FILTRATIONS ON THE KNOT CONTACT HOMOLOGY OF TRANSVERSE KNOTS

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ABSTRACT. We construct a new invariant of transverse links in the standard contact structure on $\mathbb{R}^3$. This invariant is a doubly filtered version of the knot contact homology differential graded algebra (DGA) of the link, see [4, 14]. Here the knot contact homology of a link in $\mathbb{R}^3$ is the Legendrian contact homology DGA of its conormal lift into the unit cotangent bundle $S^*\mathbb{R}^3$ of $\mathbb{R}^3$, and the filtrations are constructed by counting intersections of the holomorphic disks of the DGA differential with two conormal lifts of the contact structure. We also present a combinatorial formula for the filtered DGA in terms of braid representatives of transverse links and apply it to show that the new invariant is independent of previously known invariants of transverse links.

1. INTRODUCTION

Constructing effective invariants of transverse knots in contact 3–manifolds that go beyond the obvious homotopy invariants has been a notoriously difficult problem. For knots transverse to the standard contact structure on $\mathbb{R}^3$, the first such invariant proven to be effective was constructed only recently, using combinatorial knot Heegaard Floer homology [15]; a related transverse invariant in the Heegaard Floer homology of a general compact contact 3–manifold was subsequently constructed in [10]. In this paper we define and combinatorially present a new invariant for transverse links in the standard contact $\mathbb{R}^3$. The new invariant is a filtration on the knot contact homology differential graded algebra (DGA) [4, 14] for transverse links and it appears to be quite different from the Heegaard Floer invariants. The construction of the invariant can be applied more generally to produce invariants of transverse links in any contact 3–manifold and might even say interesting things in higher dimensions, but the details are considerable more difficult, as are the computations. So we defer the discussion of the general case to a future paper.

1.1. **Transverse knot contact homology.** In [14], the third author constructs a combinatorial DGA associated to a framed knot $K \subset \mathbb{R}^3$ and shows that its homology, combinatorial knot contact homology, is an invariant of framed knots. This invariant detects the unknot and encodes the Alexander polynomial, among other things.

The current authors prove in [4] that the combinatorial DGA is (stable tame) isomorphic to the Legendrian contact homology DGA of the conormal lift of a knot K and also generalize the combinatorial description of knot contact homology to many-component links. Here the conormal lift of a link K is a union Λ_K of Legendrian tori in the unit cotangent bundle $S^*\mathbb{R}^3$ of $\mathbb{R}^3$ with the contact structure given as the kernel of the canonical 1-form. The Legendrian contact homology DGA of Λ_K is an algebra generated by its Reeb chords with differential defined by a count of holomorphic disks in $S^*\mathbb{R}^3 \times \mathbb{R}$ in the spirit of symplectic field theory [8]. The calculation of it in [4] uses the contactomorphism $S^*\mathbb{R}^3 \approx T^*S^2 \times \mathbb{R}$ to transfer the holomorphic disk count in $S^*\mathbb{R} \times \mathbb{R}$ to T^*S^2. The disk count is then carried out using the relation between holomorphic disks and Morse flow trees [2].

We will call the Legendrian invariant of Λ_K defined using holomorphic disks the *knot contact homology* of K. We denote the underlying DGA by $(KC\mathcal{A}(K), \partial)$ and its homology by $KCH(K)$. The algebra $KC\mathcal{A}(K)$ is a free graded tensor algebra over the coefficient ring $R = \mathbb{Z}[H_1(\Lambda_K)] =$

$\mathbb{Z}[\lambda_1^{\pm 1}, \mu_1^{\pm 1}, \dots, \lambda_r^{\pm 1}, \mu_r^{\pm 1}]$, where r is the number of components of K; the differential ∂ depends on K as well, but we suppress this dependence to simplify notation.

Let (x_1, x_2, x_3) be coordinates on $\mathbb{R}^3$ and let $\xi_0 = \ker(dx_3 - x_1\, dx_2 + x_2\, dx_1)$ denote the standard contact structure on $\mathbb{R}^3$ (which is tight and invariant under rotations about the x_3 axis). For links K transverse to ξ_0, we extend the coefficient ring of the knot contact homology DGA to $R[U, V]$, where U and V are two formal variables which encode intersections of holomorphic disks with the natural lifts of ξ_0 that correspond to its two coorientations. The resulting DGA will be denoted $(KC\mathcal{A}^-(K), \partial^-)$ and has a double filtration: by positivity of intersections the differential ∂^- does not decrease the exponents of U or V. Our main result is as follows.

Theorem 1.1. *The filtered stable tame isomorphism type of $(KC\mathcal{A}^-(K), \partial^-)$, and hence its homology, is an invariant of the transverse knot or link K in $(\mathbb{R}^3, \xi_0)$. The DGA can be computed from a braid representative of K as described in Theorem 1.3 below. Moreover, the invariant distinguishes transverse knots with the same self-linking number and with the same Heegaard Floer invariants.*

The explicit formula for $(KC\mathcal{A}^-(K), \partial^-)$ is somewhat involved and is therefore presented separately in Subsection 1.2. It should however be noted that our method for obtaining the combinatorial formula relies heavily on the explicit description of holomorphic disks used in the computation of knot contact homology in [4]. An alternate and purely combinatorial approach to Theorem 1.1 is worked out in [11]. It begins with the combinatorial description of Theorem 1.3 in terms of a braid representation of the knot and proves invariance under transverse isotopy, without any reference to the underlying geometry, via a study of effects of Markov moves. See [11] also for various algebraic properties of the invariant and more detailed calculations which demonstrate its effectiveness.

