CONTACT SURGERY NUMBERS

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Abstract. It is known that any contact 3-manifold can be obtained by rationally contact Dehn surgery along a Legendrian link L in the standard tight contact 3-sphere. We define and study various versions of contact surgery numbers, the minimal number of components of a surgery link L describing a given contact 3-manifold under consideration.

In the first part of the paper, we relate contact surgery numbers to other invariants in terms of various inequalities. In particular, we show that the contact surgery number of a contact manifold is bounded from above by the topological surgery number of the underlying topological manifold plus three.

In the second part, we compute contact surgery numbers of all contact structures on the 3-sphere. Moreover, we completely classify the contact structures with contact surgery number one on $S^1 \times S^2$, the Poincaré homology sphere and the Brieskorn sphere $\Sigma(2,3,7)$. We conclude that there exist infinitely many non-isotopic contact structures on each of the above manifolds which cannot be obtained by a single rational contact surgery from the standard tight contact 3-sphere. We further obtain results for the 3-torus and lens spaces.

As one ingredient of the proofs of the above results we generalize computations of the homotopical invariants of contact structures to contact surgeries with more general surgery coefficients which might be of independent interest.

1. Introduction

A fundamental result due to Lickorish–Wallace says that any 3-manifold can be obtained from S^3 by a finite sequence of Dehn surgeries. Moreover, it is known that any 3-manifold can be obtained from S^3 by performing only **integer** Dehn surgeries (*i.e.* the surgery coefficients are integers) or by only **even** Dehn surgeries [Ka79] (*i.e.* the surgery coefficients are all even integers) or by only **simple** Dehn surgeries (*i.e.* all components of the surgery link in S^3 are unknots). These special surgery diagrams have interesting geometric meanings. A surgery diagram with only integer surgery coefficients specifies a simply connected closed 4-manifold bounded by the surgered 3-manifold and a surgery diagram with only even coefficients specifies a parallelization of the surgered 3-manifold [Ka79], *cf.* [DGGK18].

For a given 3-manifold M, Auckly [Au97] defined the **surgery number** $\mathrm{s}(M)$ of M as the minimal number of Dehn surgeries required to obtain M from S^3. It is natural to define surgery numbers for more restricted classes of surgeries. Let $*$ be an extra property of a Dehn surgery. In this work we will consider integer surgeries ($*=\mathbb{Z}$), simple surgeries ($*=U$) or both ($*=\mathbb{Z},U$). Then the $*$ **surgery number** $\mathrm{s}_*(M)$ is defined to be the minimal number of components of a link in S^3 needed to describe M as Dehn surgery with property $*$ along the link.

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While upper bounds on surgery numbers can be provided by constructing explicit Dehn surgery presentations of a given manifold, it is much harder to find lower bounds. Lower bounds on surgery numbers can be given for example by the rank of the first homology or the fundamental group or induced from the linking pairing [Au97]. More advanced obstructions (that can yield lower bounds on homology spheres) can be obtained from gauge theory [Au97], from Heegaard–Floer homology [HKL16, HL18] or from the $SU(2)$ character variety of the fundamental group [SZ19]. In general, the known lower bounds do not coincide with the number of components in explicit surgery diagrams and therefore it is in general hard to compute surgery numbers. At the moment there is no homology sphere known to have surgery number larger than 2.

In the present paper we consider analogous questions in contact geometry and study contact surgery numbers of contact 3-manifolds. To define contact surgery numbers let us first briefly recall contact surgery. Let K be a Legendrian knot in a contact 3-manifold (M, ξ). Then there is the classical construction to do Dehn surgery along K with respect to the contact structure. The result is that for any non-vanishing **contact surgery coefficient** (*i.e.* measured with respect to the contact longitude of K, obtained by pushing K into the Reeb-direction), there exist finitely many tight contact structures on the newly glued-in solid torus that fit together with the old contact structure to give a global contact structure on the surgered manifold.

If the contact surgery coefficient is of the form $1/n$, for $n \in \mathbb{Z}$, the contact structure on the newly glued-in solid torus is unique. Therefore, one often restricts to contact (± 1)-surgeries. However, these diagrams usually tend to be very complicated. Here we want to allow more general coefficients and adapt the convention that if we say that some contact manifold (M, ξ) can be obtained by contact r-surgery, we mean there exist one choice of the above finitely many tight contact structures on the newly glued-in solid torus such that we get (M, ξ) after the surgery.

The following is the generalization of the Lickorish–Wallace theorem to contact geometry.

Theorem 1.1 (Ding–Geiges [DG04])**.** *Let (M, ξ) be a contact 3-manifold. Then (M, ξ) can be obtained by rationally contact Dehn surgery along a Legendrian link in (S^3, ξ_{st}). Moreover, one can assume all contact surgery coefficients to be of the form ± 1 and all components of the Legendrian link to be unknots with Thurston–Bennequin numbers $\text{tb} = -1$ or -2.*

The addendum that the Legendrian link can be assumed to consists of Legendrian unknots with Thurston–Bennequin invariants $\text{tb} = -1$ or -2 is due to Avdek [Av13, Theorem 1.7]. A further generalization, observed in [DGS04], is that one can present any contact manifold in a surgery diagram with only one $+1$ surgery coefficient and all other coefficients negative.

With this Theorem 1.1 in mind it is natural to ask what is the simplest Legendrian link describing a given contact 3-manifold. In this paper we propose to study various versions of contact surgery numbers, measuring the complexity of surgery links L of a given contact 3-manifold in terms of the number of components of L.

Definition 1.2 Let (M, ξ) be a contact 3-manifold. We define the **contact surgery number** $\text{cs}(M, \xi)$ to be the minimal number of components of a Legendrian link in (S^3, ξ_{st}) needed to describe (M, ξ) as a rationally contact surgery along the link (with non-vanishing contact surgery coefficients).

The definition can be extended by requiring additional properties of the surgeries. Let $*$ be an additional property of a contact surgery. In this article, we will consider the following properties:

- $* = \mathbb{Z}$: All contact surgery coefficients are non-vanishing integers.
- $* = 1/\mathbb{Z}$: All contact surgery coefficients are of the form $\pm 1/n$.
- $* = \pm 1$: All contact surgery coefficients are ± 1.
- $* = U$: Any component of the Legendrian surgery link is a Legendrian realization of the unknot.
- $* = L$: All contact surgery coefficients are negative except for at most one surgery coefficient which is $+1$. (L stands for Legendrian surgery.)

Sometimes we will consider combinations of these properties. For any of the above properties we define the $*$ **contact surgery number** $\mathrm{cs}_*(M, \xi)$ to be the minimal number of components of a Legendrian link with property $*$ in (S^3, ξ_{st}) needed to describe (M, ξ) as contact Dehn surgery along that link.

1.1. **Inequalities.** Upper bounds to contact surgery numbers and relations to other invariants in terms of inequalities can be obtained by explicit constructions of surgery diagrams. In Section 2 and 3 we discuss contact Kirby moves (that is, modifications of contact surgery diagrams not changing the contactomorphism type of the surgered manifold), which we then use to relate various versions of contact surgery numbers to each other.

Our first result in that direction roughly says that if we can obtain a contact manifold via a single contact surgery then we can also obtain it via a surgery along a 3-component link in which every component is a Legendrian realization of the unknot.

Theorem 5.1. *Let (M, ξ) be a contact 3-manifold. Then the following inequalities hold true*

$$\begin{aligned}
\mathrm{cs}_U(M, \xi) &\leq 3\,\mathrm{cs}(M, \xi),\\
\mathrm{cs}_{U,\mathbb{Z}}(M, \xi) &\leq 3\,\mathrm{cs}_{\mathbb{Z}}(M, \xi),\\
\mathrm{cs}_{U,1/\mathbb{Z}}(M, \xi) &\leq 3\,\mathrm{cs}_{1/\mathbb{Z}}(M, \xi),\\
\mathrm{cs}_{U,\pm 1}(M, \xi) &\leq 3\,\mathrm{cs}_{\pm 1}(M, \xi).
\end{aligned}$$

We emphasize that contact surgery numbers are bounded from below by topological surgery numbers, since every contact surgery induces a topological surgery on the underlying manifolds. In Section 6 we present various general upper bounds on contact surgery numbers of contact manifolds that depend only on the topological surgery numbers of the underlying topological manifold. Our main result is as follows.

Theorem 6.9. *Let (M, ξ) be a contact manifold. Then*

$$\begin{aligned}
\mathrm{cs}_{\pm 1}(M, \xi) &\leq \mathrm{s}_{\mathbb{Z}}(M) + 3,\\
\mathrm{cs}_{L,\pm 1}(M, \xi) &\leq \mathrm{s}_{\mathbb{Z}}(M) + 4,\\
\mathrm{cs}(M, \xi) &\leq \mathrm{s}(M) + 3.
\end{aligned}$$

Similar bounds hold true for the U-versions of the surgery numbers, see Section 6 for the details.

Our method of proof is constructive, but in concrete situations the bounds can often be improved. In the following we will describe examples of contact manifolds (M,ξ) where the difference $\mathrm{cs}(M,\xi)-\mathrm{s}(M)$ is 0, 1 and 2, we do not know if there exist examples where where the difference is 3.

Question 1.3. *Does there exists a contact manifold (M,ξ) such that*

$$\mathrm{cs}(M,\xi)-\mathrm{s}(M)=3?$$

1.2. **Computations of contact surgery numbers.** In the second part of the article we explicitly compute contact surgery numbers for some contact manifolds. In particular, we compute the contact surgery numbers of all contact structures on S^3 and we classify all contact structures with $\mathrm{cs}=1$ on the Poincaré homology sphere P, on the Brieskorn homology sphere $\Sigma(2,3,7)$, and on $S^1\times S^2$.

1.2.1. *The* 3*-sphere.* We first state the results for S^3. In order to do so, we first need to recall the classification of contact structures on S^3 by Eliashberg [El89, El92] and introduce some notation.

Let ξ_{st} be the unique tight contact structure on S^3. This is the unique contact manifold with vanishing contact surgery number. Overtwisted contact structures on S^3, up to isotopy, are in one-to-one correspondence with plane fields, up to homotopy; which in turn can be indexed by the integers. This indexing can be done with Gompf's d_3-invariant from [Go98] (where it is denoted by θ). Here we choose the normalization of the d_3-invariant such that $\mathrm{d}_3(\xi_{\mathrm{st}})=0$.

With this normalization the d_3-invariants of contact structures on homology spheres take values in the integers. We denote the unique overtwisted contact structure on a homology sphere M with d_3-invariant equal to $n\in\mathbb{Z}$ by ξ_n.

Theorem 7.16. *An overtwisted contact structure on S^3 has $\mathrm{cs}=1$ if and only if its d_3-invariant is of the form*

$$k(q+qk-2z)$$

for $q\geq 1$, $k\geq 1$ and $z=0,1,\dots q-1$, or

$$qk(k+1)+2k+1$$

for $q\leq -1$, $k\geq 0$, or

$$qk(k-1)+1$$

for $q\leq -1$, $k\geq 0$. All other overtwisted contact structures on S^3 have $\mathrm{cs}=2$.

In Section 7 we also derive similar results for $\mathrm{cs}_{\pm 1}$, $\mathrm{cs}_{1/\mathbb{Z}}$ and $\mathrm{cs}_{\mathbb{Z}}$. We formulate some direct corollaries, see Section 7 for more applications.

Corollary 7.17. *There exist infinitely many non-isotopic contact structures on S^3 which cannot be obtained by a single rationally contact surgery from (S^3,ξ_{st}). As a concrete example we see that $\mathrm{cs}_{\pm 1}(S^3,\xi_0)=2$, see Figure 1 (ii) for a surgery description of (S^3,ξ_0) along a 2-component link.*

The next corollary say that (S^3,ξ_1) has a unique contact (± 1)-surgery diagram along a single Legendrian knot in (S^3,ξ_{st}).

Corollary 7.2. *If (S^3,ξ_1) is obtained by a single contact (± 1)-surgery along a Legendrian knot K in (S^3,ξ_{st}), then K has to be the (unoriented) Legendrian unknot with Thurston–Bennequin invariant $\mathrm{tb}=-2$ and rotation number $|\mathrm{rot}|=1$ and the contact surgery coefficient has to be $(+1)$, see Figure 1 (i).*

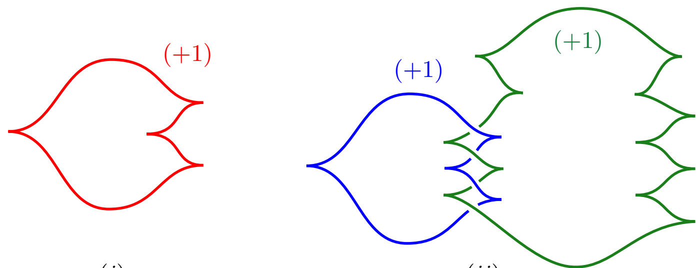

FIGURE 1. (i) This is the only contact structure on S^3 with $\mathrm{cs}_{\pm 1} = 1$ and it is the unique contact (± 1)-surgery description along a single Legendrian knot of (S^3, ξ_1). (ii) This surgery diagram is a contact (± 1)-surgery description of (S^3, ξ_0) along a 2-component link with a vanishing d_3-invariant.

We also have results about the Legendrian contact surgery numbers of contact structures on S^3.

Theorem 7.6. *A contact structure on S^3 has $\mathrm{cs}_L = 1$ if and only if it is isotopic to ξ_1. Moreover, there exist infinite families of contact structures on S^3 with Legendrian contact surgery number equal to two. As concrete examples we have*

(1) $\mathrm{cs}_{L,\pm 1}(S^3, \xi_{2k}) = 2$ *for $k \in \mathbb{Z}$, and*
(2) $\mathrm{cs}_L(S^3, \xi_{1-2k}) = 2$ *for $k \in \mathbb{N}$.*

1.2.2. *The Poincaré homology sphere and the Brieskorn sphere $\Sigma(2,3,7)$.* In Sections 7.2 and 7.3, we study the contact surgery numbers of the the Poincaré homology sphere P and the Brieskorn homology sphere $\Sigma(2,3,7)$.

Theorem 7.8. *A contact structure on P has $\mathrm{cs} = 1$ if and only if its d_3-invariant is of the form*

$$m(3-m) - 1$$

where m is an arbitrary integer with $m \geq 3$. Furthermore, we have $\mathrm{cs}_{\mathbb{Z}}(P, \xi) \leq 3$ for any contact structure on P.

Corollary 7.9. *There exists infinitely many contact structures on P with $\mathrm{cs} = 2$.*

Corollary 7.11. *The unique tight contact structure ξ_{st} on P cannot be obtained by a rationally contact surgery along a single Legendrian knot from (S^3, ξ_{st}). More concretely,*

$$2 \leq \mathrm{cs}(P, \xi_{\mathrm{st}}) \leq \mathrm{cs}_{\pm 1}(P, \xi_{\mathrm{st}}) \leq 3.$$

However, (P, ξ_{st}) can be obtained by a single Legendrian surgery (i.e. contact (-1)-surgery) along a Legendrian knot in an overtwisted S^3.

In particular, we have an example of a tight manifold, (P, ξ_{st}), whose contact surgery number differs from the topological surgery number of the underlying topological manifold. We note that tight contact structures are often attained in simple surgery diagrams. For example all tight contact structures on lens spaces occur from a rational contact surgery along a Legendrian realizations of the unknot [Gir00, Ho00]. However, the (P, ξ_{st})-example shows that this simplicity is not always reflected in the number of components of the surgery link. Nonetheless, (P, ξ_{st}) can be obtained by a contact surgery along a 3-chain link.

Theorem 7.13. *The unique tight contact structure* ξ_{st} *on* $\Sigma(2,3,7)$ *can be obtained by a single Legendrian surgery along a right-handed Legendrian trefoil and thus* $\text{cs}_{\pm 1}(\Sigma(2,3,7),\xi_{\text{st}}) = 1$. *An overtwisted contact structure on* $\Sigma(2,3,7)$ *has* $\text{cs} = 1$ *if and only if its* d_3*-invariant is of the form*

$$l(3-l)-1 \text{ or } m(m-1)$$

where m, l *are arbitrary integers with* $l \geq 0$ *and* $m \geq 2$. *Furthermore, we have* $\text{cs}_{\mathbb{Z}}(\Sigma(2,3,7),\xi) \leq 3$ *for any contact structure on* $\Sigma(2,3,7)$.

It is known that $\Sigma(2,3,7)$ can be obtained by a topological $(+1)$-surgery on the figure eight knot, however we see that is not the case for $(\Sigma(2,3,7),\xi_{\text{st}})$.

Corollary 7.14. *One cannot obtain* $(\Sigma(2,3,7),\xi_{\text{st}})$ *by a rational contact surgery along a Legendrian realization of the figure eight knot.*

From Theorem 7.13, $(\Sigma(2,3,7),\xi_{\text{st}})$ can be obtained by a surgery on the right-handed trefoil which is in fact the unique knot with this property.

Corollary 7.15. *There is a unique Legendrian surgery description of* $(\Sigma(2,3,7),\xi_{\text{st}})$ *along a single Legendrian knot, the contact* (-1)*-surgery along the unique (unoriented) Legendrian right-handed trefoil knot with* $\text{tb} = 0$ *and* $|\text{rot}| = 1$.

1.2.3. $S^1 \times S^2$. We now consider contact 3-manifolds with non-trivial homology. Specifically we study the contact surgery numbers of $S^1 \times S^2$ which has a unique tight contact structure ξ_{st}. All the remaining contact structures on $S^1 \times S^2$ are overtwisted and by Eliashberg's classification of overtwisted contact structures, they only depend on the algebraic topology of the underlying tangential 2-plane field. Since $H_1(S^1 \times S^2) = \mathbb{Z}$, there is another invariant of the tangential 2-plane field, namely its *spin*^{c} structure, see Sections 4 and 7.5 for more discussion.

Theorem 7.20. *There exists exactly one contact structure in every spin*^{c} *structure of* $S^1 \times S^2$ *which can be obtained by a contact surgery along a single Legendrian knot in* (S^3,ξ_{st}). *In particular, no overtwisted contact structure on* $S^1 \times S^2$ *with trivial Euler class can be obtained by a surgery along a single Legendrian knot in* (S^3,ξ_{st}).

Proposition 7.19. *A contact structure* ξ *on* $S^1 \times S^2$ *has* $\text{cs}_{\pm 1}(S^1 \times S^2,\xi) = 1$ *if and only if* $(S^1 \times S^2,\xi)$ *is contactomorphic to* $(S^1 \times S^2,\xi_{\text{st}})$.

Moreover, contact $(+1)$*-surgery along the Legendrian unknot with* $\text{tb} = -1$ *and* $\text{rot} = 0$ *is the unique contact* (± 1)*-surgery diagram of* $(S^1 \times S^2,\xi_{\text{st}})$ *along a single Legendrian knot in* (S^3,ξ_{st}).

Corollary 7.21. *There exist infinitely many contact structures on* $S^1 \times S^2$ *with* $\text{cs} = 2$.

1.3. **Miscellaneous results.** In Section 7.6 we study the contact surgery number of the infinite family of tight structures on T^3. We will see that any tight contact structure ξ on T^3 satisfies

$$cs_{\pm}(T^3,\xi) \leq 5.$$

As a curious corollary, we find an infinite family of distinct non-loose Legendrian knots with the same classical invariants in a fixed overtwisted contact structure η on $S^1 \times S^2 \# S^1 \times S^2$. Recall, a Legendrian knot in an overtwisted contact structure is non-loose if the complement of a standard neighborhood of the Legendrian knot is tight. We will show there is a family $L_n, n \in \mathbb{N}$ of Legendrian knots in $(S^1 \times S^2 \# S^1 \times S^2, \eta)$ having the same classical invariants, such that Legendrian surgery on L_n yields the tight contact structure on T^3 with Giroux torsion $n-1$. This is the such infinite family.

In the following subsection we study contact surgery numbers of contact lens spaces. We give specific examples of tight contact structures with $\mathrm{cs}_\pm = 1$ and a criteria for a tight contact structure to have $\mathrm{cs}_\pm \geq 2$.

In the last section of the paper we study Legendrian surgeries between overtwisted contact structures on S^3 and show that there are many constraints on such surgeries.

Conventions

Throughout this paper, we assume the reader to be familiar with Dehn surgery and contact topology on the level of [PS97, GS99, Ge08, OS04].

We work in the smooth category. All manifolds, maps, etc. are assumed to be smooth. We assume all 3-manifolds to be connected closed oriented and all contact structures to be positive and coorientable. Legendrian links in (S^3, ξ_{st}) are always presented in their front projection.

We choose the normalization of the d_3-invariant such that $\mathrm{d}_3(S^3, \xi_{\mathrm{st}}) = 0$. Although, this normalization differs from the normalization used for example in [Go98, DGS04, DK16], we believe it to be more natural, since then the d_3-invariants of contact structures on homology spheres take values in the integers and since the d_3-invariant behaves additive under connected sum.

