support norm of $\xi_{\mathbf{n},m}$ is one when $m \leq 0$ and $M_{\mathbf{n},m}$ is hyperbolic. If the support norm were not one, then it would be zero (the support norm must be non-negative since $\xi_{\mathbf{n},m}$ is not the tight contact structure on S^3). But the only surface with boundary which has Euler characteristic zero is the annulus, and the only 3-manifolds with open book decompositions whose pages are annuli are lens spaces. $\square$

It is natural to ask if the difference between $\mathrm{sn}(\xi)$ and $2\,\mathrm{sg}(\xi)+\mathrm{bn}(\xi)-2$ can be arbitrarily large. While we cannot answer this question we do note the following.

Theorem 3. *For a fixed* $\mathbf{n}$ *the difference between* $\mathrm{sn}(\xi_{\mathbf{n},m})$ *and* $2\,\mathrm{sg}(\xi_{\mathbf{n},m})+\mathrm{bn}(\xi_{\mathbf{n},m})-2$ *is bounded independent of* $m<0$.

Before we prove this theorem we estimate the binding numbers $\mathrm{bn}(\xi_{\mathbf{n},m})$ in some special cases.

Proposition 4. *The binding number of* $\xi_{(1),-1}$ *satisfies* $3 \leq \mathrm{bn}(\xi_{(1),-1}) \leq 9$. *For each* $m<-1$, *the binding number of* $\xi_{(1),m}$ *satisfies* $4 \leq \mathrm{bn}(\xi_{(1),m}) \leq 9$.

The manifold $M_{(1),-1}$ is the Brieskorn sphere $\Sigma(2,3,7)$. Since $\Sigma(2,3,7)$ is not a lens space, it does not admit an open book decomposition with planar pages and two or fewer binding components. Therefore, $\mathrm{bn}(\xi_{(1),-1}) \geq 3$. It is well-known that the only exceptional surgeries on the figure eight are integral surgeries [6]. Therefore, $E_{(1)}=\{-1,0,1\}$. So, from Theorem 2, we know that $\mathrm{bn}(\xi_{(1),m})>3$ for all $m<-1$. To prove Proposition 4, we construct an open book decomposition of $M_{(1),m}$ with planar pages and nine binding components and we show that it supports $\xi_{(1),m}$ for $m<0$.

Recall that overtwisted contact structures on a 3-manifold M are isotopic if and only if they are homotopic as 2-plane fields. Moreover, the homotopy type of a 2-plane field ξ is uniquely determined by its induced $Spin^c$ structure $\mathbf{t}_\xi$ and its *3-dimensional invariant* $d_3(\xi)$. Therefore, in order to show that the open book decomposition we construct actually supports $\xi_{(1),m}$ (for $m \leq 0$), we need only prove that the contact structure it supports is overtwisted and has the same 3-dimensional invariant as $\xi_{(1),m}$ (their $Spin^c$ structures automatically agree since $H_1(M_{(1),m};\mathbb{Z})=0$). Below, we describe how to compute these invariants from supporting open book decompositions. For more details, see the exposition in [3].

Suppose that ϕ is a product of Dehn twists around homologically non-trivial curves $\gamma_1,\ldots,\gamma_k$ in some genus g surface S with n boundary components. The open book (S,id) supports the unique tight contact structure on $\#^{2g+n-1}(S^1\times S^2)$, and the γ_i may be thought of as Legendrian curves in this contact manifold. The contact manifold (M,ξ) supported by the open book (S,ϕ) bounds an achiral Lefschetz fibration X, which is constructed from $\natural^{2g+n-1}(S^1\times D^3)$ by attaching 2-handles along these Legendrian curves. Each 2-handle is attached with contact framing ± 1 depending on whether the corresponding Dehn twist in ϕ is left- or right-handed, respectively. As long as $c_1(\mathbf{t}_\xi)$ is torsion in $H^2(M;\mathbb{Z})$, $d_3(\xi)$ is an element of $\mathbb{Q}$ and may be computed according to the formula,

$$d_3(\xi)=\frac{1}{4}(c^2(X)-2\chi(X)-3\sigma(X))+q. \tag{1}$$

Here, q is the number of left-handed Dehn twists in the factorization ϕ. The number $c^2(X)$ is the square of the class $c(X)\in H^2(X;\mathbb{Z})$ which is Poincaré dual to