Those familiar with the algebraic formalism of Heegaard Floer homology will notice that we can construct several other invariants from $(KC\mathcal{A}^-(K), \partial^-)$. Two simpler invariants $(\widehat{KC\mathcal{A}}(K), \widehat{\partial})$ and $\left(\widehat{\widehat{KC\mathcal{A}}}(K), \widehat{\widehat{\partial}}\right)$ arise by setting $(U, V) = (0, 1)$ and $(0, 0)$, respectively. Setting $(U, V) = (1, 1)$, on the other hand, reduces $(KC\mathcal{A}^-(K), \partial^-)$ to the original knot contact homology DGA, $(KC\mathcal{A}(K), \partial)$. Alternatively, one can tensor $(KC\mathcal{A}^-(K), \partial^-)$ with an extended coefficient ring $R[U^{\pm 1}, V^{\pm 1}]$, to obtain a DGA, $(KC\mathcal{A}^\infty(K), \partial^\infty)$. It turns out that $(KC\mathcal{A}^\infty(K), \partial^\infty)$ depends on only the topological isotopy class of K, rather than its transverse isotopy class.

Theorem 1.2. *If K is a transverse knot, then as a DGA over $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}, U^{\pm 1}, V^{\pm 1}]$, $(KC\mathcal{A}^\infty(K), \partial^\infty)$ is an invariant, up to stable tame isomorphism, of the topological knot underlying K.*

In fact, it equals the Legendrian contact homology with coefficients in a relative homology group, see Subsection 3.7 and also [11]. A statement similar to Theorem 1.2 holds for general transverse links but will not be proven here.

1.2. **Calculating $(KC\mathcal{A}(K)^-, \partial^-)$ from a braid presentation.** We now turn to the computation of our transverse link invariant. To this end we first need to introduce some notation. Our notation and conventions differ slightly from those of [11], in which the equivalence of conventions is discussed.

The unit circle U in the plane $\{x_3 = 0\}$ is transverse to the contact structure defined by the 1-form $dx_3 - x_2\, dx_1 + x_1\, dx_2$. By work of Bennequin [1], any transverse link K can be isotoped so that it is braided around U, i.e., contained in a tubular neighborhood $U \times D^2$ of U so that it is transverse to the disk fibers. So K can be represented as the closure of a braid B in $U \times D^2 = S^1 \times D^2$. The braid B is an element of the braid group B_n for some n and can be expressed as a word in the standard generators of B_n, σ_j $(j = 1, \dots, n-1)$ intertwining strands j and $j+1$, along with their inverses.

The Legendrian contact homology DGA of Λ_K is a free associative non-commutative unital DGA over the group ring of $H_1(\Lambda_K; \mathbb{Z})$. Fixing a framing on each component of K (e.g., the standard

topological 0-framing) yields a distinguished basis $\{\lambda_1, \mu_1, \dots, \lambda_r, \mu_r\}$ of $H_1(\Lambda_K; \mathbb{Z})$, where r is the number of components of K. More precisely, by identifying $T\mathbb{R}^3$ with $T^*\mathbb{R}^3$, we can identify Λ_K with the boundary of a tubular neighborhood of K, which consists of r disjoint tori. Each component of K has a natural orientation given by the positive coorientation of the contact structure ξ_0. Then μ_j, λ_j are the meridian and framed longitude of the j-th torus.

Now, as described above, suppose that K is the closure of a braid $B \in B_n$ in a tubular neighborhood of the standard transverse unknot U. Let $\mathcal{A}_n$ denote the graded unital algebra over $\mathbb{Z}[\lambda_1^{\pm 1}, \mu_1^{\pm 1}, \dots, \lambda_r^{\pm 1}, \mu_r^{\pm 1}]$ freely generated by

$$\begin{aligned}
&\{a_{ij}\}_{1\le i,j\le n;\, i\ne j} \text{ in degree } 0,\\
&\{b_{ij}\}_{1\le i,j\le n;\, i\ne j} \text{ in degree } 1,\\
&\{c_{ij}\}_{1\le i,j\le n} \text{ in degree } 1, \text{ and}\\
&\{e_{ij}\}_{1\le i,j\le n} \text{ in degree } 2,
\end{aligned}$$

where the degrees of $\lambda_j^{\pm 1}$ and $\mu_j^{\pm 1}$ equal 0 for $j = 1, \dots, r$. (When there is only one link component, we drop the subscripts on λ_j, μ_j.) Let $\mathcal{A}_n^{(0)}$ denote the subalgebra of $\mathcal{A}_n$ of elements of degree 0.