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2. Contact Kirby moves

To establish explicit upper bounds on contact surgery numbers we will need several modifications on contact surgery diagrams not changing the contactomorphism type of the resulting contact manifold. In this section we will discuss these modifications but first we introduce the following notations. Let K be a Legendrian knot in (M, ξ), then we will write $(M_K(r), \xi_K(r))$ for a contact manifold obtained by contact r-surgery along K where it is important to remember that the contact structure $\xi_K(r)$ is in general not unique. We often write as short notation

$$K(r) := (M_K(r), \xi_K(r)),$$

if the manifold M is implicitly understood. For K together with a Legendrian push-off of K we write $K \times K$. We denote a copy of K with n extra stabilizations by K_n, if this knot K_n is again stabilized m-times this is denoted by $K_{n,m}$.

The following three lemmas due to Ding and Geiges are fundamental in the study of contact Dehn surgeries along Legendrian links.

Lemma 2.1 (Cancellation Lemma). *Contact $(1/n)$-surgery ($n \in \mathbb{Z} \setminus \{0\}$) along a Legendrian knot K in (M, ξ) and contact $(-1/n)$-surgery along a Legendrian push-off of K cancel each other, i.e. the result is contactomorphic to (M, ξ). In the notation introduced above this reads*

$$K\left(\frac{1}{n}\right) \times K\left(-\frac{1}{n}\right) = (M, \xi).$$

Lemma 2.2 (Replacement Lemma). *Contact $(\pm 1/n)$-surgery ($n \in \mathbb{N}$) along a Legendrian knot K in (M, ξ) yields the same contact manifold as contact (± 1)-surgeries along n Legendrian push-offs of K, i.e.*

$$K\left(\pm\frac{1}{n}\right) \cong K(\pm 1) \times \cdots \times K(\pm 1).$$

Lemma 2.3 (Transformation Lemma). *Let K be a Legendrian knot in (M, ξ) with given contact surgery coefficient $r \in \mathbb{Q} \setminus \{0\}$.*

(1) *Then*

$$K(r) \cong K\left(\frac{1}{k}\right) \times K\left(\frac{1}{\frac{1}{r} - k}\right)$$

holds for all integers $k \in \mathbb{Z}$.

(2) *If the contact surgery coefficient r is negative, one can write r uniquely as*

$$r = [r_1 + 1, r_2, \ldots, r_n] := r_1 + 1 - \cfrac{1}{r_2 - \cfrac{1}{\cdots - \cfrac{1}{r_n}}} \tag{1}$$

with integers $r_1, \ldots, r_n \leq -2$ and we have

$$K(r) \cong K_{|2+r_1|}(-1) \times K_{|2+r_1|,|2+r_2|}(-1) \times \cdots \times K_{|2+r_1|,\ldots,|2+r_n|}(-1).$$

For a proof of the Cancellation Lemma and the Replacement Lemma we refer to [DG01], cf. [DGS04]. The proof of the transformation lemma is given in [DG04], cf. [DGS04].

Remark 2.4 We make the following two remarks.

(1) The first part of the Transformation Lemma in [DG04] and [DGS04, Section 1] is only formulated for natural numbers $k \in \mathbb{N}$ and negative contact surgery coefficients $r < 0$, but the proof given there works exactly the same for all integers $k \in \mathbb{Z}$ and all surgery coefficients $r \in \mathbb{Q} \setminus \{0\}$.

(2) Observe that by choosing k such that $\frac{1}{r} - k$ is negative one can use the transformation lemma to change any contact r-surgery (for $r \neq 0$) into a sequence of contact (± 1)-surgeries along a different Legendrian link. Moreover in [DG04] and [DGS04] it is shown that all the different choices of stabilizations in this Legendrian link correspond exactly to the different tight contact structures one can choose on the new glued-in solid torus in the contact r-surgery. So any choice of contact structure $\xi_K(r)$ on $M_K(r)$ corresponds to exactly one choice of stabilizations in the transformation lemma.

For any negative rational number $r < 0$ there exists a unique continued fraction expansion as in part (2) of Lemma 2.3. Moreover, we observe that if $-(n+1) \leq r < -n$ for some $n \in \mathbb{N}$, then $n+1+r \in [0,1)$, and it follows that the first coefficient r_1 in Equation (1) is $r_1 = -(n+2)$. Using this observation we can prove the following general lemma, which we will use later.

Lemma 2.5. *Let $r > 0$ be a positive contact surgery coefficient, which is not the reciprocal of an integer. Then any contact r-surgery $K(r)$ along K is equivalent to a contact $(1/n)$-surgery along K, for some $n \in \mathbb{N}$, followed by a negative contact r'-surgery, for some $r' < 0$, along a Legendrian push-off K' of K with at least one extra stabilization, as in Figure 2.*

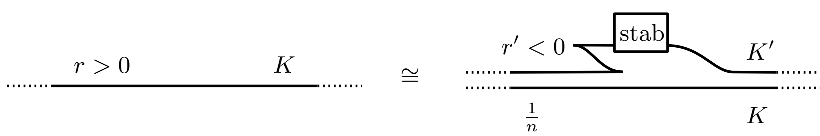

FIGURE 2. An application of the Transformation Lemma. The knot K' is a Legendrian push-off of K away from the shown segment. The box represents an unspecified number of stabilizations of arbitrary sign.

Proof. With the above discussion it is enough to find an $n \in \mathbb{N}$ such that

$$-2 \leq r' = \frac{1}{\frac{1}{r} - n} < 0.$$

Then the first coefficient r_1 in the continuous fraction expansion of r' is smaller than -2 and therefore, a stabilization is needed. An easy calculation shows that any $n \in \mathbb{N}$ with

$$\frac{1}{r} + \frac{1}{2} \leq n$$

suffice. □

From Lemma 2.2 and 2.3 the following specification for integer contact surgeries can be deduced.

Lemma 2.6. *Let K be a Legendrian knot and n be a positive integer. Then the following contactomorphisms hold true:*

$$K(-n) \cong K_{n-1}(-1) \tag{2}$$

$$K(+n) \cong K(+1) \times K\left(-\frac{n}{n-1}\right) \tag{3}$$

$$K\left(-\frac{n}{n-1}\right) \cong K_1\left(-\frac{1}{n-1}\right) \tag{4}$$

$$K(+n) \cong K(+1) \times K_1\left(-\frac{1}{n-1}\right) \tag{5}$$

Proof. Equation (2) follows from Lemma 2.3 Part (2) and the trivial continued fraction expansion $-n = [-n]$.

To prove the other equations, we first notice that Equation (3) is a direct consequence of Lemma 2.3 Part (1) (for $k = 1$) and that Equation (5) follows from Equation (3) and (4). A straightforward induction argument verifies the continued fraction expansion

$$-\frac{n}{n-1} = [\underbrace{-2, \dots, -2}_{n-1}].$$

Equation (3) follows from Lemma 2.3 Part (2) together with Lemma 2.2. □

We will often concentrate on positive integer contact surgeries, i.e. contact surgeries for surgery coefficient $n \in \mathbb{N}$. The above lemma says that there are exactly two contact surgeries corresponding to an integer contact surgery, depending on the sign of the stabilization.

We will also need the following version of a handle slide for contact surgery which is due to Ding and Geiges [DG09], cf. [CEK21] for the generalization to other contact surgery coefficients.

Lemma 2.7 (Contact handle slide)**.** *The two blue Legendrian knots in Figure 3 in the exterior of the red Legendrian knot along which we perform a contact (-1)-surgery are isotopic.*

If the blue Legendrian knot comes equipped with a contact surgery coefficient $r \in \mathbb{Q} \setminus \{0\}$ then the contact surgery coefficient is not changing under the isotopy.

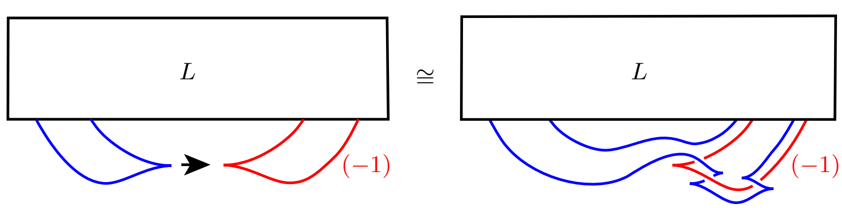

FIGURE 3. A version of the contact handle slide

As a direct application of a contact handle slide we obtain the following generalization of a slam dunk [DG09].

Lemma 2.8 (Contact slam dunk)**.** *Let K be a Legendrian knot and U be a Legendrian meridian of K with $\mathrm{tb} = -1$. Then contact (-1)-surgery along K followed by contact $(+1)$-surgery along U cancel each other.*

All moves so far preserve the number of (±1)-framed Legendrian knots mod 2. But in fact, there are moves where this number is not preserved. The most important one which we want to use later is the following, first obtained by Lisca and Stipsciz for contact (±1)-surgery in [LS11].

Lemma 2.9 (Lantern destabilization)**.** *Let K be a Legendrian knot. Then*

$$K_1(+1) \times K_{1,1}\left(-\frac{1}{n}\right) \times K_{1,1,s}\,(r) \cong K(+1) \times K_1\left(-\frac{1}{n-1}\right) \times K_{1,s}\,(r)$$

holds for all $n, s \in \mathbb{N}_0$ and $r \in \mathbb{Q} \setminus \{0\}$, where the stabilizations of K_1 and $K_{1,1}$ are chosen with the same sign, the remaining stabilizations are arbitrary, see Figure 4.

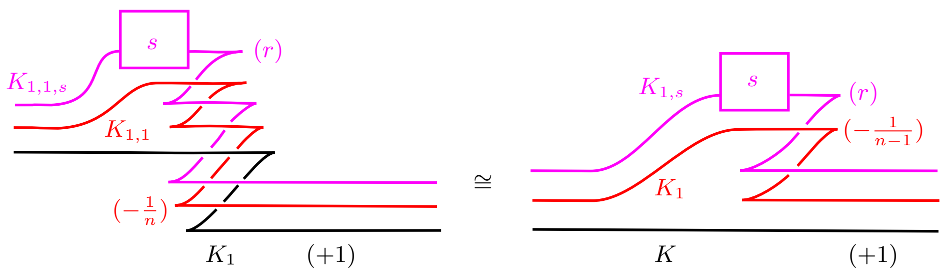

FIGURE 4. Lantern destabilization.

For a proof we refer to [LS11] where the equivalence of the surgery diagrams is shown by translating them into open books and then using the lantern relation and a destabilization (which explains the name). In [LS11] the proof is only given for the case of $r = \infty$. It is straightforward that their proof also works with the additional surgery curve $K_{1,1,s}$.

3. Tightness and overtwistedness

Next, we will need to understand how the properties of the contact structures behave under contact surgery. Since contact (-1)-surgery, also called Legendrian surgery, corresponds to a symplectic handle attachment (see [DG04, Section 3]), contact (-1)-surgery preserves Stein fillability [El90], strong symplectic fillability [We91], weak symplectic fillability [EH01, Lemma 2.4] and the non-vanishing of the contact element in Heegaard–Floer homology [OS05].

Wand [Wa15] showed that Legendrian surgery also preserves tightness for closed contact manifolds. And because of the Replacement Lemma 2.2 and the Transformation Lemma 2.3 all these properties are also preserved for any negative contact surgery coefficient. For this reason in this section we will concentrate on positive contact surgery. First, we study contact $(1/n)$-surgeries, for $n \in \mathbb{N}$. Note that in this case the resulting contact structure is unique.

Lemma 3.1. *If $K(1/n_0)$ is overtwisted for $n_0 \in \mathbb{N}$, then $K(1/n)$ is overtwisted for every $n \geq n_0$.*

Proof. Follows directly from Wand's theorem and the Cancellation and Replacement Lemmas. □

Lemma 3.2. *If $K(1/n)$ is tight, then $K(r)$ is tight for any r with*

$$n-1<\frac{1}{r}\leq n.$$

Proof. The Transformation Lemma yields

$$K(r)=K\left(\frac{1}{n}\right)\times K\left(\frac{1}{\frac{1}{r}-n}\right).$$

Since $K(1/n)$ is tight and $\frac{1}{\frac{1}{r}-n}<0$ the claim follows from Wand's theorem. □

Next we will analyze the case of stabilized Legendrian knots, for which we can completely determine the overtwistedness/tightness of the manifolds obtained by contact Dehn surgery along them.

Theorem 3.3 (Ozbagci [Oz06])**.** *Let K in (M,ξ) be a Legendrian knot.*

(1) *If K is a stabilization, then for any surgery coefficient $r>0$ there exists at least one contact r-surgery such that $K(r)$ is overtwisted. Specifically if the first stabilization in the Transformation Lemma has an opposite sign to the stabilization of K then every positive contact r-surgery is overtwisted. Otherwise, the result may be overtwisted or tight.*
(2) *If K is 2-times stabilized, one time positive and one time negative, then, for $r>0$, every contact r-surgery is overtwisted.*

Theorem 3.3 (1) is due to Ozbagci who obtained the statement by translating the contact surgery diagrams into open books [Oz06]. Lisca and Stipsicz [LS11] gave a direct proof of Theorem 3.3 (1) for contact $(+n)$-surgeries. Here we extend the proof from [LS11] to slightly more general statements.

Proof of Theorem 3.3. Let K be a stabilized Legendrian knot. First, we want to show that $K(1/n)$ is overtwisted for all $n\in\mathbb{N}$. For this consider Figure 5 (or in case of the stabilization with the other sign consider the mirrored figure) where the Legendrian knot K is presented in a neighborhood of the stabilization. Now consider a Legendrian knot L in the complement of K which is given by n-parallel Legendrian push-offs of K away from the stabilization, but looks near the stabilization as in Figure 5. The Legendrian knot L represents, seen as a curve on a tubular neighborhood νK of K, a curve isotopic to $\mu+n\lambda_C$ and therefore bounds a meridional disk of the newly glued-in solid torus in the surgered manifold.

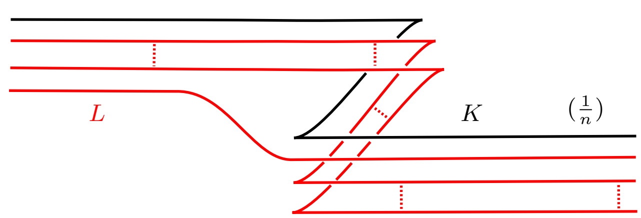

FIGURE 5. L bounds an overtwisted disk in the surgered manifold.

Next, we want to show that the Legendrian unknot L seen as Legendrian knot in $K(1/n)$ has vanishing Thurston–Bennequin invariant tb_{new} and therefore bounds an overtwisted disk. To this end, we use the formula from [Ke18, Ke17] to compute the new Thurston–Bennequin invariant tb_{new} of L in $K(1/n)$ out of the algebraic surgery data to be vanishing.

The case of contact r-surgery, with $r > 0$, follows similarly. By Lemma 2.5 there exists a choice of stabilizations in the Transformation Lemma, such that $K(r)$ is equivalent to the second contact surgery diagram shown in Figure 6 (or the mirrored figure in case of the stabilization with opposite sign).

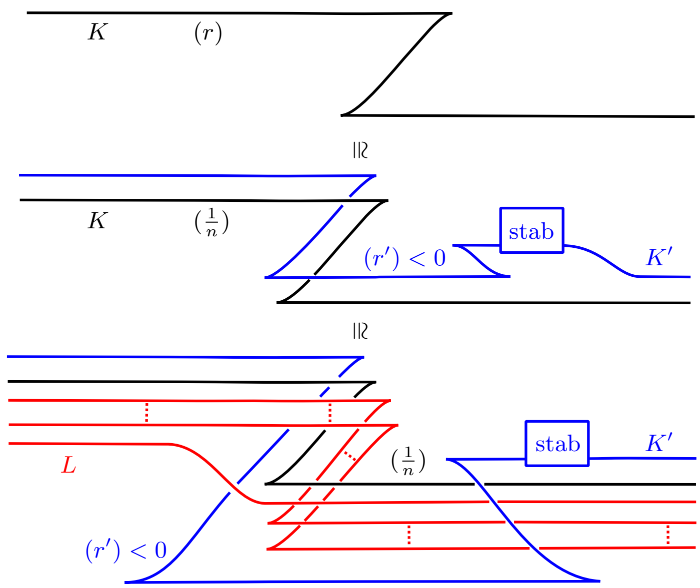

FIGURE 6. L bounds an overtwisted disk in the surgered manifold.

The third surgery diagram in Figure 6 is by a Legendrian isotopy equivalent to the second one. Consider the Legendrian knot L in this surgery diagram, as in the first case. If we see K' as a Legendrian knot in $K(1/n)$, we see that K' does not intersect the Seifert disk of L in $K(1/n)$ and therefore the overtwisted disk bounded by L survives after the contact r'-surgery along K' in $K(1/n)$. So, the first part of Theorem 3.3 follows. The contact surgeries where the stabilizations of K' are all chosen with the same sign as that of K can be tight follows from the next theorem.

Part (2) follows directly. □

The case of integer contact surgeries along knots stabilized only with one sign are summarized in the following theorem due to Lisca and Stipsicz, whose proof follows from the lantern destabilization (Lemma 2.9). We refer to [LS11, Section 2] for the details.

Theorem 3.4 (Lisca–Stipsicz [LS11])**.** *Let L be a Legendrian knot which cannot be destabilized and let $K = L_s$ be an s-times stabilization of L with all signs of the stabilizations equal. The contact n-surgeries (again with all stabilizations with the same sign) along K are determined as follows.*

$$\text{For } n < s+1 : K(+n) = L_{s-n+1}(+1) \text{ and}$$

$$\text{for } n \geq s+1 : K(+n) = L(+1) \times L_1\left(-\frac{1}{n-1-s}\right).$$

In particular, it follows that for $n < s+1$ the contact manifold $K(n)$ is overtwisted and that for $n \geq s+1$ the contact manifold $K(n)$ is tight if $L(+1)$ is tight.

For more general knot types, one can use Theorem 3.4 to trace most information about tightness back to understanding the tightness/overtwistedness of its non-destabilizable Legendrian realizations. As an example we can determine tightness/overtwistedness for most surgery slopes along Legendrian unknots.

Corollary 3.5. *Let U be a Legendrian realization of the unknot in (S^3, ξ_{st}).*

1. *If U is 2-times stabilized, one time positive and one time negative, then any positive surgery along U, is overtwisted.*
2. *We denote by U_s the Legendrian unknot with $\mathrm{tb} = -s$ and $|\mathrm{rot}| = s-1$, i.e. U_s is obtained by $(s-1)$ stabilizations of the same sign from the unique Legendrian unknot $U = U_1$ with $\mathrm{tb} = -1$. If $n < s+1$ any contact $(+n)$-surgery along U_s is overtwisted. If $n \geq s+1$ then exactly one contact $(+n)$-surgery along U_s is overtwisted and the other is tight.*

Proof. Part (1) follows from Theorem 3.3 Part (2) and part (2) from Theorem 3.4. □

Remark 3.6 An alternative approach to determine if a given (rationally) contact surgery diagram represents a tight manifold is to use the contact class in Heegaard–Floer homology, see for example [LS04, Go15, MT18].

4. Computing the homotopical invariants after surgery

Later we will also need to compute the algebraic invariants of the underlying tangential 2-plane field of a contact structure. We are mainly interested in the Euler class and Gompf's d_3-invariant [Go98]. For contact $(\pm 1/n)$-surgeries this can be done relatively straightforward with the following result from [DK16].

Lemma 4.1. *Let $L = L_1 \sqcup \ldots \sqcup L_k$ be an oriented Legendrian link in (S^3, ξ_{st}) and denote by (M, ξ) the contact manifold obtained from S^3 by contact $(\pm 1/n_i)$-surgeries along L ($n_i \in \mathbb{N}$). We write tb_i, rot_i for the Thurston–Bennequin invariant and the rotation number of L_i, and l_{ij} for the linking between L_i and L_j. We denote the topological surgery coefficient of L_i by $r_i = p_i/q_i = \pm 1/n + \mathrm{tb}_i$ and define the generalized linking matrix as*

$$Q := \begin{pmatrix} p_1 & q_2 l_{12} & \cdots & q_n l_{1n} \\ q_1 l_{21} & p_2 & & \\ \vdots & & \ddots & \\ q_1 l_{n1} & & & p_n \end{pmatrix}.$$

1. *The Poincaré dual of the Euler class is given by*

$$\mathrm{PD}\left(e(\xi)\right) = \sum_{i=1}^{k} n_i \,\mathrm{rot}_i\, \mu_i \in H_1(M).$$

The first homology group $H_1(M)$ of M is generated by the meridians μ_i and the relations are $Q\mu = 0$ where μ is the vector with entries μ_i.

(2) *The Euler class $e(\xi)$ is torsion if and only if there exists a rational solution $\mathbf{b} \in \mathbb{Q}^k$ of $Q\mathbf{b} = \mathbf{rot}$. In this case, the* d_3*-invariant is well defined and computed as*

$$\mathrm{d}_3 = \frac{1}{4}\left(\sum_{i=1}^{k} n_i b_i \operatorname{rot}_i + (3-n_i)\operatorname{sign}_i\right) - \frac{3}{4}\sigma(Q)$$

where sign_i *denotes the sign of the contact surgery coefficient of L_i and $\sigma(Q)$ denotes the signature of Q. (In the proof of Theorem 5.1. in* [DK16] *it is shown that the eigenvalues of Q are all real and thus the signature of Q is well-defined although Q is in general a non-symmetric matrix.)*

In the rest of this chapter we will present similar formulas for integer surgeries and then consider more general rational cases. Later, in the presence of 2-torsion, we will also need to consider Gompf's half Euler class Γ [Go98]. Thus, we will shortly discuss how to compute Γ for more general contact surgeries.