$$\sum_{i=1}^{k} \mathrm{rot}(\gamma_i) C_i \in H_2(X, M; \mathbb{Z}),$$

where C_i is the cocore of the 2-handle attached along γ_i, and $\mathrm{rot}(\gamma_i)$ is the rotation number of γ_i. The class $c(X)$ restricts to $c_1(\mathbf{t}_\xi)$ in $H^2(M;\mathbb{Z})$. Since we have assumed that $c_1(\mathbf{t}_\xi)$ is torsion, some multiple $k \cdot c(X)$ is sent to zero by the map $i^* : H^2(X;\mathbb{Z}) \to H^2(M;\mathbb{Z})$, and, hence, comes from a class $c_r(X)$ in $H^2(X, M;\mathbb{Z})$, which can be squared. So, by $c^2(X)$, we mean $\frac{1}{k^2} c_r^2(X)$.

Lemma 5. *For $m < 0$, the 3-dimensional invariant $d_3(\xi_{(1),m}) = 1/2$.*

Proof of Lemma 5. Let $M_{(1),m}$ denote the result of $-\frac{1}{m}$ surgery on the figure eight knot. It is therefore a rational homology 3-sphere, and the 3-dimensional invariant $d_3(\xi_{(1),m})$ is a well-defined element of $\mathbb{Q}$. Observe that the Dehn twist D_δ is isotopic to the composition $(D_x D_y)^6$. As described above, the contact manifold supported by the open book $(\Sigma, \phi_{(1),m})$ bounds an achiral Lefschetz fibration X, constructed from $\sharp^2(S^1 \times D^3)$ by attaching $12|m| + 2$ 2-handles corresponding to the Dehn twists in the factorization

$$\phi_{(1),m} = (D_x D_y)^{6m} \cdot D_x D_y^{-1}.$$

From the discussion in [3, Section 6.1], it follows that $\mathrm{rot}(x) = \mathrm{rot}(y) = 0$; hence, $c(X) = 0$. Moreover, $\chi(X) = 12|m| + 1$ and $q = 12|m| + 1$. Therefore, the formula in (1) gives

$$d_3(\xi_{(1),m}) = \frac{12|m| + 1}{2} - \frac{3\sigma(X)}{4}.$$

The achiral Lefschetz fibration associated to the monodromy $(D_x D_y)^{2m}$ gives a well-known Milnor fiber with the reverse orientation. Its signature is $8|m|$. One may easily check (via Kirby calculus or gluing formulas for the signature or computations of the degree of related Heegaard-Floer contact invariants) that $\sigma(X) = 8|m|$. Thus $d_3(\xi_{(1),m)}) = 1/2$. □

Proposition 4 follows if we can find a planar open book with nine binding components which supports an overtwisted contact structure on $M_{(1),m}$ with $d_3 = 1/2$. The figure eight knot K is pictured in Figure 2. K can be embedded as a homologically non-trivial curve on the surface S obtained by plumbing together two positive Hopf bands and two negative Hopf bands, as shown on the left in Figure 3.

Topologically, S is an embedded copy of the planar surface P with five boundary components shown on the right in Figure 3. Moreover, S is a page of the open book decomposition of $M_{(1),0} \cong S^3$ given by (P, ϕ), where ϕ is the product of right-handed Dehn twists around the curves γ_1 and γ_3 and

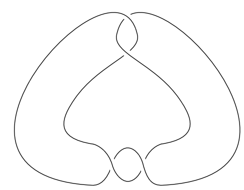

FIGURE 2. The figure eight, drawn here as a twist knot.

left-handed Dehn twists around the curves γ_2 and γ_4. The knot K is the image, under this embedding, of the curve $r \subset P$.

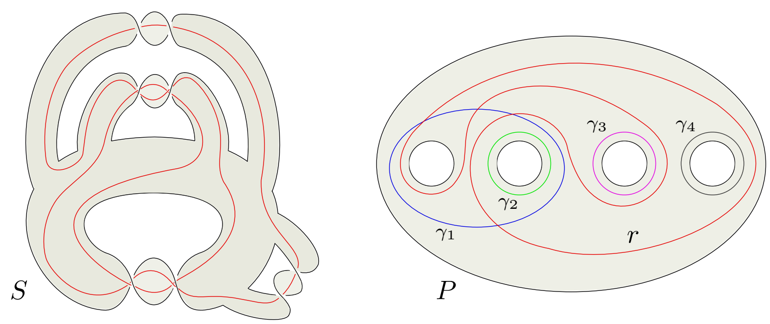

FIGURE 3. The figure eight knot, embedded on a planar surface with five boundary components.