We define an automorphism $\phi_B \colon \mathcal{A}_n^{(0)} \to \mathcal{A}_n^{(0)}$ as follows. Introduce auxiliary variables $\tilde{\mu}_1, \dots, \tilde{\mu}_n$ of degree 0, and write $\tilde{\mathcal{A}}_n^{(0)}$ for the unital algebra over $\mathbb{Z}[\tilde{\mu}_1^{\pm 1}, \dots, \tilde{\mu}_n^{\pm 1}]$ freely generated by $\{a_{ij}\}_{1\le i,j\le n; i\ne j}$. For $k = 1, \dots, n-1$, let $\phi_{\sigma_k} : \tilde{\mathcal{A}}_n^{(0)} \to \tilde{\mathcal{A}}_n^{(0)}$ be given by

$$\begin{aligned}
\phi_{\sigma_k}(a_{ij}) &= a_{ij} && i, j \ne k, k+1\\
\phi_{\sigma_k}(a_{k\,k+1}) &= -a_{k+1\,k}\\
\phi_{\sigma_k}(a_{k+1\,k}) &= -\tilde{\mu}_k \tilde{\mu}_{k+1}^{-1} a_{k\,k+1}\\
\phi_{\sigma_k}(a_{i\,k+1}) &= a_{ik} && i \ne k, k+1\\
\phi_{\sigma_k}(a_{k+1\,i}) &= a_{ki} && i \ne k, k+1\\
\phi_{\sigma_k}(a_{ik}) &= a_{i\,k+1} - a_{ik} a_{k\,k+1} && i < k\\
\phi_{\sigma_k}(a_{ik}) &= a_{i\,k+1} - \tilde{\mu}_k \tilde{\mu}_{k+1}^{-1} a_{ik} a_{k\,k+1} && i > k+1\\
\phi_{\sigma_k}(a_{ki}) &= a_{k+1\,i} - a_{k+1\,k} a_{ki} && i \ne k, k+1\\
\phi_{\sigma_k}(\tilde{\mu}_k^{\pm 1}) &= \tilde{\mu}_{k+1}^{\pm 1}\\
\phi_{\sigma_k}(\tilde{\mu}_{k+1}^{\pm 1}) &= \tilde{\mu}_k^{\pm 1}\\
\phi_{\sigma_k}(\tilde{\mu}_i^{\pm 1}) &= \tilde{\mu}_i^{\pm 1} && i \ne k, k+1.
\end{aligned}$$

Write B in terms of braid group generators, $B = \sigma_{i_0}^{m_0} \dots \sigma_{i_l}^{m_l} \in B_n$, and let $\phi_B = (\phi_{\sigma_{i_0}})^{m_0} \circ \dots \circ (\phi_{\sigma_{i_l}})^{m_l}$. Then ϕ descends to a homomorphism from B_n to the automorphism group of $\tilde{\mathcal{A}}_n^{(0)}$; in particular, ϕ satisfies the braid relations.

For $j = i, \dots, n$, let $\alpha(i) \in \{1, \dots, r\}$ be the number of the link component of K corresponding to the i-th strand of B. Then ϕ_B can be viewed as an automorphism of $\mathcal{A}_n^{(0)}$ by setting $\tilde{\mu}_i = \mu_{\alpha(i)}$ for all i and having ϕ_B act as the identity on λ_i for all i. As an automorphism of $\mathcal{A}_n^{(0)}$, ϕ_B acts as the identity on μ_i as well.

For convenient notation we assemble the generators of $\mathcal{A}_n$ into $(n \times n)$-matrices. Writing $\mathbf{M}_{ij}$ for the element in position ij in the $(n \times n)$-matrix $\mathbf{M}$, we define the $(n \times n)$-matrices

$$\mathbf{A}: \begin{cases} \mathbf{A}_{ij} = a_{ij} & \text{if } i < j, \\ \mathbf{A}_{ij} = \mu_{\alpha(j)} a_{ij} & \text{if } i > j, \\ \mathbf{A}_{ii} = 1 + \mu_{\alpha(i)}, \end{cases} \qquad \mathbf{B}: \begin{cases} \mathbf{B}_{ij} = b_{ij} & \text{if } i < j, \\ \mathbf{B}_{ij} = \mu_{\alpha(j)} b_{ij} & \text{if } i > j, \\ \mathbf{B}_{ii} = 0, \end{cases}$$

$$\mathbf{C}: \begin{cases} \mathbf{C}_{ij} = c_{ij}, \end{cases} \qquad \mathbf{E}: \begin{cases} \mathbf{E}_{ij} = e_{ij}. \end{cases}$$

We also associate $(n \times n)$-matrices with coefficients in $\mathcal{A}_n^{(0)}$ to the braid B as follows. Let $\phi_B(\mathbf{A})$ be the matrix defined by $\big(\phi_B(\mathbf{A})\big)_{ij} = \phi_B(\mathbf{A}_{ij})$. Then there are invertible matrices Φ_B^L and Φ_B^R so that

$$\phi_B(\mathbf{A}) = \Phi_B^L \cdot \mathbf{A} \cdot \Phi_B^R.$$

More specifically, we define these matrices by setting B' to be the $(n+1)$-braid obtained by adding an extra strand labeled 0 to B (that is, viewing the word defining B as a word in the $(n+1)$-strand braid group generated by $\sigma_0, \dots, \sigma_{n-1}$). Let ϕ'_B be the corresponding automorphism of the $\{a_{ij}\}_{0 \le i,j \le n; i \ne j}$. Then define Φ_B^L and Φ_B^R by

$$\phi'_B(a_{i0}) = \sum_{j=1}^{n} \left(\Phi_B^L\right)_{ij} a_{j0} \quad \text{and} \quad \phi'_B(a_{0j}) = \sum_{i=1}^{n} a_{0i} \left(\Phi_B^R\right)_{ij};$$

see also [4, 12, 14]. (Note that since the 0-th strand does not interact with the others, $\tilde{\mu}_0$ does not appear anywhere in the expressions for $\phi'_B(a_{i0})$ and $\phi'_B(a_{0j})$, and so Φ_B^L, Φ_B^R have coefficients in $\mathcal{A}_n^{(0)}$.)