4.1. d_3**-invariants of integer surgeries.** Combining Lemmas 4.1 and 2.6 we present a formula for computing the possible d_3-invariants of integer contact surgeries along a single Legendrian knot. In this paper, we mainly use its Corollaries 4.3 and 4.4 in the case that the underlying topological surgery coefficient is ±1 to give lower bounds on contact surgery numbers, we hope that this more general formulation will be useful independently.

Theorem 4.2. *Let K be a Legendrian knot in (S^3, ξ_{st}) with Thurston–Bennequin invariant t and rotation number r and let $n \in \mathbb{N}$ be a positive integer. Then the possible values for the* d_3*-invariants of the integer contact surgeries along K can be computed as follows.*

If $t - n \neq 0$, $K(-n)$ is a rational homology sphere. Thus the d_3*-invariants of its contact structures are well defined and take the values*

$$\mathrm{d}_3\big(K(-n)\big) = \mathrm{d}_3\big(K_{n-1}(-1)\big) = \frac{1}{4}\left(\frac{r'^2}{t-n} - 2 - 3\operatorname{sign}(t-n)\right),$$

where r' denotes the rotation number $\operatorname{rot}(K_{n-1})$ *of the stabilized knot.*

If $t + n \neq 0$, $K(+n)$ is a rational homology sphere. Thus the d_3*-invariants of its contact structures are well defined and take the values*

$$\begin{aligned}\mathrm{d}_3\big(K(+n)\big) =& \mathrm{d}_3\left(K(+1) \times K_1\left(-\frac{1}{n-1}\right)\right)\\ =& \frac{1}{4}\left(\frac{(1\pm r)^2 + (t-n)(1\pm r) - tn}{t+n} \mp r - 2 + n - 3\sigma\right),\end{aligned}$$

where the upper line of signs in the equation corresponds to the positive stabilization of K_1 and the lower line of signs corresponds to the negative stabilization of K_1 and σ is the signature of the generalized linking matrix, which takes the values

$$\sigma = \begin{cases} 0, & \text{if } \operatorname{sign}(t+n) = +1,\\ +2, & \text{if } \operatorname{sign}(t+n) = -1 \text{ and } t \geq 1,\\ -2, & \text{if } \operatorname{sign}(t+n) = -1 \text{ and } t \leq 0.\end{cases}$$

Proof. First we remark that the condition $t - n \neq 0$ in the case of negative surgeries and the condition $t + n \neq 0$ in the case of positive surgeries ensure that the topological surgery coefficient (measured with respect to the Seifert framing) is non-vanishing implying that the surgered manifold has torsion first homology. Thus [Go98] shows that d_3-invariant is well defined. Next we use the formula from Lemma 4.1 to compute the d_3-invariants.

We start with the case of a negative contact surgery $K(-n) = K_{n-1}(-1)$. The topological surgery coefficient is $t-n$ and thus the linking matrix is $Q = (t-n)$ with signature $\operatorname{sign}(t-n)$. The solution b of $Qb = r'$ is given by

$$b = \frac{r'}{t-n}.$$

Plugging everything into the formula from Lemma 4.1 yields the claimed result.

The case of positive surgery is more complicated since

$$K(+n) = K(+1) \times K_1\left(-\frac{1}{n-1}\right).$$

We first build the generalized linking matrix as

$$Q = \begin{pmatrix} 1+t & tn-t \\ t & tn-t-n \end{pmatrix}$$

with determinant $\det(Q) = -(t+n)$ and trace $\operatorname{tr}(Q) = 1 + n(t-1)$. If $\det(Q) < 0$ the signature σ of Q is 0, in the case of positive determinant the signature σ is ± 2 depending on the sign of the trace. By observing that when $\det(Q) > 0$, we have $\operatorname{tr}(Q) > 0$ if and only if $t \geq 1$ the claimed formula for the signature follows.

Next, we solve the equation

$$Q\mathbf{b} = \begin{pmatrix} r \\ r \pm 1 \end{pmatrix}$$

for $\mathbf{b}$ and obtain

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \mp 1 + \frac{n(r\pm1)-tnr+tn(r\pm1)}{t+n} \\ -\frac{r\pm1\pm t}{t+n} \end{pmatrix}.$$

Plugging it in the formula from Lemma 4.1 we get the claimed values for the d_3-invariants. □

We will mostly use this result in the case that the topological surgery coefficient is ±1 and thus the surgered manifold is an integer homology sphere. For easier readability we formulate these important cases in Corollaries 4.3 and 4.4 as independent results.

Corollary 4.3. *Let K in (S^3, ξ_{st}) be a Legendrian knot with Thurston–Bennequin invariant t and rotation number r. Then we can compute the possible d_3-invariants of $K(1-t)$ as follows.*

If $t \geq 2$, the d_3-invariant of its contact structures take the values

$$\mathrm{d}_3\left(K(1-t)\right) = \mathrm{d}_3\left(K_{t-2}(-1)\right) = \frac{1}{4}\left(r'^2 - 5\right),$$

where r' denotes the rotation number $\operatorname{rot}(K_{t-2})$ of the stabilized knot.

If $t \leq 0$, the d_3-invariant of its contact structures take the values

$$\begin{aligned}\mathrm{d}_3\left(K(1-t)\right) =&\mathrm{d}_3\left(K(+1)\times K_1\left(-\frac{1}{-t}\right)\right)\\ =&\frac{1}{4}\left((t\pm r)^2-1\right),\end{aligned}$$

where the upper line of signs in the equation corresponds to the positive stabilization of K_1 and the lower line of signs corresponds to the negative stabilization of K_1.

Corollary 4.4. *Let K in (S^3, ξ_{st}) be a Legendrian knot with Thurston–Bennequin invariant t and rotation number r. Then we can compute the d_3-invariants of $K(-1-t)$ as follows.*

If $t \geq 0$, the d_3-invariant of its contact structures take the values

$$\mathrm{d}_3\left(K(-1-t)\right) = \mathrm{d}_3\left(K_t(-1)\right) = \frac{1}{4}\left(1-r'^2\right),$$

where r' denotes the rotation number $\mathrm{rot}(K_t)$ of the stabilized knot.

If $t \leq -2$, the d_3-invariant of its contact structures take the values

$$\begin{aligned}\mathrm{d}_3\left(K(-1-t)\right) =&\mathrm{d}_3\left(K(+1)\times K_1\left(-\frac{1}{-t}\right)\right)\\ =&\frac{1}{4}\left(1-(t\pm r)^2\right)-(t\pm r),\end{aligned}$$

where the upper line of signs in the equation corresponds to the positive stabilization of K_1 and the lower line of signs corresponds to the negative stabilization of K_1.

Remark 4.5 We observe, that in Corollary 4.3, $\mathrm{tb}(K_{t-2}) = 2$ is even and thus r' is odd, so $r'^2 - 5$ is always divisible by 4. In fact, the first formula in Corollary 4.3 yields integer d_3-invariants. Similarly, we see that $t\pm r$ is always odd, thus $(t\pm r)^2-1$ is divisible by 4 and thus the second equation in Corollary 4.3 yields integer values as well.

Similarly, we observe that in Corollary 4.4, $1 - r'^2$ is always divisible by 4 and that $t \pm r$ is always odd, thus $1 - (t \pm r)^2$ is divisible by 4.

In particular, we get the following two key results completely characterizing the possible values of the d_3-invariants obtained by contact surgery along different Legendrian realizations of the same smooth knot type. To formulate the results compactly we first introduce notation.

Let K in S^3 be a smooth, unoriented knot with two oriented Legendrian realizations L^+_{max} and L^-_{max} of K (which are allowed to be isotopic) such that $\mathrm{TB} = \mathrm{tb}(L^+_{max}) = \mathrm{tb}(L^-_{max})$ and $\mathrm{ROT} = \mathrm{rot}(L^+_{max}) = -\mathrm{rot}(L^-_{max}) \geq 0$ such that any other oriented Legendrian realization L of K has classical invariants $\mathrm{tb}(L)$ and $\mathrm{rot}(L)$ such that $\mathrm{tb}(L)$ and $\mathrm{rot}(L)$ are also obtained as classical invariants of a stabilization of L^+_{max} or L^-_{max}. We define the integer

$$T := \frac{\mathrm{TB}+\mathrm{ROT}-1}{2}.$$

Corollary 4.6. *Let K be a knot as described above and M be the 3-manifold obtained by topological $(+1)$-surgery along K. Then the possible values of the d_3-invariants of contact structures on M obtained by integer contact surgery (corresponding to topological $(+1)$-surgery) along Legendrian realizations of K are*

$$m(m-1),$$

where m is an integer such that $m \geq -T$ and if $\mathrm{TB} \geq 2$ *then the* d_3*-invariant can be in addition*

$$(T-m)^2 + (T-m) - 1,$$

where $m = 1, 2, \dots, \mathrm{TB} - 1$.

Proof. Let L be a Legendrian realization of K in (S^3, ξ_{st}) with Thurston–Bennequin invariant $t = \mathrm{tb}(L)$. Since L has the same classical invariants as a stabilization of L_{max} or $-L_{max}$ the possible values for the rotation number $r = \mathrm{rot}(L)$ are

$$r = \pm \mathrm{ROT} + \mathrm{TB} - t - 2k, \tag{6}$$

for $k = 0, 1, \dots, \mathrm{TB} - t$. From which we conclude the possible values for $t \pm r$ as:

$$t \pm r = \mathrm{TB} \pm \mathrm{ROT} - 2k, \tag{7}$$

for $k = 0, 1, \dots, \mathrm{TB} - t$.

We now want to use Corollary 4.3 to compute the possible d_3-values. Therefore, we need to distinguish two cases. We start with the case that $\mathrm{TB} \geq t \geq 2$. Then we know by Corollary 4.3 that

$$\mathrm{d}_3\big(L(1-t)\big) = \frac{r'^2 - 5}{4},$$

where r' is the rotation number of the stabilized knot L_{t-2}. Since $\mathrm{tb}(L_{t-2}) = 2$ we know from Equation (6) that

$$r' = \pm(\mathrm{TB} + \mathrm{ROT} - 2 - 2k),$$

for $k = 0, 1, \dots, \mathrm{TB} - 2$. Setting $m = k+1$ and plugging everything into the equation of the d_3-invariant yields the second set of possible values.

In the case $t \leq 0$, we have from Corollary 4.3 that

$$\mathrm{d}_3\big(L(1-t)\big) = \frac{(t \pm r)^2 - 1}{4}.$$

Plugging in Equation (7) yields

$$\mathrm{d}_3\big(L(1-t)\big) = \frac{(\mathrm{TB} \pm \mathrm{ROT})^2 - 1}{4} - k(\mathrm{TB} \pm \mathrm{ROT}) + k^2.$$

Next, we write $\mathrm{TB} \pm \mathrm{ROT} = 2T_\pm + 1$ and thus get

$$\mathrm{d}_3\big(L(1-t)\big) = (T_\pm - k)^2 + (T_\pm - k).$$

By setting $m = k - T_\pm$ we get $m^2 - m$ for $m = -T_\pm, -T_\pm + 1, \dots, -T_\pm + \mathrm{TB} - t$. But since $T = T_+ \geq T_-$ and $-t$ can be arbitrarily large we see that $m \geq -T$. □

The analogous result for topological (-1)-surgery is as follows.

Corollary 4.7. *Let M be the* 3*-manifold obtained by topological* (-1)*-surgery along K. Then the possible values of the* d_3*-invariants of contact structures on M obtained by integer contact surgery (corresponding to topological* (-1)*-surgery) along Legendrian realizations of K are*

$$m(3-m) - 1,$$

where m is an integer such that $m \geq -T$ and if $\mathrm{TB} \geq 0$ *then the* d_3*-invariant can be in addition*

$$-(m-T)^2 + (m-T),$$

where $m = 0, 1, \dots, \mathrm{TB}$.

Proof. The proof works analogous as in the proof of Corollary 4.6. In the case of $\mathrm{TB} \geq t \geq 0$ we know by Corollary 4.4 that

$$\mathrm{d}_3\left(L(1-t)\right) = \frac{1-{r'}^2}{4},$$

where r' is the rotation number of the stabilized knot L_t with $\mathrm{tb}(L_t) = 0$. Together with Equation (6) we get the second set of values for the d_3-invariants.

The first set of values for d_3, we get by combining Corollary 4.4 and Equation (7) in the case $t \leq -2$. □

4.2. **d_3-invariants of rational surgeries.** In this section we will discuss some special cases of rational contact surgeries, which we will need later in Section 7.4 to compute rational contact surgery numbers of contact structures on S^3. In this section K will always denote a topological knot in S^3 with a unique maximal Legendrian representative L_{max} with Thurston–Bennequin invariant $\mathrm{TB} = -1$ and $\mathrm{ROT} = 0$, such that any other Legendrian realization L of K is obtained by stabilizations from L_{max}. (In other words, K has the same range of classical invariants as the unknot.)

The goal is to analyze the d_3-invariants of contact surgeries along Legendrian realizations of K corresponding to a topological $(1/q)$-surgery. For that we consider some Legendrian realization L of K with Thurston–Bennequin invariant $t \leq -1$, rotation number r and some integer $q \in \mathbb{Z} \setminus \{0\}$. The goal is to analyze

$$L\left(\frac{1}{q} - t\right) = L\left(\frac{1-qt}{q}\right),$$

where we assume $(t,q) \neq (-1,-1)$. (Otherwise, the contact surgery coefficient would vanish.) First we need the analog of Lemma 2.6 in this setting.

Lemma 4.8. *If $t = -1$ and $q \leq -2$ we have*

$$L\left(\frac{1-qt}{q}\right) = L\left(\frac{1+q}{q}\right) = L\left(\frac{1}{2}\right) \times L_1\left(-\frac{1}{-q-2}\right). \tag{8}$$

If $t \leq -2$ and $q \leq -1$ we have

$$L\left(\frac{1-qt}{q}\right) = L\left(+1\right) \times L_1\left(-\frac{1}{-t-2}\right) \times L_{1,1}\left(-\frac{1}{-q-1}\right). \tag{9}$$

If $t \leq -1$ and $q \geq 1$ we have

$$L\left(\frac{1-qt}{q}\right) = L\left(+1\right) \times L_1\left(-\frac{1}{-t-1}\right) \times L_{1,q-1}\left(-1\right). \tag{10}$$

Proof of Lemma 4.8. We start with the case $t = -1$ and $q \leq -2$, in which we have

$$1 < \frac{q}{1+q} \leq 2.$$

From Lemma 2.3 we obtain

$$L\left(\frac{1+q}{q}\right) = L\left(\frac{1}{2}\right) \times L\left(-\frac{1+q}{2+q}\right) = L\left(\frac{1}{2}\right) \times L_1\left(-\frac{1}{-q-2}\right),$$

where the last equality is exactly Lemma 2.6(4) (for $n = -(1+q)$).

For the case $t \leq -2$ and $q \leq -1$ we observe

$$0 < \frac{q}{1-qt} \leq 1$$

and thus get from Lemma 2.3

$$L\left(\frac{1-qt}{q}\right) = L\,(+1) \times L\left(-\frac{qt-1}{q(t+1)-1}\right).$$

Now an easy induction proves the continued fraction expansion of the last surgery coefficient to be

$$-\frac{qt-1}{q(t+1)-1} = \big[\underbrace{-2,\dots,-2}_{-t-2},-3,\underbrace{-2,\dots,-2}_{-q-2}\big]$$

and from Lemma 2.3 and Lemma 2.2 the claimed result follows.

Finally, in the case $t \leq -1$ and $q \geq 1$ we have

$$0 < \frac{q}{1-qt} < 1$$

and thus the following continuous fraction expansion proves the lemma:

$$-\frac{qt-1}{q(t+1)-1} = \big[\underbrace{-2,\dots,-2}_{-t-1},-q-1\big].$$

□

Next, we compute the possible values of the d_3-invariants corresponding to the three cases of Lemma 4.8.

Lemma 4.9. *For $t=-1$ and $q \leq -2$ we have*

$$\mathrm{d}_3\left(L\left(\frac{1+q}{q}\right)\right) = 1.$$

Proof. From Lemma 4.8(8) we know that

$$L\left(\frac{1+q}{q}\right) = L\left(\frac{1}{2}\right) \times L_1\left(-\frac{1}{-q-2}\right)$$

with generalized linking matrix

$$Q = \begin{pmatrix} -1 & q+2 \\ -2 & 2q+3 \end{pmatrix}$$

whose signature is $\sigma(Q) = -2$. Solving

$$Q\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}$$

yields $b_1 = \mp(q+2)$ and $b_2 = \mp 1$ and plugging everything into the formula from Lemma 4.1 gives the claimed value. □

Lemma 4.10. *For $t \leq -2$ and $q \leq -1$ we have*

$$\mathrm{d}_3\left(L\left(\frac{1-qt}{q}\right)\right) = q\frac{(t\pm r)^2+3}{4} + q(t\pm r) + 1 \quad \textit{or}$$

$$\mathrm{d}_3\left(L\left(\frac{1-qt}{q}\right)\right) = q\frac{(t\pm r)^2-1}{4} - (t\pm r)$$

depending on the choices of stabilizations in the expression of the surgery in terms of reciprocal integer contact surgeries from Lemma 4.8.

Proof of Lemma 4.10. From Lemma 4.8(9) we know that

$$L\left(\frac{1-qt}{q}\right) = L\,(+1) \times L_1\left(-\frac{1}{-t-2}\right) \times L_{1,1}\left(-\frac{1}{-q-1}\right)$$

with generalized linking matrix

$$Q = \begin{pmatrix} 1+t & -t^2-2t & -qt-t \\ t & 1-t^2-t & -qt-t+q+1 \\ t & -t^2-t+2 & 1-qt-t+2q \end{pmatrix}$$

whose signature is $\sigma(Q) = -3$. Solving

$$Q\mathbf{b} = \mathbf{r}$$

yields

$$\begin{aligned} b_1 =& r_1 + tr_3 - tr_1 + tqr_3 - 2qtr_2 + t^2qr_1 - t^2qr_2 \\ b_2 =& r_3 - r_2 + qr_3 - 2qr_2 + tqr_1 - tqr_2 \\ b_3 =& 2r_2 - r_3 + tr_2 - tr_1. \end{aligned}$$

Now we know that $(r_1, r_2, r_3) = (r, r \pm 1, r)$ or $(r_1, r_2, r_3) = (r, r \pm 1, r \pm 2)$. By plugging in $(r_1, r_2, r_3) = (r, r\pm 1, r)$ into the formula from Lemma 4.1 we get the first claimed equation for the d_3-invariant and by plugging in $(r_1, r_2, r_3) = (r, r\pm 1, r\pm 2)$ we get the second equation. □

Lemma 4.11. *For $t \leq -1$ and $q \geq 1$ we have*

$$\mathrm{d}_3\left(L\left(\frac{1-qt}{q}\right)\right) = q\frac{(t\pm r)^2 - 1}{4} + (t \pm r + 1)z,$$

where $z = 0, 1, 2, \dots, q-1$, is determined by $z = (q \pm r \mp r')/2$ with r' denoting the rotation number of $L_{1,q-1}$ from Lemma 4.8.

Remark 4.12 We remark that for $q = \pm 1$ we recover the formulas from Corollaries 4.3 and 4.4.

Proof of Lemma 4.11. From Lemma 4.8(10) we know that

$$L\left(\frac{1-qt}{q}\right) = L\,(+1) \times L_1\left(-\frac{1}{-t-1}\right) \times L_{1,q-1}\,(-1)$$

with generalized linking matrix

$$Q = \begin{pmatrix} 1+t & -t^2-t & t \\ t & -t^2 & t-1 \\ t & 1-t^2 & t-q-1 \end{pmatrix}$$

whose signature is $\sigma(Q) = -1$. Solving

$$Q\mathbf{b} = \mathbf{r}$$

yields

$$\begin{aligned} b_1 =& r_1 + tr_3 - tr_1 - qtr_2 + qt^2r_1 - qt^2r_2 \\ b_2 =& r_3 - r_2 - qr_2 + qtr_1 - qtr_2 \\ b_3 =& -r_2 + tr_1 - tr_2. \end{aligned}$$

Now we know that $r_1 = r$, $r_2 = r\pm 1$ and $r_3 = r'$ is the rotation number of $L_{1,q-1}$. By plugging everything into the formula from Lemma 4.1 we get the claimed equation for the d_3-invariant. □

4.3. **The Euler class of integer surgeries.** If M has nontrivial homology we get another invariant of tangential 2-plane fields, namely its underlying *spin*^{c} structure. The set of *spin*^{c} structures on M is in one-to-one correspondence with the elements of $H^2(M) = H_1(M)$ (although this correspondence is not natural). If there is no 2-torsion in the first homology, two *spin*^{c} structures are equal if and only if their Euler classes agree. (Recall, that the Euler class of an oriented 2-plane field is always an even class.) Moreover, two tangential 2-plane fields with the same underlying *spin*^{c} structure only differ by a connected sum with some, in general non-unique, (S^3, ξ_n) for appropriate $n \in \mathbb{Z}$. For more details we refer to [DGS04, Go98].