Since the Seifert framing of K agrees with the framing induced by S, $(P, D_r^m \cdot \phi)$ is an open book decomposition for $-\frac{1}{m}$ surgery on K. Let ξ_m denote the contact structure on $M_{(1),m}$ which is supported by this open book. It is easy to check that the open book $(P, D_r^m \cdot \phi)$ is not right-veering for $m \leq 0$. (This can be seen by taking, for example, the horizontal arc connecting the right most boundary components of the surface on the right of Figure 3.) Therefore, the corresponding ξ_m are overtwisted [4].

Lemma 6. *For $m \leq 0$, the 3-dimensional invariant $d_3(\xi_m) = 3/2$.*

Proof of Lemma 6. Figure 4 shows another illustration of P, on the left; the four topmost horizontal segments are identified with the four bottommost horizontal segments to form 1-handles. As discussed above, we can think of these curves as knots in $\#^4(S^1 \times S^2) = \partial(\sharp^4(S^1 \times D^3))$. The contact manifold supported by $(P, D_r^m \cdot \phi)$ bounds the achiral Lefschetz fibration X obtained from $\sharp^4(S^1 \times D^3)$ by attaching 2-handles along the curves γ_2 and γ_4 with framing $+1$, along the curves γ_1 and γ_3 with framing -1, and along $|m|$ parallel copies of r (with respect to the blackboard framing) with framing $+1$, as indicated on the right in Figure 4.

Let X, Y, Z and W denote the 1-handles attached to D^4 to form $\sharp^4(S^1 \times D^3)$, as shown in Figure 4. Furthermore, let $S_{\gamma_1}, \dots, S_{\gamma_4}$ and $S_{r_1}, \dots, S_{r_{|m|}}$

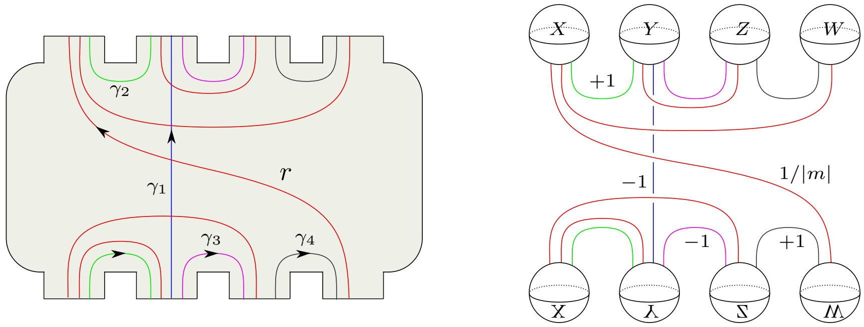

FIGURE 4. On the left, the surface P. On the right, a Kirby diagram for the achiral Lefschetz fibration corresponding to the open book $(P, D_r^m \cdot \phi)$. The label $1/|m|$ indicates that we attach 2-handles along $|m|$ parallel copies of the curve r with framing $+1$.

denote the cores of the 2-handles attached to the curves $\gamma_1, \ldots, \gamma_4$ and the $|m|$ parallel copies $r_1, \ldots, r_{|m|}$ of r, and let $C_{\gamma_1}, \ldots, C_{\gamma_4}$ and $C_{r_1}, \ldots, C_{r_{|m|}}$ denote the cocores of these 2-handles. These cores form a basis for the group of 2-chains $C_2(X;\mathbb{Z})$; X, Y, Z and W for a basis for the 1-chains $C_1(X;\mathbb{Z})$; and the boundary map $d_2 : C_2(X;\mathbb{Z}) \to C_1(X;\mathbb{Z})$ sends

$$\begin{aligned}
d_2(S_{\gamma_1}) &= Y, \\
d_2(S_{\gamma_2}) &= X - Y, \\
d_2(S_{\gamma_3}) &= Y - Z, \\
d_2(S_{\gamma_4}) &= Z - W, \\
d_2(S_{r_i}) &= -Z + Y + W.
\end{aligned}$$