Also, define an $(n \times n)$ coefficient matrix $\boldsymbol{\lambda}$ as follows. Among the strands $1, \dots, n$ of the braid, call a strand *leading* if it is the first strand belonging to its component. Then let $\boldsymbol{\lambda}$ be the diagonal matrix defined by

$$\boldsymbol{\lambda}: \begin{cases} \boldsymbol{\lambda}_{ij} = 0 & \text{if } i \ne j, \\ \boldsymbol{\lambda}_{ii} = \lambda_{\alpha(i)} \mu_{\gamma(i)}^{w(i)} & \text{if the } i^{\text{th}} \text{ strand is leading}, \\ \boldsymbol{\lambda}_{ii} = 1 & \text{otherwise}, \end{cases}$$

where $w(i)$ is the writhe (algebraic crossing number) of the i-th component of the braid, considered by itself. Finally, in order to capture the two filtrations, we define the following additional matrices:

$$\mathbf{A}^U: \begin{cases} \mathbf{A}^U_{ij} = U a_{ij} & \text{if } i < j, \\ \mathbf{A}^U_{ij} = \mu_{\alpha(j)} a_{ij} & \text{if } i > j, \\ \mathbf{A}^U_{ii} = U + \mu_{\alpha(i)}, \end{cases} \qquad \mathbf{B}^U: \begin{cases} \mathbf{B}^U_{ij} = U b_{ij} & \text{if } i < j, \\ \mathbf{B}^U_{ij} = \mu_{\alpha(j)} b_{ij} & \text{if } i > j, \\ \mathbf{B}^U_{ii} = 0, \end{cases}$$

$$\mathbf{A}^V: \begin{cases} \mathbf{A}^V_{ij} = a_{ij} & \text{if } i < j, \\ \mathbf{A}^V_{ij} = \mu_{\alpha(j)} V a_{ij} & \text{if } i > j, \\ \mathbf{A}^V_{ii} = 1 + \mu_{\alpha(i)} V, \end{cases} \qquad \mathbf{B}^V: \begin{cases} \mathbf{B}^V_{ij} = b_{ij} & \text{if } i < j, \\ \mathbf{B}^V_{ij} = \mu_{\alpha(j)} V b_{ij} & \text{if } i > j, \\ \mathbf{B}^V_{ii} = 0. \end{cases}$$

Theorem 1.3. *The filtered knot contact homology DGA, $(KC\mathcal{A}(K)^-, \partial^-)$ of a transverse link K represented as a braid B on n strands is given by the DGA, $(\mathcal{A}_n, \partial^-)$, with the differential $\partial^- \colon \mathcal{A}_n \to \mathcal{A}_n$ defined by the following matrix equations:*

$$\begin{aligned} \partial^- \mathbf{A} &= 0, \\ \partial^- \mathbf{B} &= -\boldsymbol{\lambda}^{-1} \cdot \mathbf{A} \cdot \boldsymbol{\lambda} \ + \ \Phi_B^L \cdot \mathbf{A} \cdot \Phi_B^R, \\ \partial^- \mathbf{C} &= \mathbf{A}^V \cdot \boldsymbol{\lambda} \ + \ \mathbf{A}^U \cdot \Phi_B^R, \\ \partial^- \mathbf{E} &= \mathbf{B}^V \cdot (\Phi_B^R)^{-1} \ + \ \mathbf{B}^U \cdot \boldsymbol{\lambda}^{-1} \ - \ \Phi_B^L \cdot \mathbf{C} \cdot \boldsymbol{\lambda}^{-1} \ + \ \boldsymbol{\lambda}^{-1} \cdot \mathbf{C} \cdot (\Phi_B^R)^{-1}, \end{aligned}$$

where if $\mathbf{M}$ *is an* $(n \times n)$*-matrix, the matrix* $\partial^- \mathbf{M}$ *is defined by* $(\partial^- \mathbf{M})_{ij} = \partial^-(\mathbf{M}_{ij})$.

The rest of the paper is organized as follows. We provide some general geometric background in Section 2 before defining the transverse invariant $(KC\mathcal{A}^-(K), \partial^-)$ and proving invariance in Section 3. In Section 4, we derive the combinatorial formula for $(KC\mathcal{A}^-(K), \partial^-)$ by proving Theorem 1.3, and present examples in Section 5.

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2. Geometric Constructions

In this section we recall the conormal construction and see how it can be used to construct invariants of smooth embeddings using contact geometry. We then discuss filtrations on the Legendrian contact homology DGA induced by complex hypersurfaces with certain properties.