Our first result is a formula for the Euler class for a single positive integer contact surgery. Results for surgeries along links with arbitrary integers can easily be derived from this.

Proposition 4.13. *Let L be a Legendrian knot in (S^3, ξ_{st}) and $n \in \mathbb{N}$, with Thurston–Bennequin invariant t and rotation number r. Then we can compute the Euler class of $L(n) = (M, \xi)$ as*

$$\mathrm{PD}\left(\mathrm{e}(\xi)\right) = \pm(n \mp r - 1)\mu \in H_1(M) = \langle \mu \,|\, -(t+n)\mu \rangle \cong \mathbb{Z}_{t+n},$$

where the first row of signs correspond to a positive stabilization in $L(n)$ and the second to a negative stabilization.

Proof. From Lemma 2.6 we can write

$$(M, \xi) = L(n) = L(1) \times L_1\left(-\frac{1}{n-1}\right)$$

and we get the generalized linking matrix as

$$Q = \begin{pmatrix} 1+t & t(n-1) \\ t & tn-t-n \end{pmatrix}.$$

The first homology group of M is generated by the meridians of the surgery curves and the relations are given by Q. From Lemma 4.1 we obtain the claimed description of the Euler class. □

As a corollary, we see how the Euler class changes under stabilizing the knot L and adjusting the contact surgery coefficient appropriately such that the underlying smooth manifold is still the same.

Corollary 4.14. *In the same notation as in Proposition 4.13, we consider the contact surgery diagram given by*

$$L_s(n+s) = L_s(+1) \times L_{s,1}\left(-\frac{1}{n+s-1}\right),$$

where the first s-stabilizations are all performed with the same sign, but the last stabilization of $L_{s,1}$ has the opposite sign. Then the above surgery diagram presents again a contact structure ξ' on M, whose Euler classes are related by

$$\mathrm{PD}(\mathrm{e}(\xi')) = \mathrm{PD}(\mathrm{e}(\xi)) \pm 2s\mu \in H_1(M) = \langle \mu \,|\, -(t+n)\mu \rangle \cong \mathbb{Z}_{t+n}.$$

It follows that we can obtain contact structures on M via integer contact surgery along a stabilization of L realizing any possible Euler class of 2-plane fields on M.

If we have more than one surgery curve, we perform the same construction along every surgery curve and obtain integer contact surgery diagrams of contact structures of any possible Euler class along a stabilization of that Legendrian link.

4.4. **Gompf's Γ invariant.** In the case that a 3-manifold has no 2-torsion in its first homology, we know that the Euler class of a contact structure ξ on M uniquely determines the underlying *spin*^{c} structure. However, in the presence of 2-torsion this is not true anymore. On the other hand, Gompf defined a refined invariant, the Γ **invariant** of a contact structure which resolves that ambiguity. In some sense Γ can be through of as a half Euler class of the contact structure. It is defined as follows. Let (M,ξ) be a contact manifold and V be a vector field in ξ such that the zero set of V is given by 2γ for a 1-cycle γ. Then V determines a unique *spin* structure $\mathfrak{s}$ and $\Gamma(\xi,\mathfrak{s})$ is defined to be

$$\Gamma(\xi,\mathfrak{s}) = [\gamma] \in H_1(M).$$

In [Go98] it is shown that $\Gamma(\xi,\mathfrak{s})$ is well-defined and only depends on ξ and $\mathfrak{s}$. Furthermore, it behaves natural under connected sums and coverings, classifies tangential 2-plane fields on $M \setminus \{pt\}$ and it fulfills $2\Gamma(\xi,\mathfrak{s}) = \mathrm{PD}(\mathrm{e}(\xi))$ for any $\mathfrak{s}$.

Here, we want to generalize Gompf's formula for computing $\Gamma(\xi,\mathfrak{s})$ from contact (-1)-surgery diagrams to general contact (±1)-surgery diagrams. For that we first recall how to present *spin* structures in surgery diagrams [GS99]. Let $L = L_1 \sqcup \ldots \sqcup L_k$ be a topological integer surgery diagram of a 3-manifold M (in particular, the framings of L_i are measured with respect to the Seifert framing). We write $\mathrm{lk}(L_i,L_i)$ for the framing of L_i. A sublink $(L_j)_{j\in J}$ for some subset $J \subset \{1,2,\ldots,k\}$ is called **characteristic sublink** if for any component L_i of L we have

$$\mathrm{lk}(L_i,L_i) \equiv \sum_{j\in J} \mathrm{lk}(L_i,L_j) \mod(2).$$

The set of characteristic sublinks of L is in bijection with the *spin* structures of M which is in (non natural) bijection to $H_1(M;\mathbb{Z}_2)$. Thus we can describe a given *spin* structure of M via a characteristic sublink of L.

Lemma 4.15. *Let $L = L_1 \sqcup \ldots \sqcup L_k$ be an oriented Legendrian link in (S^3,ξ_{st}) and let (M,ξ) be the contact manifold obtained from S^3 by contact (±1)-surgeries along L. We write r_i for the rotation number of L_i, we denote the topological surgery coefficient of L_i by $\mathrm{lk}(L_i,L_i)$ and write Q for the linking matrix. Moreover, we describe a spin structure $\mathfrak{s}$ of M via a characteristic sublink $(L_j)_{j\in J}$ of L. Then*

$$\Gamma(\xi,\mathfrak{s}) = \frac{1}{2}\left(\sum_{i=1}^{k} r_i\mu_i + \sum_{j\in J}(Q\mu)_j\right),$$

where $(Q\mu)_j$ denotes the j-th entry of $Q\cdot(\mu_1,\ldots,\mu_k)^t$.

Since $Q\mu = 0$ in $H_1(M)$ we see directly that $2\Gamma(\xi,\mathfrak{s})$ is Poincaré dual to the Euler class.

Proof. If all contact surgery coefficients are (-1) Theorem 4.12 in [Go98] implies that

$$\Gamma(\xi,\mathfrak{s}) = \sum_{i=1}^{k}\frac{1}{2}\left(r_i + \sum_{j\in J}\mathrm{lk}(L_i,L_j)\right)\mu_i,$$

but since

$$Q\mu = \left(\sum_{i=1}^{k}\mathrm{lk}(L_i,L_j)\mu_i\right)_{1\leq j\leq k}$$

this directly implies the formula for contact (-1)-surgeries as stated in the lemma.

Next, we claim that if some of the surgeries are contact $(+1)$-surgeries the formula does not change (but the homology does change of course). In the proof of Lemma 3 of [GO15] it is shown that a vector field V on $S^3 \setminus \mathring{\nu L}$ tangent to ξ which is given by the Legendrian ruling curves on $\partial\nu L$ extends without zeros over the surgered manifolds for both surgery coefficients, (-1) and $(+1)$. Thus $\Gamma(\xi, \mathfrak{s})$ is given by the same homology class in the complement of L and is therefore represented by the same formula. □

As done in [DK16] for the Euler class one can similarly deduce from this lemma a general formula for Γ of contact $(1/n)$-surgeries. But since we will not need this here, we describe instead an example how to compute Γ directly for a single contact $(1/n)$-surgery along a single Legendrian knot K.

Example 4.16 Let K be a Legendrian knot in (S^3, ξ_{st}) with Thurston–Bennequin invariant t and rotation number r. Let n be a positive integer then

$$K(\pm 1/n) = \underbrace{K(\pm 1) \times \ldots \times K(\pm 1)}_{n}$$

and the first homology is generated by $\mu := \mu_1 = \cdots = \mu_n$ with the relation $(nt \pm 1)\mu = 0$. Since the *spin* structures are in bijection with the first homology group with $\mathbb{Z}_2$-coefficients, we see that there is a unique *spin* structure if $nt \pm 1$ is odd and that there exists exactly two spin structures if $nt \pm 1$ is even. We observe that in both cases the empty sublink is characterizing defining a *spin* structure $\mathfrak{s}_0$. In the case that $nt \pm 1$ is even the whole surgery link is also characterizing defining a *spin* structure $\mathfrak{s}_1$. Then Lemma 4.15 readily implies that

$$\Gamma\big(K(\pm 1/n), \mathfrak{s}_0\big) = \frac{nr}{2}\mu$$

and if $nt \pm 1$ is even we have in addition

$$\Gamma\big(K(\pm 1/n), \mathfrak{s}_1\big) = n\frac{r + nt \pm 1}{2}\mu.$$

Next, we study the case of general positive integer contact surgeries. The following two results generalize the results from Proposition 4.13 and Corollary 4.14 for the Euler class to Gompf's Γ invariant. Again, similar results hold for positive integer surgeries along links with more components.

Proposition 4.17. *Let L be a Legendrian knot in (S^3, ξ_{st}) and n a positive integer. Then $L(n)$ is given by the contact (± 1)-surgery description*

$$L(+1) \times \underbrace{L_1(-1) \times \cdots \times L_1(-1)}_{n-1}. \tag{11}$$

The first copy of L in the above surgery description is a characterizing sublink and thus defines a spin structure $\mathfrak{s}$. Then

$$\Gamma\big(L(n), \mathfrak{s}\big) = \left(-\frac{t + r \pm 1}{2} + \frac{\pm 1 - 1}{2}n\right)\mu.$$

Proof. From the surgery description (11) we compute the linking matrix to be

$$Q := \begin{pmatrix} t+1 & t & t & \cdots & t \\ t & t-2 & t-1 & \cdots & t-1 \\ t & t-1 & t-2 & & t-1 \\ \vdots & \vdots & & \ddots & \\ t & t-1 & \cdots & t-1 & t-2 \end{pmatrix}.$$

This gives us the presentation of the first homology of $L(n)$ as

$$H_1\big(L(n)\big) = \langle \mu | (-(n+t)\mu = 0\rangle \cong \mathbb{Z}_{n+t},$$

where $\mu := \mu_2 = \cdots = \mu_n$ and $\mu_1 = -n\mu$. Moreover, it is easy to compute that L, the first surgery curve in (11), is really a characterizing sublink of the surgery description (11). (In case that $t+n$ is even there is another *spin* structure which corresponds to the whole surgery link. However, we will not need that *spin* structure and therefore do not consider it here.) Then a straightforward computation with Lemma 4.15 yields the claimed value for the Γ invariant. □

Next, we want to see how the Γ invariant changes under stabilization and under appropriately changing the surgery coefficients.

Proposition 4.18. *In the same notation as in Proposition 4.17, we consider the contact surgery diagram given by*

$$L_s(n+s) = L_s(+1) \times \underbrace{L_{s,1}(-1) \times \cdots \times L_{s,1}(-1)}_{n+s-1},$$

where the first s-stabilizations are all performed with the same sign, but the last stabilization of $L_{s,1}$ has the opposite sign. Then the above surgery diagram presents again a contact structure ξ' on $M = L(n) = L_s(n+s)$, whose Γ invariants are related by

$$\Gamma\big(L_s(n+s), \mathfrak{s}\big) - \Gamma\big(L(n), \mathfrak{s}\big) = \pm s\mu \in H_1(M) = \langle \mu | -(t+n)\mu\rangle \cong \mathbb{Z}_{t+n}.$$

It follows that for any given first homology class c of M we can find a contact structure on M obtained via integer contact surgery along a stabilization of L whose Γ invariant equals c.

Proof. First we need to show that the corresponding characteristic sublinks and thus the *spin* structures are mapped to each other under the Kirby moves relating the underlying integer surgery descriptions. For that we need to understand how to keep track of *spin* structures (or equivalently) characteristic sublinks through Kirby moves [GS99]. We think of a characteristic sublink as colored in a different color than the other link components. If we slide L_i over L_j then L_j changes the color if and only if L_i is in the characteristic sublink. Moreover, if K is a blown up curve then K belongs to the characteristic sublink if and only if $\sum_{j\in J} \operatorname{lk}(K, L_j) \equiv 0 \mod(2)$.

Figure 7 shows a sequence of Kirby moves relating the two surgery descriptions while keeping track of the characteristic sublink defining $\mathfrak{s}$. It follows that we can apply directly Proposition 4.17 and thus get the claimed change of Γ invariant. □

Next, we consider rational surgeries. For more general rational contact surgery coefficients $r \in \mathbb{Q}$ there is no general formula, since the results will heavily depend on the continued fraction expansion of r. However, we can still determine how the Γ invariant changes under stabilization by generalizing Proposition 4.18 and in

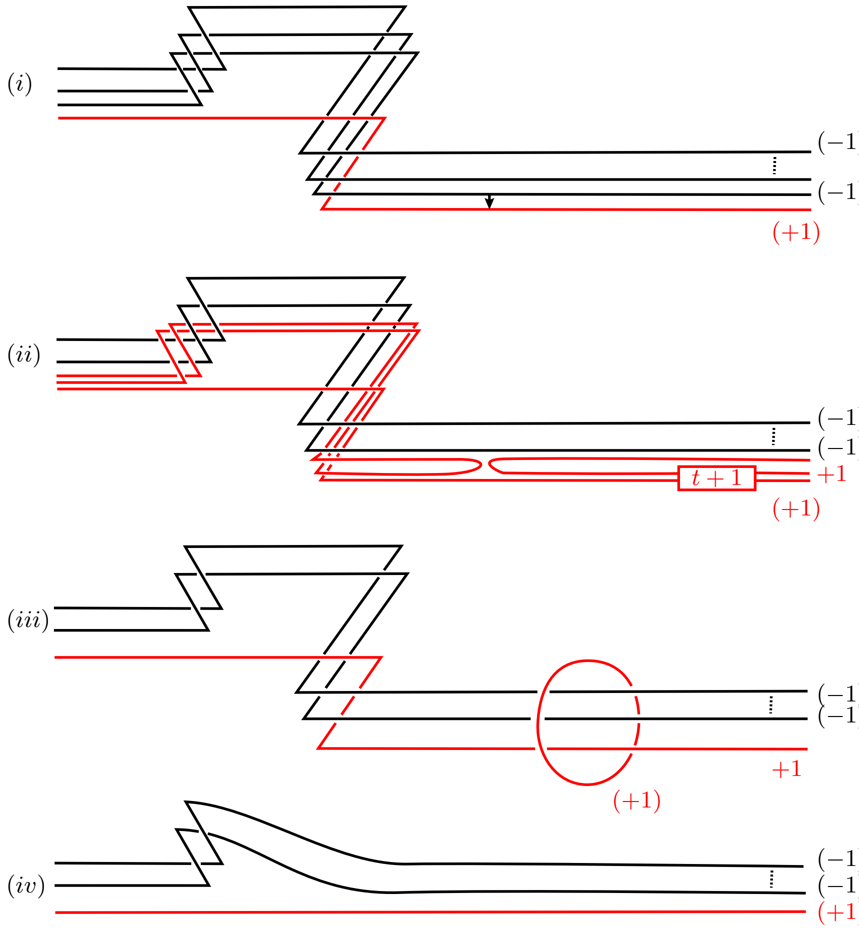

FIGURE 7. Figures (i) shows the characteristic sublinks of $L_1(n+1)$ and Figure (iv) the characteristic sublink of $L(n)$ in red. To see that they induce the same *spin* structure on M, we first perform a handle slide as indicated with the black arrow in Figure (i) to get Figure (ii). The new red link belongs then to the characteristic sublink. Since it is not a Legendrian link anymore we measure its framing with respect to the Seifert framing (as indicated by omitting the parentheses). To get Figure (iii) we apply an isotopy. Blowing down the (-1)-framed unknot yields Figure (iv). The case for general s follows by induction.

particular we will see that we get any possible Γ invariant by stabilizing the surgery link.

Theorem 4.19. *Let L be a Legendrian knot in (S^3, ξ_{st}) and we write (M, ξ) for a contact manifold from $L(r)$, $r > 1$. For any integer $s \geq 1$, there exist contact manifolds (M, ξ') in $L_s(r+s)$ and a spin structure $\mathfrak{s}$ on M such that*

$$\Gamma\big(\xi', \mathfrak{s}\big) - \Gamma\big(\xi, \mathfrak{s}\big) = \pm s\mu \in H_1(M).$$

In particular, for any given first homology class c of M there is a contact structure on M obtained via rational contact surgery along a stabilization of L whose Γ invariant equals c. As a direct corollary, we also obtain contact structures on M via contact surgery along a stabilization of L realizing any given even homology class as Euler class.

Proof. Let $r = p/q$, $p > q > 0$, be the contact surgery coefficient such that $L(r) = (M, \xi)$ and let

$$-\frac{p}{p-q} = [r_1, \dots, r_n] \tag{12}$$

be the negative continued fraction expansion. Then the Transformation Lemma 2.3 implies that

$$L(r) = L(1) \times L_{s_1}(-1) \times \dots \times L_{s_1,\dots,s_n}(-1), \tag{13}$$

where the sequence of the stabilization s_i is determined by the r_i as in the Transformation Lemma. We choose an orientation on the surgery link from Equation (13) that is inherited from the orientation of L.

By induction it is enough to prove the theorem for $s = 1$ and thus we consider the contact manifold $L_1(r+1) = (M, \xi')$. Then we get

$$-\frac{p+q}{(p+q)-q} = [-2, r_1, r_2, \dots, r_n],$$

where $[r_1, \dots, r_n]$ is the continued fraction expansion from Equation (12). Thus we obtain from the Transformation Lemma 2.3 a contact (± 1)-surgery description of $L_1(r+1)$ as

$$L_1(+1) \times L_{1,1}(-1) \times L_{1,1,s_1-1}(-1) \times \dots \times L_{1,1,s_1-1,s_2\dots s_n}(-1), \tag{14}$$

where we consider the case that the two stabilizations of $L_{1,1}$ have different signs. All the other stabilizations of the $L_{1,1,s_1-1,s_2,\dots,s_i}$ have the same sign as the original stabilizations of $L_{s_1,\dots,s_i}$ in Equation (13).

We write

$$\begin{aligned} \mathrm{rot}_1 &:= \mathrm{rot}(L), \\ \mathrm{rot}_{i+1} &:= \mathrm{rot}(L_{s_1,\dots,s_i}), \text{ for } i = 1, \dots, n, \end{aligned}$$

for the rotation numbers of the Legendrian knots from (13) and $\mu_1, \dots, \mu_{n+1}$ for their meridians. Similarly, we write

$$\begin{aligned} \mathrm{rot}'_0 &:= \mathrm{rot}(L_{1,1}) = \mathrm{rot}_1, \\ \mathrm{rot}'_1 &:= \mathrm{rot}(L_1) = \mathrm{rot}_1 \pm 1, \\ \mathrm{rot}'_{i+1} &:= \mathrm{rot}(L_{1,1,s_1-1,s_2,\dots,s_i}) = \mathrm{rot}_{i+1} \pm 1, \text{ for } , i = 1, \dots, n, \end{aligned}$$

for the rotation numbers of the surgery description (14) and $\mu'_0, \dots, \mu'_{n+1}$ for their meridians. We also can express the linking matrix Q' of (14) in terms of the linking matrix Q of (13) as follows

$$Q' = \begin{pmatrix} t-2 & t-1 & \cdots & t-1 \\ t-1 & & & \\ \vdots & & Q & \\ t-1 & & & \end{pmatrix} - \begin{pmatrix} 1 & \dots & 1 \\ 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}.$$

As in Figure 7 we can transform the surgery description (14) into the surgery description (13) by performing a single handle slide and then blowing down. Thus we observe that if $J \subset \{1, \dots, n+1\}$ represents a characteristic sublink of (13) then it also represents a characteristic sublink of (14) and so we choose a characteristic sublink given by J for both surgery descriptions that represents a *spin* structure $\mathfrak{s}$ on M.

Finally, we relate the homology classes of the μ_i and the μ_i', by following the smooth handle slides as in Figure 7 relating the two surgery descriptions. Using Lemma 4.15 we observe that the difference $\Gamma(\xi', \mathfrak{s}) - \Gamma(\xi, \mathfrak{s})$ is given by $\pm\mu$, where μ is the meridian of L generating $H_1(M)$ and the sign is given by the sign of the stabilizations in (14). □

5. Inequalities between contact surgery numbers

In this section we start analyzing contact surgery numbers. We first discuss general inequalities between various contact surgery numbers.

Directly from the definitions we get many inequalities between the different versions of contact surgery numbers, for example we have

$$\mathrm{cs} \leq \mathrm{cs}_{\mathbb{Z}}, \text{ and } \mathrm{cs}_{1/\mathbb{Z}} \leq \mathrm{cs}_{\pm 1} \leq \mathrm{cs}_{U,\pm 1}.$$

The main result of this section are the following non-trivial inequalities.

Theorem 5.1. *Let (M, ξ) be a contact 3-manifold. Then the following inequalities hold true*

$$\begin{aligned} \mathrm{cs}_U(M,\xi) &\leq 3\,\mathrm{cs}(M,\xi), \\ \mathrm{cs}_{U,\mathbb{Z}}(M,\xi) &\leq 3\,\mathrm{cs}_{\mathbb{Z}}(M,\xi), \\ \mathrm{cs}_{U,1/\mathbb{Z}}(M,\xi) &\leq 3\,\mathrm{cs}_{1/\mathbb{Z}}(M,\xi), \\ \mathrm{cs}_{U,\pm 1}(M,\xi) &\leq 3\,\mathrm{cs}_{\pm 1}(M,\xi). \end{aligned}$$

Before we discuss the proof we review the following analogous result by Guo and Yu [GY10] in the topological category.