The homology $H_2(X;\mathbb{Z})$ is therefore generated by $h_1, \ldots, h_{|m|}$, where

$$h_i = S_{r_i} + S_{\gamma_4} - S_{\gamma_1}.$$

By construction, X may also be obtained from D^4 by attaching 2-handles along $|m|$ parallel copies of the figure eight with framing $+1$, so the intersection matrix Q_X is simply the $|m| \times |m|$ identity matrix with respect to the corresponding basis. Since the curves r_i are parallel, it is clear that $h_i \cdot h_j = 0$ for $i \neq j$. It follows that $h_i \cdot h_i = 1$ for $i = 1, \ldots, |m|$.

Recall that the class $c(X)$ is Poincaré dual to

$$\sum_{i=1}^{4} \mathrm{rot}(\gamma_i) \cdot C_{\gamma_i} + \sum_{i=1}^{|m|} \mathrm{rot}(r_i) \cdot C_{r_i}.$$

Via the discussion in [3, Section 3.1], we calculate that $\mathrm{rot}(\gamma_4) = \mathrm{rot}(\gamma_2) = \mathrm{rot}(\gamma_3) = -1$ and $\mathrm{rot}(\gamma_1) = \mathrm{rot}(r_i) = 0$. Therefore, $\langle c(X), h_i \rangle = -1$ for

$i = 1, \ldots, |m|$. So, thought of as a class in $H^2(X, \partial X; \mathbb{Z})$, $c(X)$ is Poincaré dual to

$$-h_1 - \cdots - h_{|m|}.$$

Hence, $c^2(X) = |m|$. In addition, $\chi(X) = 1 + |m|$, $\sigma(X) = |m|$ and $q = 2 + |m|$. From the formula in (1), we have

$$d_3(\xi_m) = \frac{1}{4}(|m| - 2(1 + |m|) - 3|m|) + 2 + |m| = 3/2.$$

This completes the proof of Lemma 6. □

Proof of Proposition 4. Recall that

$$d_3(\xi \# \xi') = d_3(\xi) + d_3(\xi') + 1/2$$

for any two contact structures ξ and ξ'. Let $m < 0$. Since $d_3(\xi_{(1),m}) = 1/2$, $d_3(\xi_m) = 3/2$, and both $\xi_{(1),m}$ and ξ_m are overtwisted, it follows that $\xi_{(1),m}$ is isotopic to $\xi_m \# \xi'$, where ξ' is the unique (overtwisted) contact structure on S^3 with $d_3(\xi') = -3/2$. In [3], Ozbagci and the second author show that $\mathrm{bn}(\xi') \leq 5$. In particular, ξ is supported by the open book $(P', D_b^{-1} D_a^{-1} \cdot \psi)$, where a and b are the curves on the surface P' shown in Figure 5 and ψ is a composition of right-handed Dehn twists around the four unlabeled curves.

Then, the planar open book $(P \#_b P', D_r^m \cdot \phi \cdot D_b^{-1} D_a^{-1} \cdot \psi)$ with nine binding components supports $\xi_{(1),m} \simeq \xi_m \# \xi'$, and the proof of Proposition 4 is complete. (Here, $\#_b$ denotes boundary connected sum.) □

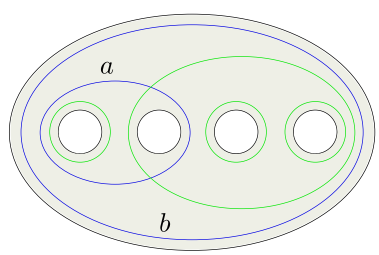

FIGURE 5. The surface P'.

The table below summarizes what we know of the support genus, binding number and support norm for the contact structures $\xi_{(1),m}$.