2.1. **Legendrian contact homology.** In [8], the Legendrian contact homology $LCH(\Lambda)$ of a Legendrian submanifold Λ of a contact manifold (V, ξ) was introduced. The analytic underpinnings were worked out in detail in [7] for a fairly general and useful case (but under the simplifying assumption that the chosen Reeb field of ξ has no closed orbits, see below). In this case the Legendrian contact homology $LCH(\Lambda)$ is the homology of a DGA, $(LC\mathcal{A}(\Lambda), \partial)$, over a fixed ring, which changes by a particular type of quasi-isomorphism, called a stable tame isomorphism, as Λ changes by Legendrian isotopy. Thus, the stable tame isomorphism class of $(LC\mathcal{A}(\Lambda), \partial)$ might be considered to be the actual Legendrian invariant underlying $LCH(\Lambda)$.

We briefly sketch the definition of this DGA in the case handled in [7] for the convenience of the reader and to establish some notation; for a more complete definition, see [7] and for generalizations see [8].

Let P be a manifold with exact symplectic form $d\lambda$. The manifold $P \times \mathbb{R}$ has a natural contact structure $\xi = \ker(dz - \lambda)$ where z is the coordinate on $\mathbb{R}$. The Reeb vector field of this form is ∂_z and consequently there are no closed Reeb orbits. Consider the projection

$$\pi_{\mathbb{C}} : P \times \mathbb{R} \to P.$$

The algebra $LC\mathcal{A}(\Lambda)$ is the free tensor algebra generated over $\mathbb{Z}[H_1(\Lambda)]$ by the double points of $\pi_{\mathbb{C}}(\Lambda)$. Notice that the double points are in one to one correspondence with "Reeb chords", that is, flow lines of the Reeb vector field that begin and end on Λ. Thus we will frequently refer to double points as Reeb chords. For the double points we choose "capping paths" in Λ: that is, paths in Λ that connect any Reeb chord endpoint to a fixed base point in its connected component, and fixed paths connecting the base points of distinct components; together, these give paths that connect the two points in Λ which project to a double point in $\pi_{\mathbb{C}}(\Lambda)$. At a double point p there are two point in Λ that project to it. We label the one with larger z coordinate z^+ and the other z^-. The projection of a neighborhood of z^+ in Λ to P will be called the *upper sheet at* z and the projection of a neighborhood of z^- will be called the *lower sheet at* z. We then can define a Maslov type index $|c|$ and $|A|$ of Reeb chords c and homology classes $A \in H_1(\Lambda)$ to provide a grading on $LC\mathcal{A}(\Lambda)$. To define the differential we fix an almost complex structure J on P (which can be thought of as an almost complex structure on ξ using the isomorphism $d\pi_{\mathbb{C}}|_\xi$). For this almost complex structure the differential is determined by counting (pseudo-)holomorphic disks mapped into P with boundary on $\pi_{\mathbb{C}}(\Lambda)$. Given a Reeb chord a, a (noncommutative) word of Reeb chords $\mathbf{b} = b_1 \cdots b_m$, and a homology class $A \in H_1(\Lambda)$, we define a moduli space

$$\mathcal{M}_A(a; \mathbf{b}) \tag{2.1}$$

of holomorphic disks $u\colon D \to P$ with: boundary on Λ; one positive[1] puncture at a and negative punctures at $b_1, \dots, b_m$ in the order given by the boundary orientation; and the homology class A given by the lift of $u(\partial D)$ to Λ together with the chosen capping paths. For a generic almost complex structure, this moduli space is a manifold of dimension $|a| - |\mathbf{b}| - |A| - 1$, where $|\mathbf{b}| = \sum_{i=1}^{m} |b_i|$. Furthermore, the space has a natural compactification which consists of (several level) broken curves and which admits the structure of a manifold with boundary with corners. The moduli spaces can be coherently oriented provided the Legendrian submanifold Λ is spin.

Define the differential on the generators of $LC\mathcal{A}(\Lambda)$ by

$$\partial a = \sum_{\{u \in \mathcal{M}_A(a;\mathbf{b}) \mid |a|-|\mathbf{b}|-|A|-1=0\}} (-1)^{|a|+1} \sigma(u) e^A \mathbf{b}, \tag{2.2}$$

where $\sigma(u) \in \{\pm 1\}$ is determined by the moduli space orientation. The differential is then extended to all of $LC\mathcal{A}(L)$ by the graded product rule and linearity.

2.2. **Almost complex hypersurfaces and filtration on Legendrian contact homology.** We now show how to use a complex hypersurface to add a "filtration" to the Legendrian contact homology DGA. With the notation above, suppose that $\overline{H}$ is a submanifold of P such that

(1) $\overline{H}$ is an almost complex hypersurface ($J(T\overline{H}) = T\overline{H}$ and the (real) codimension of $\overline{H}$ equals 2) and
(2) $\Lambda \cap \overline{H} = \varnothing$.