Theorem 5.2 (Guo–Yu [GY10])**.** *Let M be a 3-manifold. Then the following inequalities hold true*

$$\begin{aligned} \mathrm{s}_{\mathbb{Z},U}(M) &\leq 3\,\mathrm{s}_{\mathbb{Z}}(M) \\ \mathrm{s}_U(M) &\leq 3\,\mathrm{s}(M) \end{aligned}$$

The first inequality is stated in their paper [GY10], the second inequality follows by similar methods.

For completeness (and since we want to use similar ideas later in the proof of Theorem 5.1) we shortly summarize a variation of their proof and also indicate why the second inequality holds true.

Proof of Theorem 5.2. The main ingredient in the proof is the next lemma due to Guo and Yu [GY10].

Lemma 5.3 (Guo–Yu [GY10])**.** *Any oriented knot K in S^3 admits a skein move transforming it into a two component link L consisting of two unknots.*

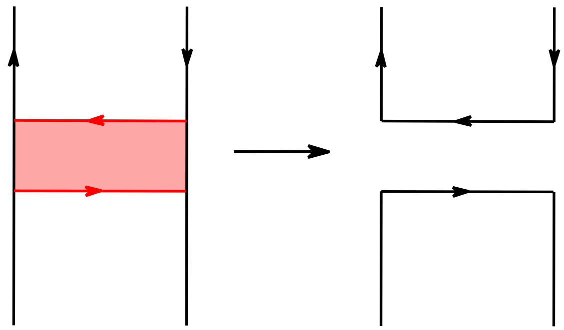

FIGURE 8. A skein move

A **skein move** is the following operation on an oriented knot K. We take an oriented band that meets the knot K in two disjoint arcs where the orientations of K and the band disagree. Then we remove the arcs from the knot and connect the endpoints via the remaining boundary components of the band, see Figure 8. Notice that if one takes a thin annulus with one boundary component being K, and attaches the band to it, we have a surface Σ and $\partial\Sigma$ is $K \cup K_1 \cup K_2$ where K_i are the unknots associated to the skein move.

Now let K be a component in a minimal surgery description of M. By Lemma 5.3 we can find a diagram of the surgery link in which there is a band as in Figure 8 such that performing a skein move on the band will transform K into a link consisting of two unknots $K_1 \cup K_2$. We illustrate this for a knot K in Figure 9(i). If the surgery coefficient of K is not an integer we use the standard method to change the surgery diagram into a diagram with only integer coefficients. (We perform inverse slam dunks, as indicated in Figure 9(ii), with surgery coefficients on the chain of unknots given as the entries in the continued fraction expression of the old surgery coefficient.)

Next, we introduce curves into the diagram K_1 (one of the components of the link formed by the skein move) with any framing q and a zero framed meridian K_0, see Figure 9(ii). A slam dunk move shows that this new surgery diagram is equivalent to the original one. In Figure 9(iii) we have performed a handle slide of K over K_1. The number of twists l and the new surgery coefficient a_1' of K_2 will depend on q, a_1 and the linking number of K and K_1. After an isotopy and possibly slam dunking the chain of unknots away we get a surgery presentation of the same 3-manifold M consisting of three unknots, see Figure 9(iv). By doing the same construction for every component of the original surgery diagram the result follows. □

Remark 5.4 We do not necessarily need to deform the surgery coefficient of K into an integer by inverse slam dunks. We can see the handle slide also as a purely 3-dimensional operation: For that we consider the knot K as a knot in the manifold obtained from S^3 by surgery along K_1 and its meridian. If we move a small part of K near K_1 and slide it over the newly glued-in solid torus the knot K will deform to the knot K_2 exactly as in Figure 9, cf. the proof of the contact handle slide in [CEK21].

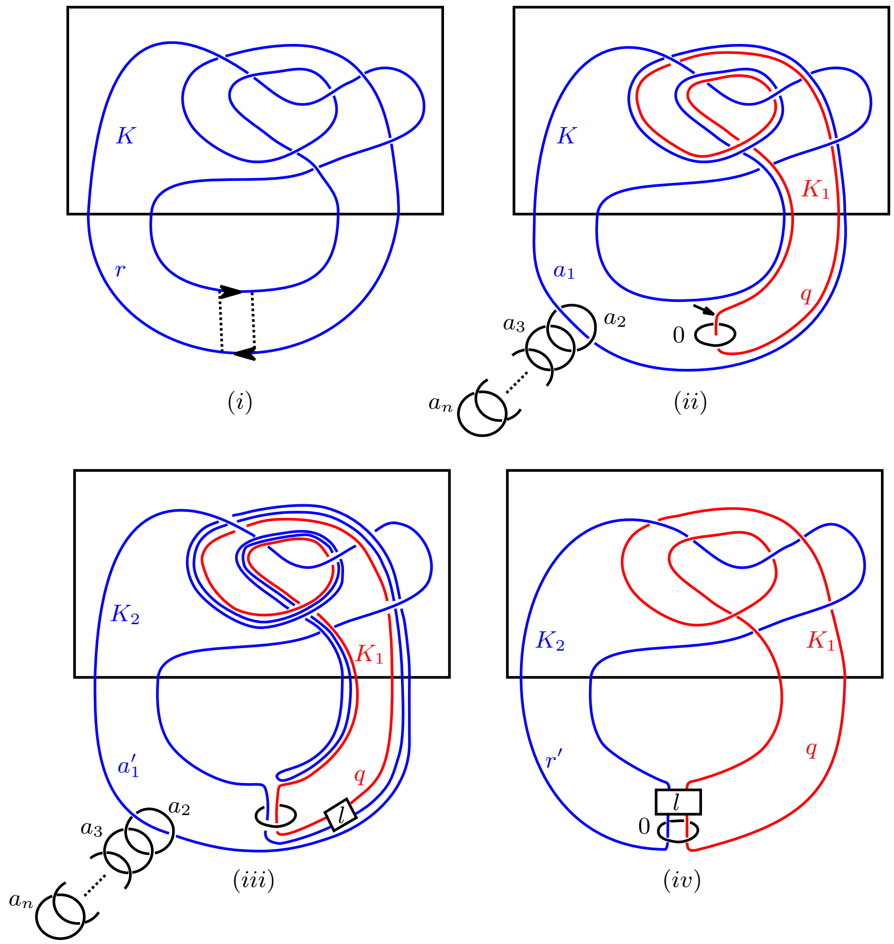

FIGURE 9. Kirby moves changing a knot into a simple 3-component link

The above slight adoption of the proof from [GY10] generalizes to the setting of contact manifolds by using the results from Section 2.

Proof of Theorem 5.1. We consider the front projection of a Legendrian knot K in a minimal contact surgery presentation of (M, ξ). By Lemma 5.3 we know that there exists a band B such that if we do a skein move along the band B we get topologically a simple 2-component link. By sliding the band along the Legendrian knot we can assume one endpoint of B to lie at a cusp of K and the other at a regular point of the front projection. If in the front projection the number of half-twists of that band is odd, we perform a Legendrian Reidemeister move I near the regular point of K where the band meets K, i.e. we can arrange a situation as in Figure 10(i). See Figure 11 for an example.

Next, we introduce two canceling surgeries along Legendrian knots $K_0 \cup K_1$ by an inverse contact slam dunk (Lemma 2.8) as in Figure 10(ii), where K_1 is the once stabilized obvious Legendrian realization of one of the unknots produced by the Skein move with contact framing (-1) and K_0 is a meridian of K_1 with $\mathrm{tb} = -1$

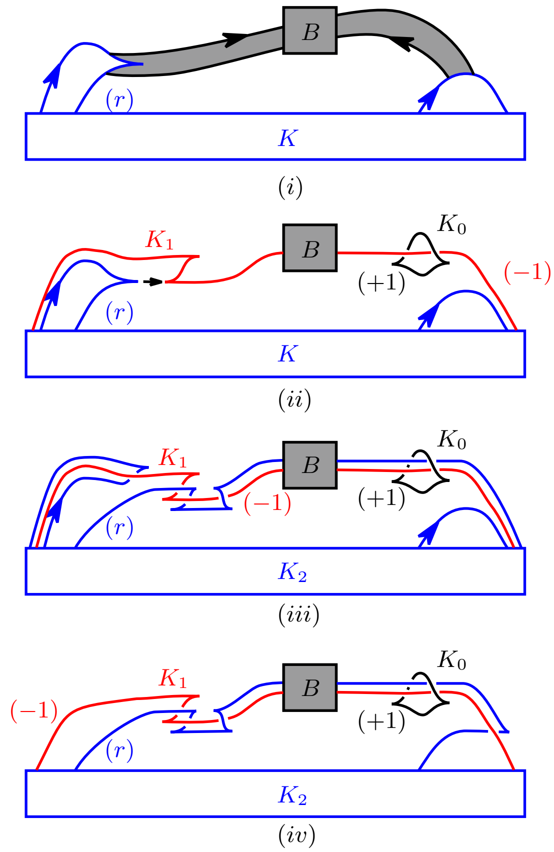

FIGURE 10. Contact Kirby moves changing a Legendrian knot into a simple 3-component Legendrian link

and contact framing $(+1)$, see Figure 10(ii). Then we perform a contact handle slide of K over K_1 and get a surgery description of the same contact manifold consisting only of Legendrian unknots, shown in Figure 10(iii) and (iv). Again, a contact handle slide can be seen as an isotopy of K in the contact manifold obtained by surgery along K_1 and thus the same proof works for other coefficients of K as well. In this setting the contact surgery coefficient stays the same, see for example [CEK21]. So we also obtain the other inequalities. □

6. UPPER BOUNDS ON CONTACT SURGERY NUMBERS

We can often bound surgery numbers from above by providing explicit constructions. We will start with a discussion of contact structures on S^3 and then upgrade this by relating contact surgery numbers of overtwisted contact manifolds to the topological surgery numbers of the underlying topological manifold. We will start with an observation of how contact surgery numbers behave under contact connected sum.

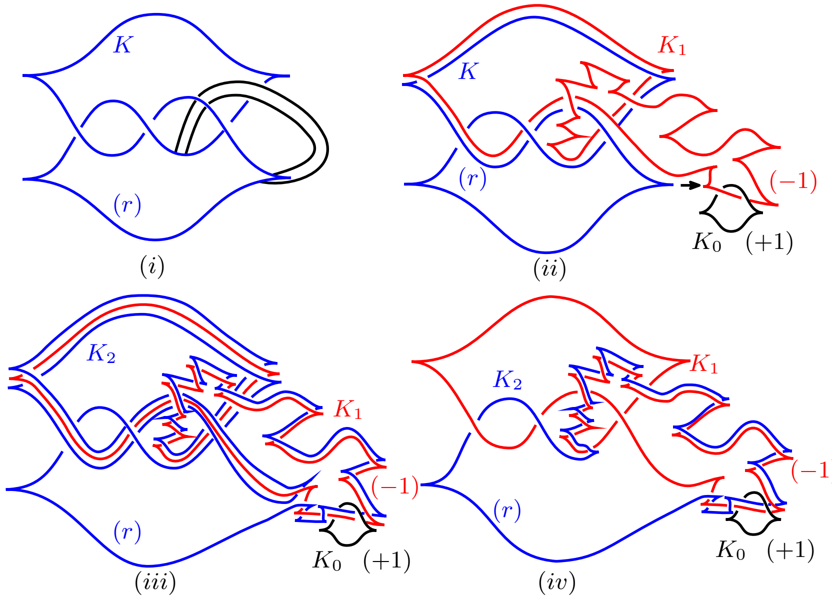

FIGURE 11. Contact Kirby moves changing a contact surgery along a Legendrian trefoil into a contact surgery along a simple 3-component Legendrian link.

Lemma 6.1. *Let (M_1,ξ_1) and (M_2,ξ_2) be contact manifolds and $(M_1\#M_2,\xi_1\#\xi_2)$ be its contact connected sum. Then their contact surgery numbers are related by*

$$\operatorname{cs}_*(M_1\#M_2,\xi_1\#\xi_2)\leq \operatorname{cs}_*(M_1,\xi_1)+\operatorname{cs}_*(M_2,\xi_2)$$

where $*$ is $\emptyset$, $\mathbb{Z}$, $1/\mathbb{Z}$, ±1, or U.

Proof. Let L_i, for $i=1,2$, be surgery diagrams of (M_i,ξ_i) with property $*$ and with minimal number of components. A surgery diagram of $(M_1\#M_2,\xi_1\#\xi_2)$ is given by the disjoint union of L_1 and L_2 and thus the claim follows. □

However the above inequality is in general not an equality. From Section 7, it will follow that there are examples of overtwisted contact manifolds (M_1,ξ_1) and (M_2,ξ_2) such that

$$\operatorname{cs}(M_1\#M_2,\xi_1\#\xi_2)<\operatorname{cs}(M_1,\xi_1)+\operatorname{cs}(M_2,\xi_2).$$

The same phenomena can also happen for tight contact manifolds. [Ya16, LS16], construct examples of Legendrian knots in (S^3,ξ_{st}) along which Legendrian surgery yields a reducible manifold.

In the rest of this article we will also use repeatedly the following two elementary observations, cf. [Ke17].

Lemma 6.2. *Let $L=L_1\cup\cdots\cup L_n\subset(S^3,\xi_{\mathrm{st}})$ be a contact surgery diagram with $(1/k_i)$-contact surgery coefficients, $i=1,\dots n$, of a contact manifold (M,ξ). Then any Legendrian knot K in (M,ξ) can be represented by a Legendrian knot in the exterior of L.*

Proof. By the classification [Gir00, Ho00] of tight contact structures on $S^1\times D^2$ all newly glued-in solid tori are standard neighborhoods of Legendrian knots L_i,

$i = 1, \dots n$, in (M, ξ) and by Lemma 2.1 contact $(-1/k_i)$-surgeries along the Legendrian knots L_i in (M, ξ) reproduces (S^3, ξ_{st}). The standard neighborhoods of the Legendrian knots L_i, used to construct (S^3, ξ_{st}), can be chosen arbitrary small. Thus, it is enough to show that an arbitrary Legendrian knot K in an arbitrary contact manifold (M, ξ) can be made disjoint from an arbitrary Legendrian link $L = L_1 \cup \dots \cup L_n$ by a Legendrian isotopy. By Darboux's theorem it is sufficient to show the same statement for Legendrian knot segments in $(\mathbb{R}^3, \xi_{\text{st}})$.

For this consider the front projection of the Legendrian knot segment of K and the Legendrian link segments L_i. By the transversality theorem K can be $\mathcal{C}^\infty$-close approximated relative to its boundary by a curve that is transverse to all L_i and represents a Legendrian knot segment in $(\mathbb{R}^3, \xi_{\text{st}})$, which is in $(\mathbb{R}^3, \xi_{\text{st}})$ disjoint from the L_i. □

Remark 6.3 On the other hand, Lemma 6.2 does not hold true for arbitrary surgery coefficients, for a concrete example see Example 4.7.2 in [Ke17]. In particular, it follows that, in general, a single contact r-surgery cannot be reversed by a single contact surgery.

We get the following application for contact surgery numbers.

Proposition 6.4. *If $cs_{1/\mathbb{Z}}(M, \xi) \leq k$ and if we can obtain another contact manifold (N, η) by contact $(\pm 1/n_i)$-surgeries along an l component Legendrian link L in (M, ξ). Then*

$$cs_{1/\mathbb{Z}}(N, \eta) \leq k + l.$$

Proof. Let J be a k-component Legendrian link in (S^3, ξ_{st}) along which contact $(\pm 1/n_i)$-surgery produces (M, ξ). By Lemma 6.2 we can present L in the exterior of J. Thus we have constructed a contact $(\pm 1/n_i)$-surgery diagram for (N, η) along a $(k + l)$-component link. □

6.1. **The 3-sphere.** We recall from the introduction, that on S^3 there is a unique tight (and in fact Stein fillable) contact structure ξ_{st} [El92], which is the unique contact manifold with vanishing contact surgery number. The overtwisted contact structures on S^3 are classified by their d_3-invariants that take integer values, [Go98]. We denote the unique overtwisted contact structure on S^3 with d_3-invariant equal to n by ξ_n.

We begin by showing that any contact structure on S^3 can be obtained from (S^3, ξ_{st}) by at most two contact (± 1)-surgeries. This improves a result from [DGS04], where they could obtain an upper bound of 3. In Section 7.1 we will actually compute all contact surgery numbers of all contact structures on S^3 and in particular, we will see that the inequality in Lemma 6.1 is not always an equality.

Proposition 6.5. *For every contact structure ξ on S^3 we have $\mathrm{cs}_{\pm 1}(S^3, \xi) \leq 2$.*

Proof. For the contact structures with odd d_3-invariant, we use a construction due to Ding, Geiges and Stipsicz [DGS04], which we briefly recall below.

First we consider the contact surgery diagram (i) in Figure 1 which smoothly represents S^3 and via Lemma 4.1 we verify that (i) yields ξ_1. Thus, we have shown that $\mathrm{cs}_{\pm 1}(S^3, \xi_1) = 1$. In Proposition 7.1 we will show that this is the only contact structure on S^3 with that property.

Next, we take an arbitrary Legendrian knot K with Thurston–Bennequin invariant t and rotation number r and we consider $K(+1) \times K_2(+1)$, where K_2 is a

2-fold stabilization of K where both stabilizations are positive, see Figure 12 (i). Since contact $(+1)$-surgery along K_2 is topologically the same as doing contact (-1)-surgery along K, by Lemma 2.1 this surgery yields a contact structure on S^3. Using Lemma 4.1 its d_3-invariant is computed as

$$\mathrm{d}_3 = -(t + r).$$

It is well known that for a knot $\mathrm{tb} + \mathrm{rot}$ is always an odd number [Ge08] and that any odd number can be realized like this and thus we have shown that any contact structure on S^3 with odd d_3-invariant has $\mathrm{cs}_{\pm 1} \leq 2$.

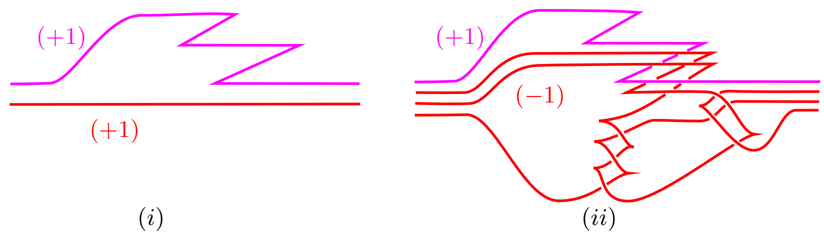

FIGURE 12. Contact (± 1)-surgery diagrams along 2-component links for all overtwisted contact structures on S^3. The construction in (i) yields odd d_3-invariants while (ii) yields even d_3-invariants. Figure (ii) is obtained by first performing Legendrian Reidemeister I moves to $\pm\Delta$ and K and then taking the appropriate connected sum.

For example, the red knot in Figure 13 (i) has $\mathrm{tb} = 2n - 1$ and $\mathrm{rot} = 0$, where $2n + 1$ are the number of self-crossings. Thus, the surgery diagram (i) yields ξ_{1-2n}. All positive odd d_3-invariants can be realized by the same construction starting with a Legendrian unknot with $\mathrm{tb} = -2n - 1$ and $\mathrm{rot} = 0$ for K.

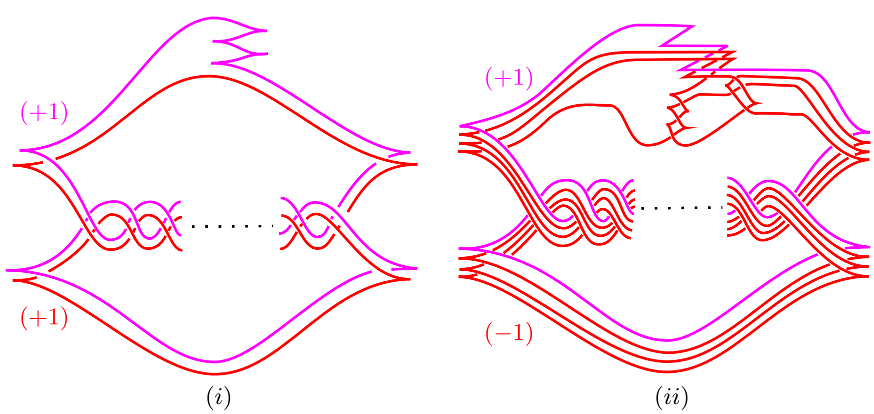

FIGURE 13. Figure (i) represents (S^3, ξ_{1-2n}) and Figure (ii) yields (S^3, ξ_{-2n}), where the purple knot has $2n + 1$ self-crossings. The same construction with the unknot yields positive d_3-invariants.