Proof of Theorem 3. We first observe that $M_{\mathbf{n},m}$ is a rational homology sphere. This can be seen by noticing that $M_{\mathbf{n},0}$ can be obtained as the 2–fold cover of S^3 branched over an alternating (non-split) link (in fact, closure of a 3–braid). Thus the determinant of the link is non-zero and hence the cardinality of the first homology of the cover is finite. Since $M_{\mathbf{n},m}$ can be obtained from $M_{\mathbf{n},0}$ by $1/m$ surgery on a null-homologous knot it has the same first homology.

	sg	bn	sn
$m > 0$	1	1	1
$m = 0$	0	1	-1
$m = -1$	0	$[3, 9]$	1
$m < -1$	0	$[4, 9]$	1

TABLE 1. Values of sg, bn and sn for $\xi_{(1),m}$.

Let K be the binding of the open book $(\Sigma, \phi_{\mathbf{n},0})$ in $M_{\mathbf{n},0}$. If we fix an overtwisted contact structure on $M_{\mathbf{n},0}$ we can find a Legendrian knot L in the knot type K with Thurston-Bennequin invariant 0 and overtwisted complement. In [5] it was shown that there is a planar open book (Σ', ϕ') for this overtwisted contact structure that contains L on a page so that the page framing is 0.

Notice that $M_{\mathbf{n},m}$ can be obtained from $M_{\mathbf{n},0}$ by composing ϕ' with a $+1$ Dehn twist along m copies of L on the page of the open book. Thus each $M_{\mathbf{n},m}$ has an overtwisted contact structure supported by a planar open book with the same number of binding components. The number of $Spin^c$ structure on $M_{\mathbf{n},m}$ is finite and independent of m. We can get from the constructed overtwisted contact structure on $M_{\mathbf{n},m}$ to an overtwisted contact structure realizing any $Spin^c$ structure by a bounded number of Lutz twists along generators of $H_1(M_{\mathbf{n},m})$ all of which lie on a page of the open book. As shown in [2] we may positively stabilize the open book a bounded number of times and then compose its monodromy with extra Dehn twists to achieve these Lutz twists. The number of these stabilizations depends on the number of $Spin^c$ structures on $M_{\mathbf{n},m}$ and thus is independent of m. We now have planar open books realizing overtwisted contact structures representing all $Spin^c$ structures on $M_{\mathbf{n},m}$ with the number of binding components bounded independent of m.

To get an open book representing any overtwisted contact structure on $M_{\mathbf{n},m}$ we can take these and connect sum with overtwisted contact structures on S^3. In [3] it was shown that all overtwisted contact structures on S^3 have bn ≤ 6. Thus we obtain a bound independent of m on the binding number for all overtwisted contact structures on $M_{\mathbf{n},m}$ and in particular on the $\xi_{\mathbf{n},m}, m < 0$. □

Remark 7. *One can also show, in a similar manner to the proof of Theorem 3, that the binding number of $\xi_{\mathbf{n},m}$ is bounded by a constant depending only on the length of* $\mathbf{n}$.

As noted in Theorem 2, $\text{sg}(\xi_{\mathbf{n},m}) = 0$ for $m < 0$ and $\text{sn}(\xi_{\mathbf{n},m}) = 1$ for all but finitely many $m < 0$; that is, these two quantities do not depend (much) on $\mathbf{n}$ or m. While we do know that the binding number of $\xi_{\mathbf{n},m}$ is bounded independent of m it could depend on (the length of) $\mathbf{n}$. This suggests the following interesting question, which we leave unanswered.

Question 8. *Does there exist, for any positive integer n, a contact structure ξ such that $2\,\mathrm{sg}(\xi)+\mathrm{bn}(\xi)-2-\mathrm{sn}(\xi)=n$?*

Note that $0\leq 2\,\mathrm{sg}(\xi_{(1),m})+\mathrm{bn}(\xi_{(1),m})-2-\mathrm{sn}(\xi_{(1),m})\leq 6$ for all $m\in\mathbb{Z}$, but we currently cannot prove that this difference is larger that 1. It would be very interesting to determine if there is an m such that this difference is greater than 1. In general, computing $2\,\mathrm{sg}(\xi_{\mathbf{n},m})+\mathrm{bn}(\xi_{\mathbf{n},m})-2-\mathrm{sn}(\xi_{\mathbf{n},m})$ could potentially provide a positive answer to this question.

Noticing that all our examples involve overtwisted contact structures we end with the following question.

Question 9. *Is there a tight contact structure ξ such that*

$$\mathrm{sn}(\xi)<2\,\mathrm{sg}(\xi)+\mathrm{bn}(\xi)-2\ ?$$

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