Given such Λ and $\overline{H}$, we can extend the base ring for the contact homology DGA of Λ from $\mathbb{Z}[H_1(\Lambda)]$ to $\mathbb{Z}[H_1(\Lambda)][U_{\overline{H}}]$, by adjoining a formal variable $U_{\overline{H}}$ and changing the definition of the boundary map, using powers of $U_{\overline{H}}$ to keep track of the number of times the holomorphic disks in the definition of the boundary map intersect $U_{\overline{H}}$. Specifically, given $u \in \mathcal{M}_A(a;\mathbf{b})$, positivity of intersection shows that the intersection number of the image of u with $\overline{H}$ is a well-defined and nonnegative integer which we denote $n_{\overline{H}}(u)$. We can now modify the differential in Equation (2.2) and define instead:

$$\partial_f a = \sum_{\{u \in \mathcal{M}_A(a;\mathbf{b}) \mid |a|-|\mathbf{b}|-|A|-1=0\}} (-1)^{|a|+1} \sigma(u) U_{\overline{H}}^{n_{\overline{H}}(u)} e^A \mathbf{b}. \tag{2.3}$$

Conditions (1) and (2) above ensure that ∂_f is a filtered differential; that is, it respects the filtration

$$LC\mathcal{A} \supset U_{\overline{H}} \cdot LC\mathcal{A} \supset U_{\overline{H}}^2 \cdot LC\mathcal{A} \supset \cdots.$$

The proof that $(LC\mathcal{A}(\Lambda), \partial)$ is invariant up to stable tame isomorphism carries over to show that the stable tame isomorphism type of $(LC\mathcal{A}(\Lambda), \partial_f)$ over $\mathbb{Z}[H_1(\Lambda)][U_{\overline{H}}]$ is an invariant of Λ under isotopies Λ_t, $0 \le t \le 1$, such that Λ_t satisfies condition (2) for all t, see Theorem 3.11 below.

2.3. **The conormal construction and knot contact homology.** Given any n-manifold M, the cotangent bundle T^*M has a canonical symplectic structure $d\lambda$ where λ is the Liouville 1–form. If we choose a metric g on M then we can consider the unit cotangent bundle S^*M. The restriction of λ to S^*M is a contact form which we denote by α, and $\xi = \ker \alpha$ is a contact structure on S^*M.

Let $\mathrm{pr}\colon T^*M \to M$ denote the natural projection. If K is a submanifold of M (of any dimension) then the unit conormal bundle

$$\Lambda_K = \{\beta \in S^*M : \mathrm{pr}(\beta) = p \in K \text{ and } \beta(v) = 0 \text{ for all } v \in T_pK\}$$

is a Legendrian submanifold of (S^*M, ξ). If we smoothly isotop K in M, it is clear that Λ_K will undergo a Legendrian isotopy in (S^*M, ξ). Thus any Legendrian isotopy invariant of Λ_K is a smooth isotopy invariant of K.

[1]Using the orientation on ∂D induced by the complex structure a puncture is positive if it maps the segment of the boundary just before the puncture to the lower sheet at the double point and the segment just after the puncture to the upper sheet. The puncture is negative if the roles of the upper and lower sheets are reversed.

In this paper, we consider conormal lifts of links $K \subset \mathbb{R}^3$ which are Legendrian submanifolds $\Lambda_K \subset S^*\mathbb{R}^3$. There is a well known contactomorphism $S^*\mathbb{R}^3 \approx J^1(S^2) = T^*S^2 \times \mathbb{R}$ and we will thus consider $\Lambda_K \subset T^*S^2 \times \mathbb{R}$ and use the version of Legendrian contact homology of Λ_K defined in [7]. In particular, we define the *knot contact homology algebra* of a link K in $\mathbb{R}^3$ to be the Legendrian contact homology algebra of $\Lambda_K \subset J^1(S^2)$. We denote it $(KC\mathcal{A}(K), \partial)$ and note that the stable tame isomorphism class of $(KC\mathcal{A}(K), \partial)$ is an isotopy invariant of K. In [4] the authors show how to compute $(KC\mathcal{A}(K), \partial)$ and demonstrate that it is equivalent to the combinatorial knot DGA introduced by the third author in [14].

We recall for later use that the projection $\pi_F : J^1(S^2) \to S^2 \times \mathbb{R}$ is called the front projection and that a generic Legendrian submanifold Λ in $J^1(S^2)$ can be recovered uniquely from $\pi_F(\Lambda) \subset S^2 \times \mathbb{R}$. Since $S^2 \times \mathbb{R}$ can be visualized as $\mathbb{R}^3 \setminus \{(0,0,0)\}$, it will frequently be useful to study a Legendrian submanifold Λ via its front projection.