For the contact structures with even d_3-invariants we proceed as follows. Let K be an arbitrary Legendrian knot with Thurston–Bennequin invariant t and rotation number r. Let K_2 be a twice positively stabilized push-off of K. We perform a contact $(+1)$-surgery along K_2 followed by a contact (-1)-surgery along

$$L = K\#\Delta\# - \Delta,$$

where Δ denotes the boundary of an overtwisted disk in $K_2(+1)$. (Recall if one performs a contact $(+1)$-surgery on a stabilized knot then the original knot bounds an overtwisted disk.) A local picture is shown in Figure 12 (ii) and an example is depicted in Figure 13 (ii).

By construction the surgery diagram yields a contact structure on S^3. Indeed, topologically L is isotopic to K in $K_2(+1)$, since L is obtained from K by connected summing two unknots. In the surgered manifold $K_2(+1)$ the (-1)-framing of L agrees with the $(+1)$-framing of K and thus the Cancellation Lemma implies that the underlying smooth manifold is S^3.

Next, it is straightforward to compute in (S^3, ξ_{st}) that $\mathrm{tb}(K_2) = t-2$, $\mathrm{rot}(K_2) = r+2$, $\mathrm{tb}(L) = t+2$, $\mathrm{rot}(L) = r$, and $\mathrm{lk}(K_2, L) = t$. Thus the linking matrix is

$$Q = \begin{pmatrix} t-1 & t \\ t & t+1 \end{pmatrix}.$$

And with Lemma 4.1 we compute the d_3-invariant to be

$$\mathrm{d}_3 = -(t + r + 1).$$

Since, $t + r$ can be any odd number, the claim follows. □

Similarly, we get upper bounds on the U-versions of contact surgery numbers.

Proposition 6.6. *For the unknot contact surgery numbers of the overtwisted contact structures of S^3 the following upper bounds hold:*

$$\mathrm{cs}_{1/\mathbb{Z},U}(S^3, \xi_n) \leq \begin{cases} 2 & \text{for } n \in 2\mathbb{Z}+1, \\ 3 & \text{for } n \in 2\mathbb{Z}, \end{cases}$$

$$\mathrm{cs}_{\pm 1,U}(S^3, \xi_n) \leq \begin{cases} 2 & \text{for } n \in 2\mathbb{N}+1, \\ 3 & \text{for } n \in 2\mathbb{N}, \\ 1 + \frac{|n-1|}{2} & \text{for } n \in -2\mathbb{N}+1, \\ 2 + \frac{|n|}{2} & \text{for } n \in -2\mathbb{N}. \end{cases}$$

Proof. First we consider the contact surgery diagram from Figure 14. By comput-

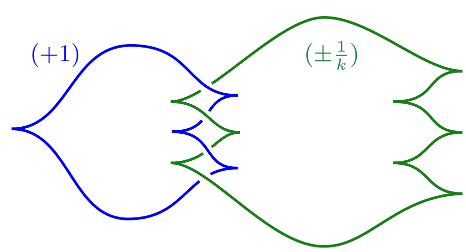

FIGURE 14. A contact surgery diagram of $(S^3, \xi_{1\pm 2k})$.

ing its d_3-invariant we see that it represents $(S^3, \xi_{1\pm 2k})$ and thus we get the first inequality on $\mathrm{cs}_{1/\mathbb{Z},U}(S^3, \xi_n)$. The other inequality is obtained via Lemma 6.1 by taking connected sums of the above surgery description with (S^3,ξ_1) (and observing via Lemma 4.1 that d_3 behaves additive under contact connected sum).

The proof of Proposition 6.5 gives the first inequality for $\mathrm{cs}_{\pm1,U}(S^3,\xi_n)$ and the second follows from connected sums with ξ_1 again. The last two inequalities follow from using the Replacement Lemma 2.2 on the surgery diagrams used to obtain the $\mathrm{cs}_{1/\mathbb{Z},U}(S^3,\xi_n)$ bounds. □

For the Legendrian surgery numbers (where only one contact $(+1)$-surgery is allowed) we have the following general upper bound.

Proposition 6.7. *For every contact structure ξ on S^3 we have* $\mathrm{cs}_{L,\pm1}(S^3,\xi)\leq 3$.

Proof. If the d_3-invariant of ξ is even, the surgery diagram from Figure 12 (ii) shows that $\mathrm{cs}_{L,\pm1}(S^3,\xi)\leq 2$. For an overtwisted contact structure ξ_{2n-1} with odd d_3-invariants we can perform a single contact $(+1)$-surgery along a Legendrian unknot with $\mathrm{tb}=-2$ in a Darboux ball in (S^3,ξ_{2n-1}) to obtain (S^3,ξ_{2n}). By the Cancellation Lemma, we find a Legendrian knot K in (S^3,ξ_{2n}) such that Legendrian surgery along K produces (S^3,ξ_{2n-1}) and thus $\mathrm{cs}_{L,\pm1}(S^3,\xi_{2n-1})\leq 3$. □

6.2. **Overtwisted contact structures.** Next, we describe some general upper bounds on contact surgery numbers of overtwisted contact structures.

Proposition 6.8. *Let (M,ξ) be an overtwisted contact manifold. Then*

$$\begin{aligned}
\mathrm{cs}_{\pm1}(M,\xi)&\leq \mathrm{s}_{\mathbb{Z}}(M)+2,\\
\mathrm{cs}_{L,\pm1}(M,\xi)&\leq \mathrm{s}_{\mathbb{Z}}(M)+3,\\
\mathrm{cs}(M,\xi)&\leq \mathrm{s}(M)+2,\\
\mathrm{cs}_{\mathbb{Z},U}(M,\xi)&\leq \mathrm{s}_{\mathbb{Z}}(M)+2,\\
\mathrm{cs}_{1/\mathbb{Z},U}(M,\xi)&\leq \mathrm{s}_{\mathbb{Z}}(M)+2,\\
\mathrm{cs}_{U}(M,\xi)&\leq \mathrm{s}_{U}(M)+2.
\end{aligned}$$

Proof. We start by proving the first inequality. Let (M,ξ) be an overtwisted contact manifold and let K be a smooth link of $\mathrm{s}_{\mathbb{Z}}(M)$-components in M that admits an integral S^3-surgery. Since (M,ξ) is overtwisted, we can change the contact framing of a loose Legendrian link in (M,ξ) arbitrarily without changing the underlying smooth knot type by stabilizing and performing connected sums with overtwisted disks. Thus, we can choose a loose Legendrian representative L of K in (M,ξ) such that contact (±1)-surgery yields a contact structure on S^3.

By Proposition 6.5 we can reach (S^3,ξ_{st}) by performing at most 2 more contact (±1)-surgeries. Since we can reverse a contact (±1)-surgery, we have constructed a Legendrian link in (S^3,ξ_{st}) with at most $(\mathrm{s}_{\mathbb{Z}}(M)+2)$-components that admits a contact (±1)-surgery to (M,ξ).

The second inequality follows along the same lines. Here we choose a loose Legendrian representative L of K in (M,ξ) such that contact $(+1)$-surgery yields a contact structure on S^3. By the Cancellation Lemma, we can obtain (M,ξ) by $\mathrm{s}_{\mathbb{Z}}(M)$-many Legendrian surgeries from an overtwisted contact structure on S^3. Thus Proposition 6.7 implies the claimed bound.

The third inequality, is more complicated, since the surgeries might not be integral and then we cannot reverse contact surgeries. Here the idea is to start with a topological surgery diagram of M with minimal number of components and deform it into contact surgery diagrams inducing any given $spin^c$ structure. By taking the connected sum with the contact structures of S^3 we get any overtwisted contact structure with that *spin*^{c} structure.

Let K in S^3 be a link giving a topological surgery diagram of M realizing $\mathrm{s}(M)$ and take a Legendrian realization L in (S^3, ξ_{st}) of K. If necessary we stabilize the components of L such that all contact surgery coefficients yielding M are larger than 1. We have constructed a rational contact surgery diagram of some contact structure ξ on M along a Legendrian link with $\mathrm{s}(M)$-components.

We fix a *spin* structure $\mathfrak{s}$ on M. Now Theorem 4.19 implies that for any given first homology class $c \in H_1(M)$ there exist a contact structure ξ_c on M which is obtained by rational surgery along a stabilization of L such that $\Gamma(\xi_c, \mathfrak{s}) = c$ and in particular we get any possible *spin*^{c} structure on M by a rational surgery along L. By taking connected sums with (S^3, ξ_n) we get surgery diagrams of all overtwisted contact structures on M by adding at most 2 more Legendrian knots to the surgery link.

The remaining three inequalities follow on exactly the same lines. Where we only need to observe that we can can always assume the knot types of the surgery knots to be unknots if we started with a smooth surgery link K that consisted only of unknots. □

6.3. **Tight contact structures.** The case of tight contact structures can now be reduced to the overtwisted contact structures.

Theorem 6.9. *Let (M, ξ) be a contact manifold. Then*

$$\begin{aligned} \mathrm{cs}_{\pm 1}(M, \xi) &\leq \mathrm{s}_{\mathbb{Z}}(M) + 3, \\ \mathrm{cs}_{L,\pm 1}(M, \xi) &\leq \mathrm{s}_{\mathbb{Z}}(M) + 4, \\ \mathrm{cs}(M, \xi) &\leq \mathrm{s}(M) + 3, \\ \mathrm{cs}_{\mathbb{Z},U}(M, \xi) &\leq \mathrm{s}_{\mathbb{Z}}(M) + 3, \\ \mathrm{cs}_{1/\mathbb{Z},U}(M, \xi) &\leq \mathrm{s}_{\mathbb{Z}}(M) + 3, \\ \mathrm{cs}_{U}(M, \xi) &\leq \mathrm{s}_{U}(M) + 3. \end{aligned}$$

Proof. Let (M, ξ) be a contact manifold. If ξ is overtwisted Proposition 6.8 implies the result. If ξ is tight we can perform a contact $(+1)$-surgery along a Legendrian unknot U in (M, ξ) with $\mathrm{tb}(U) = -2$ to get an overtwisted contact structure on M. Since we can reverse a contact $(+1)$-surgery (and the surgery dual knot of U is again an unknot) the claimed inequalities follow from Proposition 6.8. □

Remark 6.10 In all examples that we had considered the above proof could be improved. Indeed, in many cases (see for example the case of the Poincaré homology sphere or the 3-torus in Sections 7.2 and 7.6) we could identify a Legendrian knot K in (M, ξ) such that contact (± 1)-surgery along K yields a contact manifold $K(+1)$ with $\mathrm{cs}_{\pm 1}(K(+1)) = \mathrm{s}(M) - 1$ and thus Proposition 6.8 yields

$$\mathrm{cs}_{\pm 1}(M, \xi) \leq s(M) + 2.$$

In fact, we do not know a single example of a contact manifold with

$$\mathrm{cs}_{\pm 1}(M, \xi) - s(M) = 3,$$

cf. Question 1.3. However, Proposition 6.8 implies that such a manifold has to be tight.

7. Computations of contact surgery numbers

In this section, we explicitly compute contact surgery numbers for contact structures ξ on some special manifolds.

7.1. S^3 – integer surgeries.

In this section, we compute integer contact surgery numbers of all contact structures on S^3. We start with contact (± 1)-surgeries.

Proposition 7.1. *The overtwisted contact structure ξ_1 is the unique contact structure on S^3 with $\mathrm{cs}_{\pm 1} = 1$. Any other overtwisted contact structure on S^3 has $\mathrm{cs}_{\pm 1} = 2$.*

Proof. Let $L(\pm 1)$ be a contact surgery diagram of a contact structure on S^3 along a single Legendrian knot L. By a result of Gordon and Luecke [GL89] L has to be a Legendrian unknot and the topological surgery coefficient has to be of the form $1/k$, for some integer $k \in \mathbb{Z}$. Moreover, Legendrian unknots are completely classified by Eliashberg–Fraser [EF09]: Every Legendrian unknot is a stabilization of the unique Legendrian unknot U with $\mathrm{tb} = -1$ and $\mathrm{rot} = 0$. It follows that L is a Legendrian unknot with Thurston–Bennequin invariant $t \leq -1$. Thus the topological surgery coefficient corresponding to the contact (± 1)-surgery along L is $t \pm 1$, which should be of the form $1/k$. The only solution for this equation is $t = -2$, $k = -1$ and the sign of ± 1 has to be $+1$. But there is a unique (unoriented) Legendrian knot with $\mathrm{tb} = -2$ and in Section 6.1 we have seen that contact $(+1)$-surgery along it produces ξ_1. All the other overtwisted contact structures on S^3 have $\mathrm{cs}_{\pm 1} \geq 2$. The proposition now follows from the upper bounds in Proposition 6.5. □

From the proof of Theorem 7.1 we can directly conclude that ξ_1 has a unique contact (± 1)-surgery diagram along a single Legendrian knot.

Corollary 7.2. *If (S^3, ξ_1) is obtained by a single contact (± 1)-surgery along a Legendrian knot K in (S^3, ξ_{st}), then K has to be the (unoriented) Legendrian unknot with Thurston–Bennequin invariant $\mathrm{tb} = -2$ and rotation number $|\mathrm{rot}| = 1$ and the contact surgery coefficient has to be $+1$.*

Remark 7.3 The case of contact $(\pm 1/n)$-surgery works exactly the same and also yields the same result: ξ_1 is the unique contact structure on S^3 with $\mathrm{cs}_{1/\mathbb{Z}} = 1$ and all other overtwisted contact structures on S^3 have $\mathrm{cs}_{1/\mathbb{Z}} = 2$. However, in this case ξ_1 does not have a unique contact $(1/n)$-surgery diagram along a single Legendrian knot anymore, it has exactly two diagrams. Indeed, contact $(1/2)$-surgery along the Legendrian unknot with $\mathrm{tb} = -1$ also produces ξ_1.

The cases of integer (and later in Section 7.4 rational) contact surgeries are more evolved.

Theorem 7.4. *An overtwisted contact structure on S^3 has $\mathrm{cs}_{\mathbb{Z}} = 1$ if and only if its d_3-invariant is of the form*

$$l(1+l) \ \text{or} \ m(1-m)+1 \tag{15}$$

where m and l are arbitrary integers with $l \geq 1$ and $m \geq 0$. All other overtwisted contact structures on S^3 have $\mathrm{cs}_{\mathbb{Z}} = 2$.

Example 7.5 As a concrete example we have $\mathrm{cs}_{\pm 1}(\xi_0) = \mathrm{cs}_{\mathbb{Z}}(\xi_0) = 2$. In other words, we cannot obtain (S^3, ξ_0) by a single integer contact surgery along a Legendrian knot in (S^3, ξ_{st}). In Section 7.4 we will show that this is also not possible via a single rational contact surgery.

Proof. Let $L(k)$, for $k \in \mathbb{Z}\setminus\{0\}$, be an integer contact surgery diagram of a contact structure on S^3 along a single Legendrian knot L. As in the proof of Proposition 7.1 we conclude that L has to be a Legendrian unknot with Thurston–Bennequin invariant $t \leq -1$ and $k = \pm 1 + t$. Since every Legendrian unknot is a stabilization of the unique Legendrian unknot with $\operatorname{tb} = -1$ and $\operatorname{rot} = 0$, we can apply Corollaries 4.6 and 4.7 and get the claimed values for the d_3-invariants.

Note, that Corollary 4.6 indeed yields d_3-invariants of the form $l(1+l)$ with $l \geq 0$. However, by Lemma 3.4 we see that the case $l = 0$ always yields ξ_{st} and never yields ξ_0.

The second part of the theorem follows from Proposition 6.5. □

We get some partial results on the Legendrian contact surgery numbers of contact structures on S^3.

Theorem 7.6. *A contact structure on S^3 has $\operatorname{cs}_L = 1$ if and only if it is isotopic to ξ_1. Moreover, there exist infinite families of contact structures on S^3 with Legendrian contact surgery number equal to two. As concrete examples we have*

(1) $\operatorname{cs}_{L,\pm 1}(S^3, \xi_{2k}) = 2$ *for* $k \in \mathbb{Z}$, *and*
(2) $\operatorname{cs}_L(S^3, \xi_{1-2k}) = 2$ *for* $k \in \mathbb{N}$.

Proof. By Proposition 7.1 we know that the only contact structure on S^3 that can be obtained by a single contact (± 1)-surgery is ξ_1. A contact surgery diagram of (S^3, ξ_1) is contact $(+1)$-surgery along the Legendrian unknot with $\operatorname{tb} = -2$ and thus the first claim follows. Item (1) follows from the prof of Proposition 6.5, while Item (2) follows from the proof of Proposition 6.6 and the Replacement Lemma 2.2. □

7.2. **The Poincaré homology sphere.** Next, we study the case of the Poincaré homology sphere P which is the Brieskorn manifold $\Sigma(2,3,5)$. Since it is Seifert fibered with normalized Seifert invariants $(-2; 1/2, 2/3, 4/5)$ we get the surgery diagram of P shown in the middle of Figure 15. By some elementary Kirby calculus we get two simpler surgery diagrams, one along a 3-chain link of unknots and one along the left-handed trefoil knot, shown in the left and right of Figure 15, respectively.

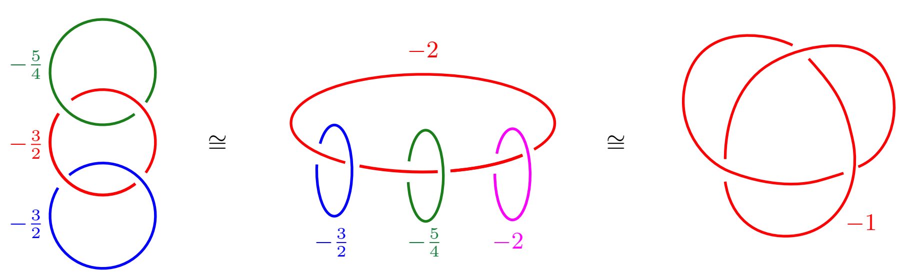

FIGURE 15. Three different surgery descriptions of the Poincaré homology sphere.

Similar to the case of S^3 we give upper bounds on contact surgery numbers by writing down explicit diagrams. For lower bounds we will use that the Poincaré homology sphere has a unique surgery diagram along a single knot which is shown on the right of Figure 15, [Gh08]. The Poincaré homology sphere P has a unique tight (and in fact, Stein fillable) contact structure ξ_{st}^P [Sc01]. Figure 16 shows a Legendrian realization of the left surgery diagram from Figure 15. Since all surgery coefficients are negative the resulting contact structure is Stein fillable and thus represents (P, ξ_{st}^P). (In this surgery diagram it is straightforward to compute $\mathrm{d}_3(\xi_{\text{st}}^P) = 2$.) In particular, we get an explicit upper bound for the contact surgery number of (P, ξ_{st}^P),

$$\mathrm{cs}_{U,1/\mathbb{Z}}(P, \xi_{\text{st}}^P) \leq 3.$$

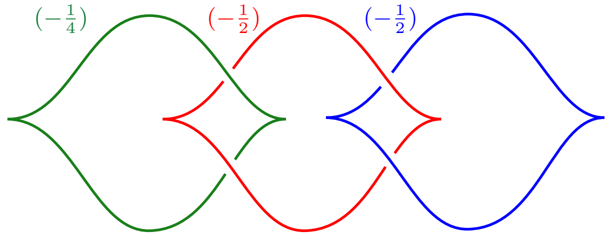

FIGURE 16. A surgery diagram of (P, ξ_{st}^P).

From Figure 16 we also get a contact surgery presentation with only (± 1)-contact surgery coefficients via Lemma 2.2, a diagram with 8 components. However, we can get an even better bound on $\mathrm{cs}_{\pm 1}(P, \xi_{\text{st}}^P)$, although we do not have an explicit description of a surgery diagram realizing this bound.

Proposition 7.7. *The contact surgery number* $\mathrm{cs}_{\pm 1}$ *of* (P, ξ_{st}^P) *can be bounded from above as*

$$\mathrm{cs}_{\pm 1}(P, \xi_{\text{st}}^P) \leq 3.$$

Proof. By [GO20, Theorem 1.5] we know that there exists a Legendrian realization K of the left-handed trefoil with $\mathrm{tb} = 0$ in (S^3, ξ_2), such that contact (-1)-surgery along K produces a tight contact structure, see Figure 5 of [GO20] for an explicit diagram. Since $\mathrm{tb}(L) = 0$ we know that contact (-1)-surgery along K produces topologically P. Since there is a unique tight contact structure on P, it follows that contact (-1)-surgery along K yields (P, ξ_{st}^P). By Propositions 6.4 and 6.5 we have

$$\mathrm{cs}_{\pm 1}(P, \xi_{\text{st}}^P) \leq \mathrm{cs}_{\pm 1}(S^3, \xi_2) + 1 = 3.$$

Note, that although $\mathrm{cs}_{\mathbb{Z}}(S^3, \xi_2) = 1$, this does not imply that $\mathrm{cs}_{\mathbb{Z}}(P, \xi_{\text{st}}^P) \leq 2$. □

Next, we turn our attention to the overtwisted contact structures on the Poincaré homology sphere P. Since P is a homology sphere the overtwisted contact structures are completely classified by their d_3-invariants taking values in $\mathbb{Z}$. Denote the unique overtwisted contact structure on P with $\mathrm{d}_3 = n$ by ξ_n. We can get P by integer surgery along the left-handed trefoil and hence by Proposition 6.8

$$\mathrm{cs}_{\pm 1}(P, \xi_n) \leq 3.$$

Now we will discuss how to obtain better lower bounds. We use these lower bounds to deduce that certain infinite families of contact structures on P having contact surgery numbers equal to 2. The strategy for getting lower bounds is similar to S^3. Here we use that P has a unique surgery diagram along a single knot, namely the left-handed trefoil with topological framing -1, [Gh08] and we use that all Legendrian realizations of left-handed trefoils in (S^3, ξ_{st}) are stabilizations of the unique representative with $\mathrm{tb} = -6$ and $|\,\mathrm{rot}\,| = 1$, [EH01].