2.4. **Transverse link invariants.** Consider a contact structure ξ on $\mathbb{R}^3$. If there is a Reeb vector field R_ξ for ξ such that the flow lines of the vector field trace out geodesics (in the flat Euclidean metric on $\mathbb{R}^3$) then we can consider the submanifolds

$$H_\xi^\pm = \{\eta \in S^*\mathbb{R}^3 : \pm\eta(R_\xi) > 0 \text{ and } \eta(v) = 0 \text{ for all } v \in \xi\}.$$

Since the Reeb flow lines of R_ξ are geodesics, $H_\xi^\pm$ are foliated by Reeb flow lines in $S^*\mathbb{R}^3$. In other words, identifying $S^*\mathbb{R}^3$ with $J^1(S^2) = T^*S^2 \times \mathbb{R}$, the projected submanifold $\overline{H}_\xi^\pm = \pi_{\mathbb{C}}(H_\xi^\pm)$ in T^*S^2 is an embedded codimension 2 submanifold. One may also choose the almost complex structure on T^*S^2 so that $\overline{H}_\xi^\pm$ is a holomorphic submanifold. Moreover, if K is a knot in $\mathbb{R}^3$ that is transverse to ξ then its conormal lift Λ_K projects to an exact Lagrangian submanifold in T^*S^2 that is disjoint from $\overline{H}_\xi^\pm$. Thus, as discussed above, we can construct an associated filtered contact homology DGA that will be an invariant of the transverse isotopy class of K.

To carry out the above construction one must choose a contact form α with $\xi = \ker(\alpha)$ so that its Reeb flow traces out geodesics. The standard contact structure does have such representatives, for example $\xi_0 = \ker(\sin x_1\, dx_2 + \cos x_2\, dx_3)$, but it is quite difficult to actually compute the filtered contact homology DGA for this contact structure. To take advantage of the computations in [4] we would prefer to work with the contact form $\ker(dx_3 + r^2\, d\theta)$, but this contact form does not have a Reeb vector field with the requisite properties. In the remainder of the paper we overcome this problem by considering a different S^2-subbundle B of $T^*\mathbb{R}^3$ instead of the unit cotangent bundle. As long as each fiber in this subbundle bounds a convex region containing the origin, we can still identify B with $J^1(S^2)$. By a judicious choice of B we will see that the projection of $H_\xi^\pm$ to T^*S^2 retains enough of the properties discussed above to allow us to explicitly calculate a filtered invariant for transverse knots in $\mathbb{R}^3$.

3. The Filtered DGA of Transverse Links in $(\mathbb{R}^3, \xi_0)$

In this section we show how the standard contact structure ξ_0 on $\mathbb{R}^3$ can be used to construct a pair of complex hypersurfaces $\overline{H}_\pm$ in T^*S^2 such that if K is any link in a sufficiently small ball around the origin which is transverse to ξ_0 then $\overline{H}_\pm \cap \pi_{\mathbb{C}}(\Lambda_K) = \varnothing$. In order to get a computable invariant we adapt the geometry and use a slightly non-standard version of $S^*\mathbb{R}^3$.

In Subsection 3.1 we describe our geometric model of $S^*\mathbb{R}^3$ and its relation to $T^*S^2 \times \mathbb{R}$. In Subsection 3.2 we show that we can control the image of the holomorphic disks used to compute the knot contact homology so that they lie near the zero section in T^*S^2 if the original link is sufficiently small. We then discuss the conormal lift of the standard contact structure on $\mathbb{R}^3$ in Subsection 3.3 and show in Subsection 3.4 that it is no restriction to assume that all transverse links and isotopies lie in a small ball around the origin. In Subsections 3.5 and 3.6, we define a suitable almost complex structure on T^*S^2 and then prove the filtered DGA of a transverse link

is well-defined and invariant up to stable tame isomorphisms under isotopies through transverse links. Finally, in Subsection 3.7, we explain why the infinity theory for knots in $\mathbb{R}^3$ is a topological invariant.

3.1. **The spherical conormal bundle and the 1-jet space of S.** Let $y = (y_1, y_2, y_3)$ be standard Euclidean coordinates on $\mathbb{R}^3$. Let S denote the smooth boundary of a (not necessarily strictly) convex subset of $\mathbb{R}^3$ which is symmetric with respect to reflection in the y_1y_2-plane and with respect to rotations about the y_3-axis. For $y \in S$, let $\nu(y)$ denote the outward unit normal to S at y. Note that the symmetries of S imply that $\nu(y_1, y_2, y_3)$ lies in the subspace spanned by the vectors $(y_1, y_2, 0)$ and $(0, 0, y_3)$. In particular, $\nu(0, 0, y_3) = (0, 0, \pm 1)$ and $\nu(y_1, y_2, 0) = \frac{1}{\sqrt{y_1^2+y_2^2}}(y_1, y_2, 0)$.

We represent the spherical cotangent bundle of $\mathbb{R}^3$ as

$$S^*\mathbb{R}^3 = \mathbb{R}^3 \times S \subset T^*\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$$

and use coordinates $(x, y) = (x_1, x_2, x_3, y_1, y_2, y_3)$ on $T^*\mathbb{R}^3$. The contact form on $S^*\mathbb{R}^3$ is the restriction of the Liouville 1–form $y \cdot dx = \sum_{j=1}^3 y_j\, dx_j$ on $T^*\mathbb{R}^3$ to $S^*\mathbb{R}^3$. We compute the Reeb vector field as follows.