Theorem 7.8. *A contact structure on P has* $\mathrm{cs} = 1$ *if and only if it has* $\mathrm{cs}_{\mathbb{Z}} = 1$ *if and only if its* d_3*-invariant is of the form*

$$m(3-m)-1 \tag{16}$$

where m is an arbitrary integer with $m \geq 3$.

Moreover, a contact structure ξ_n on P has $\mathrm{cs}_{\mathbb{Z}} = 2$ *if n cannot be written as in Equation (16) but if it can be written as the sum of a number from Equation (16) and a number from Equation (15).*

Before giving the proof of Theorem 7.8, we formulate some examples and corollaries.

Corollary 7.9. *There exist infinitely many contact structures on P with* $\mathrm{cs} = 2$*.*

Example 7.10 As concrete examples we have $\mathrm{cs}(\xi_{-1}) = 1$ and $\mathrm{cs}_{\mathbb{Z}}(\xi_0) = \mathrm{cs}(\xi_0) = 2$. Since the d_3-invariants in Equation (16) are all negative, we also have $\mathrm{cs}(\xi_n) \geq 2$ for all $n \geq 0$.

Corollary 7.11. *The unique tight contact structure ξ_{st}^P on P cannot be obtained by a rationally contact surgery along a single Legendrian knot from (S^3, ξ_{st}). In particular, we know that*

$$2 \leq \mathrm{cs}(P, \xi_{\mathrm{st}}^P) \leq \mathrm{cs}_{1/\mathbb{Z}}(P, \xi_{\mathrm{st}}^P) \leq \mathrm{cs}_{\pm 1}(P, \xi_{\mathrm{st}}^P) \leq 3.$$

However, (P, ξ_{st}^P) can be obtained by a single Legendrian surgery (i.e. contact (-1)-surgery) along a Legendrian knot in an overtwisted contact structure.

Proof. We computed $\mathrm{d}_3(\xi_{\mathrm{st}}^P) = 2$. Thus by Theorem 7.8 (P, ξ_{st}^P) cannot be obtained by a single contact surgery from a Legendrian knot in (S^3, ξ_{st}). The upper bound is obtained in Proposition 7.7 and the second part of the corollary follows directly from the proof of Proposition 7.7. □

Proof of Theorem 7.8. Let $L(r)$, for $r \in \mathbb{Q} \setminus \{0\}$, be a contact surgery diagram of a contact structure on P along a single Legendrian knot L. From the result of Ghiggini [Gh08] we conclude that L is a Legendrian left-handed trefoil and the topological surgery coefficient of L has to be -1. Now every Legendrian realization of a left-handed trefoil is a stabilization of a unique representative with $\mathrm{tb} = -6$ and $|\,\mathrm{rot}\,| = 1$ [EH01]. (Observe, that there exists two different *oriented* Legendrian realizations of the left-handed trefoil with maximal tb. But here we consider unoriented knots, since the contactomorphism type of the surgered contact manifold will not depend on the orientation of the knot.) It follows that we can apply Corollary 4.7 and get the claimed values for the d_3-invariants.

The second part of the theorem follows by taking connected sums of the contact structures from the first part and overtwisted contact structures on S^3. □

Remark 7.12 Here is another construction of an infinite family of contact structures on P with $\mathrm{cs} \leq 2$. According to [CN10] there are two maximal Legendrian Whitehead links, with Thurston-Bennequin invariants of the components equal to $(-3, -2)$ respectively $(-4, -1)$. (Note that the Whitehead link admits an isotopy interchanging the two components and therefore we do not need to distinguish the two components.) Since the two components of the Whitehead link are algebraically unlinked the computation of the d_3 invariants of a contact structure obtained by contact surgery along a Legendrian Whitehead link is the same as the d_3-invariant of the contact structure obtained by contact surgery along a Legendrian two-component unlink with the same classical invariants and contact surgery coefficients. Then we notice that we can obtain P by topological (-1)-surgery along both components of the Whitehead link. Thus there are an infinite family of contact surgeries on Legendrian realizations of the Whitehead link giving contact structures on P with $\mathrm{cs} = 2$ that have the same d_3-invariants as the contact structures on S^3 obtained by the corresponding contact surgeries along the Legendrian two-component unlink. By Theorem 7.4 the resulting values for the d_3-invariants are

$$m(1-m) + n(1-n) + 2,$$

for $m \geq 0$ and $n \geq 3$ or $m \geq 1$ and $n \geq 2$.

7.3. **The Brieskorn sphere $\Sigma(2,3,7)$.** According to [To20] the Brieskorn homology sphere $\Sigma(2,3,7)$ has a unique tight (and in fact Stein fillable) contact structure which we denote by $\xi_{\mathrm{st}}^{\Sigma}$. Moreover, denote the unique overtwisted contact structure on $\Sigma(2,3,7)$ with d_3-invariant equal to n by ξ_n. Then we have the following result about contact surgery numbers of contact structures on $\Sigma(2,3,7)$.

Theorem 7.13. *The unique tight contact structure $\xi_{\mathrm{st}}^{\Sigma}$ on $\Sigma(2,3,7)$ can be obtained by a single Legendrian surgery along a right-handed Legendrian trefoil and thus* $\mathrm{cs}_{\pm 1}(\Sigma(2,3,7), \xi_{\mathrm{st}}) = 1$.

An overtwisted contact structure on $\Sigma(2,3,7)$ has $\mathrm{cs} = 1$ if and only if it has $\mathrm{cs}_{\mathbb{Z}} = 1$ if and only if its d_3-invariant is of the form

$$l(3-l) - 1 \;\; \text{or} \;\; m(m-1) \tag{17}$$

where m, l are arbitrary integers with $l \geq 0$ and $m \geq 2$.

Moreover, a contact structure ξ_n on $\Sigma(2,3,7)$ has $\mathrm{cs}_{\mathbb{Z}} = 2$ if n cannot be written as in Equation (17) but if it can be written as the sum of a number in Equation (17) and a number from Equation (15).

Finally, we have $\mathrm{cs}_{\mathbb{Z}}(\Sigma(2,3,7), \xi) \leq 3$ for any contact structure on $\Sigma(2,3,7)$.

Proof. By the work of Ozváth and Szabó [OS19] we know that there are exactly two ways to get $\Sigma(2,3,7)$ by surgery along a single knot: $(+1)$-surgery along the figure eight knot and (-1)-surgery along the right-handed trefoil knot (both slopes measured with respect to the Seifert framing).

Now let $L(r)$, for $r \in \mathbb{Q} \setminus \{0\}$, be a contact surgery description of a contact structure on $\Sigma(2,3,7)$ along a single Legendrian knot L. Then we know that L has to be a Legendrian realization of the figure eight knot or the right-handed trefoil with the above topological surgery coefficients. Again by the work of Etnyre and Honda [EH01] we know that every Legendrian realization of the figure eight knot is a stabilization of the unique representative with $\mathrm{tb} = -3$ and $\mathrm{rot} = 0$. From Corollary 4.6 we get the second term in Equation (17).

Similarly, we know from [EH01] that every Legendrian realization of a right-handed trefoil is a stabilization of the unique representative with $\mathrm{tb} = +1$ and $\mathrm{rot} = 0$ and thus we get from Corollary 4.7 the first term in Equation (17).

However, in this case that the maximal Thurston–Bennequin invariant is equal to 1, additionally we have negative surgeries that we need to consider: the contact (-2)-surgery along the Legendrian right-handed trefoil with $\mathrm{tb} = 1$ and $\mathrm{rot} = 0$ (which is equivalent to contact (-1)-surgery along the Legendrian right-handed trefoil with $\mathrm{tb} = 0$ and $|\mathrm{rot}| = 1$). Note that negative contact surgeries preserve tightness and symplectic fillability, and thus this surgery yields the unique tight contact structure $\xi_{\mathrm{st}}^{\Sigma}$ on $\Sigma(2,3,7)$. This proves the statements about the contact surgery number of $\xi_{\mathrm{st}}^{\Sigma}$.

The last step in order to classify the contact structures with contact surgery number equals to 1, we need to argue that all of the other contact surgeries yield an overtwisted contact structure. This can be seen by computing the d_3-invariant of $\xi_{\mathrm{st}}^{\Sigma}$ to be 0 (either with Corollary 4.7 or directly with Lemma 4.1). Note that all the d_3-invariants in Equation (17) are odd or positive and thus each correspond to overtwisted contact structures.

The statement about the contact structures with contact surgery numbers equal to two, follows again by the connected sum as in the proof of Theorem 7.8. The general upper bounds is Proposition 6.8. □

The proof of Theorem 7.13 implies the following results.

Corollary 7.14. *$(\Sigma(2,3,7),\xi_{\mathrm{st}}^{\Sigma})$ cannot be obtained by rational contact surgery along a Legendrian realization of the figure eight knot.*

Corollary 7.15. *$(\Sigma(2,3,7),\xi_{\mathrm{st}}^{\Sigma})$ has a unique Legendrian surgery description along a single Legendrian knot, the contact (-1)-surgery along the unique (unoriented) Legendrian right-handed trefoil knot with $\mathrm{tb}=0$ and $|\operatorname{rot}|=1$.*

7.4. S^3 – **rational surgery.** With the same strategy as in Section 7.1 we obtain results for rational contact surgery numbers of contact structures on S^3.

Theorem 7.16. *An overtwisted contact structure on S^3 has $\mathrm{cs}=1$ if and only if its d_3-invariant is of the form*

$$k(q+qk-2z) \tag{18}$$

for $q\geq 1$, $k\geq 1$ and $z=0,1,\dots q-1$, or

$$qk(k+1)+2k+1 \tag{19}$$

for $q\leq -1$, $k\geq 0$, or

$$qk(k-1)+1 \tag{20}$$

for $q\leq -1$, $k\geq 0$.

All other overtwisted contact structures on S^3 have $\mathrm{cs}_{\mathbb{Z}}=2$.

Proof. Let $L(r)$, for $r\in\mathbb{Q}\setminus\{0\}$, be a rational contact surgery diagram of a contact structure on S^3 along a single Legendrian knot L. From [GL89] we conclude that L has to be a Legendrian unknot with Thurston–Bennequin invariant $t\leq -1$ and contact surgery coefficient $r=1/q-t$. Since every Legendrian unknot is a stabilization of the unique Legendrian unknot with $\mathrm{tb}=-1$ and $\mathrm{rot}=0$, we can get every negative odd integer as $\mathrm{tb}\pm\mathrm{rot}$ of a Legendrian unknot. We write $\mathrm{tb}+\mathrm{rot}=-1-2k$, for $k\geq 0$ and we can directly apply the results from Section 4.2 and get the claimed values for the d_3-invariants.

We check directly that the only possibility to obtain a contact structure with $\mathrm{d}_3=0$ is by setting $k=0$ in Equation (18), but in this case it follows from Lemma 3.4 that the resulting contact structure is always ξ_{st}.

The second part of the theorem follows from Proposition 6.5. □

Corollary 7.17. *There exist infinitely many non-isotopic contact structures on S^3 which cannot be obtained by a single rationally contact surgery from (S^3,ξ_{st}). As a concrete example we see that $\mathrm{cs}(S^3,\xi_0)=2$.*

As a direct corollary of the proof we recover a result from [Ke17, Ke18], which implies that Legendrian knots in (S^3, ξ_{st}) are determined by their exteriors.

Corollary 7.18. *The only Legendrian knots along which we can obtain* (S^3, ξ_{st}) *are Legendrian unknots with extremal rotation number, i.e.* $|\operatorname{rot}| = -\operatorname{tb} - 1$.

7.5. **Contact structures on $S^1 \times S^2$.** We know that $S^1 \times S^2$ has a unique tight contact structure ξ_{st}. All remaining contact structures on $S^1 \times S^2$ are overtwisted and by Eliashberg's classification of overtwisted contact structures they only depend on the algebraic topology of the underlying 2-plane field. Since $H_1(S^1 \times S^2) \cong \mathbb{Z}$ does not contain 2-torsion, two plane fields correspond to the same *spin*^{c} structure if and only if they have the same Euler class (which can be any even element in $H_1(S^1 \times S^2) \cong \mathbb{Z}$). We can get all 2-plane fields in a fixed *spin*^{c} structure by connected summing the overtwisted contact structures on S^3. The first observation is that ξ_{st} is the unique contact structure on $S^1 \times S^2$ with $\operatorname{cs}_{\pm 1} = 1$.

Proposition 7.19. *A contact structure* ξ *on* $S^1 \times S^2$ *has* $\operatorname{cs}_{\pm 1}(S^1 \times S^2, \xi) = 1$ *if and only if* $(S^1 \times S^2, \xi)$ *is contactomorphic to* $(S^1 \times S^2, \xi_{st})$. *Moreover, the contact* $(+1)$*-surgery along the Legendrian unknot with* $\operatorname{tb} = -1$ *and* $\operatorname{rot} = 0$ *is the unique contact* (± 1)*-surgery diagram of* $(S^1 \times S^2, \xi_{st})$ *along a single Legendrian knot in* (S^3, ξ_{st}).

Proof. It is well-known that $(S^1 \times S^2, \xi_{st})$ can be obtained by a single contact $(+1)$-surgery along a Legendrian unknot with $\operatorname{tb} = -1$ and thus $\operatorname{cs}_{\pm 1}(S^1 \times S^2, \xi_{st}) = 1$.

Conversely, let ξ be a contact structure on $S^1 \times S^2$ which can be obtained by a single contact ± 1 surgery along a Legendrian knot K in (S^3, ξ_{st}). By the work of Gabai [Ga87] $S^1 \times S^2$ has a unique surgery diagram along a single knot, namely the unknot with topological surgery slope 0. It follows that K is a Legendrian unknot with contact framing $-\operatorname{tb}(K)$. From the classification of Legendrian unknots we conclude that K has to be the Legendrian unknot with $\operatorname{tb}(K) = -1$ and the contact framing is $+1$ and thus $\xi = \xi_{st}$. □

Next we consider an overtwisted contact structure ξ on $S^1 \times S^2$. From Proposition 6.8 we get general upper bounds for overtwisted contact structures:

$$\operatorname{cs}_{\pm 1}(S^1 \times S^2, \xi) \leq 3$$

And similarly we can deduce general upper bounds of the U-versions of contact surgery numbers.

Since surgery along the unknot with topological surgery coefficient 0 is the unique surgery diagram of $S^1 \times S^2$ along a single knot we can again classify all contact structures on $S^1 \times S^2$ with $\operatorname{cs} = 1$.

Theorem 7.20. *There exists exactly one contact structure in every spin*^{c} *structure of* $S^1 \times S^2$ *which can be obtained by a contact surgery along a single Legendrian knot in* (S^3, ξ_{st}).

In particular, no overtwisted contact structure with trivial Euler class can be obtained by surgery along a single Legendrian knot in (S^3, ξ_{st}).

Proof. We will perform the same strategy as in the preceding subsections for S^3 and the other homology spheres. However, the difficulty here is, that $S^1 \times S^2$ is not a homology sphere and the d_3-invariant is not enough to understand the algebraic topology of 2-plane fields on $S^1 \times S^2$. In fact the d_3-invariant is only well defined when the Euler class of the contact structure is a torsion element.

To classify all contact structures on $S^1 \times S^2$ with $\mathrm{cs} = 1$ we will proceed as follows. By the above mentioned result of Gabai we know that the $\mathrm{cs} = 1$ contact structures on $S^1 \times S^2$ are exactly those which can be obtained by contact $(-t)$-surgery along a Legendrian unknot U with Thurston–Bennequin invariant t and rotation number r. Since $H^2(S^1 \times S^2) \cong \mathbb{Z}$ has no 2-torsion we know that a *spin*^{c} structure on $S^1 \times S^2$ is uniquely determined by its Euler class. We choose the explicit identification of $H^2(S^1 \times S^2)$ with $\mathbb{Z}$ by sending the Poincaré dual of the meridian μ_U of U to $1 \in \mathbb{Z}$.

Then we can describe via Proposition 4.13 the Euler classes of the contact $(-t)$-surgeries along U as

$$\mathrm{e}\left(U(-t)\right) = \mathrm{e}\left(U(+1) \times U_1\left(-\frac{1}{-t-1}\right)\right) = t \pm r + 1.$$

With Lemma 2.9, we directly see that all contact structures on $S^1 \times S^2$ with the same Euler class arising in this manner are contactomorphic. See Figure 17 for a depiction of the results. Note that the Euler classes of the contact structures on the left of Figure 17 differ from those of the right only by a sign. However, the induced contact structures are contactomorphic as can be seen by reversing the orientation of all surgery curves in one diagram (which does not change the contactomorphism type of the surgered contact manifold). On the other hand, since in all cases we have a surgery along a single unknot, the contact structures on the left are not isotopic to the contact structures on the right.

Finally, we observe that contact $(+1)$-surgery along the Legendrian unknot with $\mathrm{tb} = -1$ yields the unique tight contact structure on $S^1 \times S^2$ and thus we cannot obtain an overtwisted contact structure on $S^1 \times S^2$ with trivial Euler class by a contact surgery along a single Legendrian knot from (S^3, ξ_{st}). $\square$

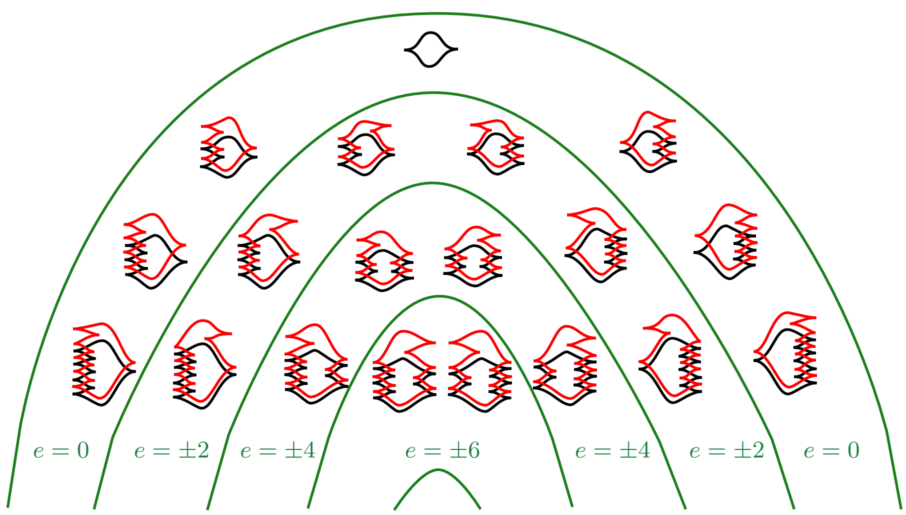

FIGURE 17. All contact structures on $S^1 \times S^2$ with $\mathrm{cs} = 1$ arise by contact $(+1)$ surgery along the black Legendrian unknots followed by a contact $(-\frac{1}{-t-1})$-contact surgery along the red knot, where all knots are oriented in the same direction. All contact structures in the regions bounded by the green arcs are contactomorphic with the indicated Euler classes.

As a corollary we get an infinite family of overtwisted contact structures on $S^1 \times S^2$ with contact surgery number two.

Corollary 7.21. *Let ξ be an overtwisted contact structure on $S^1 \times S^2$ with trivial Euler class. Then ξ has* $\mathrm{cs} = 2$ *if its* d_3*-invariant is of the form*

$$k(q + qk + 2z) - \frac{1}{2}$$

for $q \geq 1$, $k \geq 1$ and $z = 0, 1, \ldots q - 1$, or

$$qk(k+1) + 2k + \frac{1}{2}$$

for $q \leq -1$, $k \geq 0$, or

$$qk(k-1) + \frac{1}{2}$$

for $q \leq -1$, $k \geq 0$.

Remark 7.22 Recall, as discussed at the end of the introduction, that our normalization of the d_3-invariant has $d_3(S^1 \times S^2, \xi_{std}) = \frac{1}{2}$.

Proof. We first note that the standard contact structure ξ_{st} on $S^1 \times S^2$ has d_3-invariant equal to $1/2$, since it can be described by contact $(+1)$-surgery along a knot with vanishing rotation number. By Theorem 7.20 we know that any overtwisted contact structure with trivial Euler class has $\mathrm{cs} \geq 2$.

By taking a connected sum of the contact $(+1)$-surgery along the Legendrian unknot with $\mathrm{tb} = -1$ and the contact surgery diagrams of the contact structures on S^3 with $\mathrm{cs} = 1$ from Theorem 7.16 the claim follows. □

Remark 7.23 To describe the contact structures with $\mathrm{cs} = 1$ with non-trivial Euler class in terms of their algebraic topology we can choose trivializations of the exterior of U and the newly glued-in solid torus that fit together to a global trivialization of $S^1 \times S^2$. (This is possible since the topological surgery coefficient is even, cf. [DGGK18].) With respect to that fixed choice of trivialization the Hopf invariant is a complete invariant of tangential 2-plane fields on $S^1 \times S^2$ with the same Euler classes. Then we can explicitly compute the Hopf invariants of all contact structures obtained by contact $(-t)$-surgery along all realizations of Legendrian unknots to describe their algebraic topology.

Alternatively, one could work directly with the finer invariants of tangential 2-plane fields developed by Gompf [Go98] (which do not depend on a chosen trivialization of $S^1 \times S^2$).

7.6. **The 3-torus.** In this subsection we briefly study the 3-torus $T^3 = \mathbb{R}^3/(2\pi\mathbb{Z}^3)$. The tight contact structures on T^3 are classified by Kanda [Ka97] in terms of their **Giroux torsion** $n \in \mathbb{N}$: any positive tight contact structure on T^3 is contactomorphic to exactly one of

$$\xi_n := \ker \cos(n\theta)\, dx - \sin(n\theta)\, dy$$

for $n \in \mathbb{N}$ and Eliashberg showed that only ξ_1 is Stein fillable [El96]. Moreover, it is easy to see that all ξ_n are homotopic as tangential 2-plane fields.

Since $H_1(T^3) = \mathbb{Z}^3$, we know that $\mathrm{s}(T^3) \geq 3$ and since 0-surgeries on the Borromean rings produce T^3 we actually have $\mathrm{s}(T^3) = 3$. Thus Proposition 6.8 implies that any overtwisted contact structure on T^3 can be obtained by at most 5 contact (± 1)-surgeries from (S^3, ξ_{st}). A Kirby diagram of the Stein filling of (T^3, ξ_1) is shown in Figure 18 on the left. By replacing the 1-handles with (+1)-framed Legendrian unknots (see Theorem 4 of [DG09]) we get a contact (±1)-surgery diagram of (T^3, ξ_1) shown in Figure 18 on the right along a Legendrian realization of the Borromean rings and thus

$$\mathrm{cs}_{\pm1,U}(T^3, \xi_1) = 3.$$

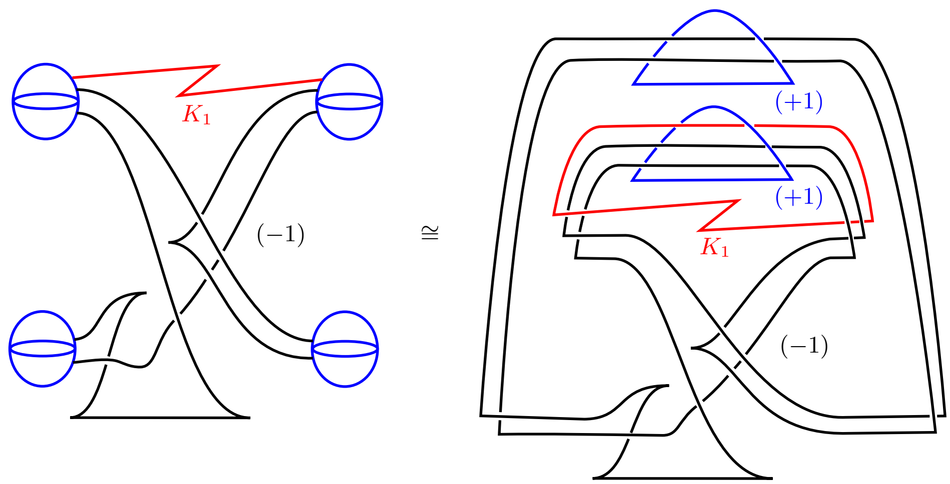

FIGURE 18. Left: A Kirby diagram of the Stein filing of (T^3, ξ_1) together with the Legendrian knot K_1. Right: A contact (±1)-surgery diagram of (T^3, ξ_1).

For the other tight contact structures we can improve the bound from Theorem 6.9 as follows.

Theorem 7.24. *For any tight contact structure ξ on T^3 we have*

$$\mathrm{cs}_{\pm1}(T^3, \xi) \leq 5.$$

The main ingredient in the above proof is the following lemma.

Lemma 7.25. *There is a fixed overtwisted contact structure ξ on $\#_2 S^1 \times S^2$, such that any tight contact 3-torus (T^3, ξ_n) admits a contact (+1)-surgery to $(\#_2 S^1 \times S^2, \xi)$; moreover, ξ has vanishing Euler class and d_3-invariant 1.*

Proof. We consider the Legendrian knot K given by

$$[0, 2\pi] \ni s \longmapsto (\theta(s), x(s), y(s)) = (0, s, 0) \in (T^3, \xi_n).$$

First we observe that K lies on the T^2_θ-fiber of height $\theta = 0$ and thus a topological 0-surgery measured with respect to the framing coming from the T^2-fiber yields $\#_2 S^1 \times S^2$.

Since the contact framing of K agrees with its T^2_0-framing it follows that contact (+1)-surgery along the once stabilized knot K_1 yields a contact structure η_n on $\#_2 S^1 \times S^2$. By Theorem 3.3 η_n is overtwisted.

Since $\#_2 S^1 \times S^2$ has no 2-torsion its $spin^c$ structure is determined by its Euler class and if the Euler class is zero then its the d_3-invariant will determine the isotopy class of η_n.

We start with the case $n = 1$. The Legendrian knot K_1 in (T^3, ξ_1) is shown in the contact surgery diagram of (T^3, ξ_1) in Figure 18. From that description we use Lemma 4.1 to compute the homtopical invariants of η_1 to be

$$\mathrm{e}(\#_2 S^1 \times S^2) = 0,$$
$$\mathrm{d}_3(\#_2 S^1 \times S^2) = 1.$$

Finally, we can obtain ξ_n from ξ_1 by cutting T^3 along the T^2_π-fiber of height $\theta = \pi$ and introduce Giroux torsion. This does not change the homotopy type of the underlying tangential 2-plane field and thus the Euler class and the d_3-invariant of ξ_n is the same as for ξ_1. Since the contact $(+1)$-surgery along K_1 happens in a region where the underlying tangential 2-plane field was not changed, the Euler class of η_n and the d_3-invariant (seen as a Hopf invariant with the appropriate trivializations of T^3 and $\#_2 S^1 \times S^2$) is the same as for η_1. □

Proof of Theorem 7.24. By Lemma 7.25 there exists a contact $(+1)$-surgery from (T^3, ξ_n) to an overtwisted contact structure on $\#_2 S^1 \times S^2$ and the latter can be obtained by at most 4 contact (± 1)-surgeries from (S^3, ξ_{st}) by Proposition 6.8. □

Remark 7.26 The surgery dual knots L_n in $(\#_2 S^1 \times S^2, \xi)$ from Lemma 7.25 are all smoothly equivalent with the same classical invariants. Moreover, they are strongly exceptional, since their contact (-1)-surgeries yield tight manifolds with Giroux torsion in their complements. Since these tight manifolds are all different the L_n are pairwise non-equivalent.

However, we do not have explicit contact surgery diagrams of the tight contact structures on T^3 or the non-loose Legendrian knots L_n in $(\#_2 S^1 \times S^2, \xi)$. But due to the work of Van Horn-Morris [VHM07] we know compatible open book decompositions. From that open book decompositions we can in principle construct contact surgery diagrams which will however have in general more than 5-surgery curves.

7.7. **Lens spaces.** The lens space $L(p, q)$ is defined to be the result of $(-p/q)$-surgery along the unknot. From Moser's classification of surgeries along torus knots [Mo71] it also follows that one can get some lens spaces by rational surgeries along torus knots.

The cyclic surgery theorem [CGLS87] implies that if K is not a torus knot, then at most two surgeries on K, which must be successive integers, can yield a lens space. A construction due to Berge gives infinite families of knots admitting surgeries yielding a lens space [Be90]. It is an open conjecture that any surgery to a lens space is of this form, cf. [Gr13].

In some situations more is known. For example it follows from Rasmussen's work [Ra07] that the only integral surgery that produces the lens space $L(4m+3, 4)$ is surgery along the negative torus knot $T_{(2,-(2m+1))}$ with coefficient $-(4m + 3)$.

The same strategy used above will also work to study contact surgery numbers of contact structures on lens spaces. (The classification of tight contact structures on lens spaces is known [Gir00, Ho00] and the classification of Legendrian torus knots is obtained in [EH01].) The only difference is that lens spaces are not homology spheres and we also need to consider the Euler classes (or in case of 2-torsion also the $spin^c$-structures) of the underlying 2-plane field. But the computations will not be much more difficult.

Instead of following this route, we determine upper and lower bounds of contact surgery numbers of the tight contact structures on the lens spaces using what is known about symplectic fillings of them. From the classification of tight contact structures on lens spaces we deduce directly that any tight contact structure on a lens space can be obtained by a single rational contact surgery along a Legendrian unknot and a Legendrian surgery along an l-component Legendrian link, where l is the length of the negative continued fraction expansion of $-p/q$. From this, one easily deduces the following.

Corollary 7.27 (Giroux [Gir00] and Honda [Ho00])**.** *Let ξ be any tight contact structure on the lens space $L(p,q)$, for $p > q \geq 1$ and we denote by* $\operatorname{length}(-p/q)$ *the length of the negative continued fraction expansion of $-p/q$. Then we know*

$$\begin{aligned} \operatorname{cs}\big(L(p,q),\xi\big) &= 1, \\ \operatorname{cs}_{\pm 1}\big(L(p,q),\xi\big) &\leq \operatorname{length}(-p/q), \\ \operatorname{cs}_{\pm 1}\big(L(p,1),\xi\big) &= 1. \end{aligned}$$

Using our knowledge about the Stein fillings of lens spaces we can also determine lower bounds. In [CY20, ER20] the classification of symplectic fillings of lens space was completed and in particular Theorem 1.9 of [ER20] says the following.

Proposition 7.28. *A contact structure on the lens space $L(nm+1, m^2)$ can be obtained from Legendrian surgery on a Legendrian realization of the $(n,-m)$–torus knot with Thurston-Bennequin invariant $-nm$; here n and m are relatively prime positive integers. In addition, a contact structure on the lens space $L(3n^2+3n+1, 3n+1)$ can be obtained from Legendrian surgery on a Legendrian realization of a Berge knot with Thurston-Bennequin invariant $-3n^2-3n$ (see* [ER20, Figure 3]*).*

Thus we know there are tight contact structures on these lens spaces with $cs_{\pm 1} = 1$. It was conjectured in [ER20] that these (together with the tight contact structures on $L(p,1)$) are the only tight contact structures on lens spaces that can be obtained from (S^3, ξ_{st}) via a single Legendrian surgery.

Recall that if $-p/q$ has continued fraction expansion $[a_1, \ldots, a_n]$ then $L(p,q)$ is obtained from surgery on a chain of n unknots with these surgery coefficients. The classification of tight contact structures implies that Legendrian surgery on all Legendrian realizations of this chain with i^{th} component having Thurston-Bennequin invariant $a_i + 1$ will produce all possible contact structures on $L(p,q)$. Given $\mathcal{C}$ such a Legendrian realization, let $\mathcal{C}_d$ be the components of the chain that have been stabilized both positively and negatively. Let D be the cardinality of $\mathcal{C}_d$. Notice that $\mathcal{C} - \mathcal{C}_d$ consists of sub-chains of $\mathcal{C}$ and each component in the sub-chain is stabilized with only one sign. Let I be the number of inconsistent sub-chains in $\mathcal{C} - \mathcal{C}_d$. An inconsistent sub-chain is a one whose first element is stabilized one way, last element is stabilized the other way, and the elements in between are not stabilized at all. We now recall Theorem 1.5 from [ER20].

Theorem 7.29. *With the notation above, if X is a Stein filling of the contact structure on $L(p,q)$ determined by $\mathcal{C}$ then its Euler characteristic satisfies*

$$\chi(X) \geq D + \lceil I/2 \rceil + 1.$$

We now have the following lower bound for some contact structures on lens spaces.

Theorem 7.30. *Let ξ be a contact structure on $L(p,q)$ defined by a chain of Legendrian unknots $\mathcal{C}$ as above. With the notation above, if $D + \lceil I/2 \rceil$ is larger than 1, then $cs_{\pm 1}\big(L(p,q),\xi\big) \geq 2$.*

For the proof we need a simple observation.

Lemma 7.31. *No tight contact structure on a lens space can be obtained from (S^3, ξ_{st}) by contact $(+1)$ surgery on a Legendrian knot.*

Proof. Suppose that $(L(p,q),\xi)$ can be so obtained and let L' be the dual of the surgery knot. We then know that (S^3,ξ_{st}) can be obtained from $(L(p,q),\xi)$ by Legendrian surgery on L'. Now $(L(p,q),\xi)$ has a simply connected Stein filling X and one can attach a 2-handle to L' and extend the Stein structure to get a Stein filling X' of (S^3,ξ_{st}). But this Stein filling will have non-trivial second homology but by work of Gromov and McDuff [Gr85, Mc90] we know that the 4-ball is the unique Stein filling of the sphere. □

Proof of Theorem 7.30. By the previous lemma we see that no tight contact structure on a lens space can be obtained from (S^3,ξ_{st}) by contact $(+1)$ surgery on a Legendrian knot. If a contact structure can be obtained from (S^3,ξ_{st}) by contact (-1) surgery on a Legendrian knot, then it has a Stein filling with Euler characteristic 2. But given the hypothesis of our theorem, Theorem 7.29 says this is not possible. □

8. Stein cobordisms between contact structures on the 3-sphere

In this section, we will classify all Stein cobordisms with second Betti number $b_2 = 1$ and no 1-handles between contact structures on the 3-sphere. For that we first recall the classification of Legendrian unknots in overtwisted contact structures up to coarse equivalence. If U is a non-loose Legendrian unknot in an overtwisted contact structure ξ on S^3. Then ξ is isotopic to ξ_1 and U is determined by its classical invariants $(\mathrm{tb}, \mathrm{rot})$ which take all values in $\{(n, \pm(n-1)) : n \in \mathbb{N}\}$ [EF09]. Surgery diagrams of all non-loose unknots are given in Figure 3 of [GO15].

On the other hand, it was shown in [Et13] that the loose Legendrian unknots in any overtwisted contact structure on S^3 are classified by their classical invariants $(\mathrm{tb}, \mathrm{rot})$ which take all values such that $\mathrm{tb} + \mathrm{rot}$ is odd. In the following we present surgery diagrams of those.

Lemma 8.1. *Figure 19 (ii) shows surgery diagrams of all loose Legendrian unknots in (S^3,ξ_1). A surgery diagram of any loose Legendrian unknot in (S^3,ξ_m) is given by taking the connected sum of a surgery diagram of (S^3,ξ_{m-1}) with the diagram of Figure 19 (ii).*

Proof. Figure 19 (i) shows a Legendrian unlink of boundaries Δ of overtwisted disks in (S^3,ξ_1) as can be seen by performing a single Rolfsen twist undoing the surgery. The Legendrian link U shown in Figure 19 (ii) is obtained from a standard Legendrian unknot with $\mathrm{tb} = -1$ and $\mathrm{rot} = 0$ by performing n_+ connected sums with Δ and n_- connected sums with $-\Delta$ followed by some number of stabilizations. Thus U represents a Legendrian unknot in (S^3,ξ_1), which is loose since we can find another overtwisted disk in its complement.

Now connected summing with $\pm\Delta$ changes the classical invariants by

$$\big(\mathrm{tb}(L\# \pm \Delta), \mathrm{rot}(L\# \pm\Delta)\big) = \big(\mathrm{tb}(L)+1, \mathrm{rot}(L) \pm 1\big)$$

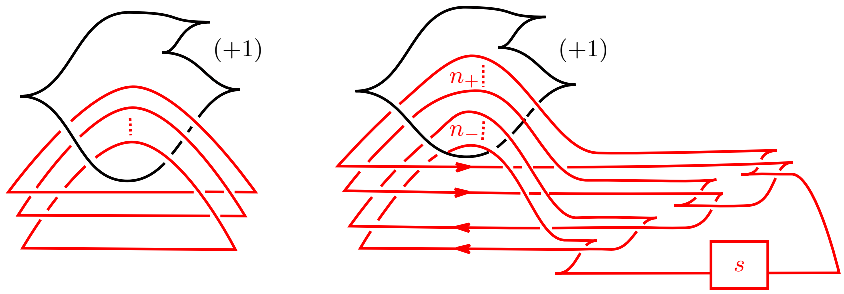

FIGURE 19. Figure (i): An unlink of overtwisted disks in (S^3, ξ_1). Figure (ii): All loose unknots in (S^3, ξ_1). The box stands for an s-fold stabilization. For $s = 0$ the unknot in (S^3, ξ_1) has classical invariants $\mathrm{tb} = n_+ + n_- - 1$ and $\mathrm{rot} = n_+ - n_-$.

and stabilization changes the classical invariants by

$$\big(\mathrm{tb}(L_{\pm 1}), \mathrm{rot}(L_{\pm 1})\big) = \big(\mathrm{tb}(L) - 1, \mathrm{rot}(L) \pm 1\big)$$

and thus we see that all classical invariants can be achieved. □

Proposition 8.2. *Let W be a Stein cobordism with $b_2(W) = 1$ and no 1-handles from (S^3, ξ_1) to another contact 3-sphere (S^3, ξ_+). Then W is obtained by attaching a Weinstein 2-handle along a Legendrian unknot U in (S^3, ξ_1). Moreover, we have that*

- *U is the non-loose unknot with $(\mathrm{tb}(U), \mathrm{rot}(U)) = (2, \pm 1)$ and ξ_+ is isotopic to ξ_{st}, or*
- *U is a loose unknot with $(\mathrm{tb}(U), \mathrm{rot}(U)) = (0, 2n - 1)$, for $n \in \mathbb{Z}$ and ξ_+ is isotopic to $\xi_{n(1-n)+1}$, or*
- *U is a loose unknot with $(\mathrm{tb}(U), \mathrm{rot}(U)) = (2, 2n - 1)$, for $n \in \mathbb{Z}$ and ξ_+ is isotopic to $\xi_{n(1-n)}$.*

In particular, there are infinitely many contact structures on S^3 that cannot be obtained from (S^3, ξ_1) by a single Legendrian surgery.

Proof. Since a Stein cobordisms has a handle decomposition without 3-handles, W consists of a single Weinstein 2-handle attached along a Legendrian knot U in (S^3, ξ_1). And thus (S^3, ξ_+) is obtained from (S^3, ξ_1) by Legendrian surgery along U. By [GL89] U has to be a Legendrian unknot with $\mathrm{tb}(U) = 0$ or $\mathrm{tb}(U) = 2$.

First, we consider the case that U is a non-loose unknot. The classification of the non-loose unknots shows that $(\mathrm{tb}(U), \mathrm{rot}(U)) = (2, \pm 1)$ and that Legendrian surgery along U yields (S^3, ξ_{st}), cf. [GO15].

If U is a loose unknot with $(\mathrm{tb}(U), \mathrm{rot}(U)) = (0, 2n-1)$ (respectively $(2, 2n-1)$) we use the surgery diagram from Lemma 8.1 to compute the d_3-invariant of the Legendrian surgery along U to be $n(1-n) + 1$ (respectively $n(1-n)$). □

Now, we are ready to prove the general statement.

Theorem 8.3. *Let W be a Stein cobordism with $b_2(W) = 1$ and no 1-handles from (S^3, ξ_-) to another contact 3-sphere (S^3, ξ_+). Then W is obtained by attaching a Weinstein 2-handle along a Legendrian unknot U in (S^3, ξ_-). Moreover, we have that*

- *ξ_- is isotopic to ξ_1, U is the non-loose unknot with* $(\mathrm{tb}(U), \mathrm{rot}(U)) = (2, \pm 1)$ *and ξ is isotopic to ξ_{st}, or*
- *ξ_- is isotopic to ξ_m, for $m \in \mathbb{Z}$, U is a loose unknot with* $(\mathrm{tb}(U), \mathrm{rot}(U)) = (0, 2n-1)$*, for $n \in \mathbb{Z}$ and ξ is isotopic to $\xi_{m+n(1-n)+1}$, or*
- *ξ_- is isotopic to ξ_m, for $m \in \mathbb{Z}$, U is a loose unknot with* $(\mathrm{tb}(U), \mathrm{rot}(U)) = (2, 2n-1)$*, for $n \in \mathbb{Z}$ and ξ is isotopic to $\xi_{m+n(1-n)}$.*

Conversely, there exists a Stein cobordism with $b_2 \leq 3$ and no 1-handles from any overtwisted contact structure on S^3 to any other contact structure on S^3.

Proof. The first part follows directly from Lemma 8.1 and Proposition 8.2 by taking connected sums.

For the second part we first observe that the statement is equivalent to finding for every overtwisted contact structure ξ_- on S^3 a Legendrian link L in (S^3, ξ_+) with at most 3 components such that contact $(+1)$-surgery along L yields (S^3, ξ_-). If the difference $\mathrm{d}_3(\xi_+) - \mathrm{d}_3(\xi_-)$ is odd, we can take L to be a link as in Figure 12 (i) contained in a Darboux ball in (S^3, ξ_+). If the difference of the d_3-invariant is even we add another disjoint copy of a Legendrian unknot with $\mathrm{tb} = -2$ in a Darboux ball. □

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