Lemma 3.1. *The Reeb vector field on $S^*\mathbb{R}^3$ is*

$$R(x, y) = |y|^{-1}(\nu(y) \cdot \partial_x) = |y|^{-1} \sum \nu_j(y)\partial_{x_j} \tag{3.1}$$

and the time t flow starting at (x, y) is

$$\Phi_R^t(x, y) = \left(x + t\left(|y|^{-1}\nu(y)\right)\, ,\, y\right). \tag{3.2}$$

Proof. If i denotes the standard complex structure on $\mathbb{C}^3 = \mathbb{R}^3 + i\mathbb{R}^3 = T^*\mathbb{R}^3$ then the Reeb field lies in the intersection of $T(S^*\mathbb{R}^3)$ and the complex tangent line at y containing $\nu(y)$. Thus up to normalization the Reeb field equals $-i\nu(y) = \nu(y) \cdot \partial_x$. The lemma follows. □

The spherical cotangent bundle $S^*\mathbb{R}^3$ can be identified with the 1–jet space $J^1(S) = T^*S \times \mathbb{R}$ as follows, where we use the flat metric on $\mathbb{R}^3$ to identify vectors and covectors.

Lemma 3.2. *The map $\phi\colon S^*\mathbb{R}^3 \to J^1(S) = T^*S \times \mathbb{R}$ given by*

$$\phi(x, y) = (y, x - (x \cdot \nu)\nu, x \cdot y).$$

is a contactomorphism

$$\phi\colon (S^*\mathbb{R}^3, y \cdot dx) \to (J^1(S), dz - p \cdot dq),$$

where $q = (q_1, q_2)$ are local coordinates on S, $p = (p_1, p_2)$ give the coordinates on the fiber of the cotangent bundle and z is the coordinate on $\mathbb{R}$.

Proof. Note that

$$\begin{aligned} \phi^*(dq) &= dy, \\ \phi^*(dz) &= x \cdot dy + y \cdot dx, \end{aligned}$$

and thus

$$\phi^*(dz - p \cdot dq) = x \cdot dy + y \cdot dx - (x - (x \cdot \nu)\nu) \cdot dy = y \cdot dx,$$

where we use $\nu \cdot dy = 0$, which holds since $y \in S$ and ν is the normal of S at y. □

3.2. **Confining holomorphic curves.** Fix $\delta > 0$, let $S \subset \mathbb{R}^3$ be as above and write $\rho_0 = \max\{|y|\colon y \in S\}$. Below we will measure lengths of cotangent vectors in T^*S using the metric coming from the one induced on $T^*\mathbb{R}^3$ by the flat metric on $\mathbb{R}^3$. If $(q,p) \in T^*S$ then we write $|p|$ for the length of the cotangent vector measured with respect to this metric.

Lemma 3.3. *If K is any link contained in $B(\delta)$, the ball of radius δ about the origin in $\mathbb{R}^3$, then*

$$\Lambda_K \subset \{(q,p,z) \in T^*S \times \mathbb{R}\colon |p| \leq \delta\}.$$

Moreover, if c is a Reeb chord of Λ_K then

$$\int_c (dz - p\,dq) \leq 2\rho_0\delta.$$

Proof. By Lemma 3.2 if $x \in K$ and $(x,y) \in \Lambda_K \subset S^*\mathbb{R}^3$ then

$$|p| = |x - (\nu(y)\cdot x)\nu(y)| \leq |x| \leq \delta.$$

For the second statement we note that Lemma 3.1 implies that a Reeb chord c of Λ_K is a lift (l_c, y) into $S^*\mathbb{R}^3$ of a line segment l_c in $\mathbb{R}^3$ with endpoints on K and that the action of the chord is

$$\int_{l_c} y \cdot dx \leq \int_{l_c} \rho_0 |dx| \leq 2\rho_0\delta.$$

□

As a consequence of Lemma 3.3 we can confine holomorphic curves with boundary on Λ_K. As in Subsection 2.3, $\pi_{\mathbb{C}} : J^1(S) \to T^*S$ will denote the projection map and $\overline{\Lambda}_K = \pi_{\mathbb{C}}(\Lambda_K)$.

Lemma 3.4. *Let J be an almost complex structure on T^*S that is compatible with the symplectic form on T^*S. Fix $\delta_0 > 0$ a constant. Then there exists $0 < \delta < \delta_0$ such that if K is a link in $B(\delta)$ then any J-holomorphic disk with boundary on $\overline{\Lambda}_K$ and one positive puncture lies in $\{(q,p) \in T^*S\colon |p| < 2\delta_0\}$.*

Proof. Consider $0 < \delta < \delta_0$. By Stokes' theorem and Lemma 3.3, the area of a disk u as described is bounded above by $2\rho_0\delta$ and its boundary is contained in the region where $|p| \leq \delta$. By monotonicity (see for example Proposition 4.3.1 [9]), there exists a constant C_0 (depending only on J) such that if u leaves the region where $|p| \leq 2\delta_0$ then the area of u is at least $C_0\delta_0^2$. Thus, taking $\delta < C_0\delta_0^2/2\rho_0$ the lemma follows. □

3.3. **A contact form on $\mathbb{R}^3$ and its spherical cotangent lifts.** Fix the contact form $\alpha_0 = dx_3 - x_2\,dx_1 + x_1\,dx_2$ on $\mathbb{R}^3$ and write, as in Section 1, $\xi_0 = \ker(\alpha_0)$ for the corresponding contact structure. Note that α_0 is invariant under rotations in the x_1x_2-plane and that the diffeomorphism

$$(x_1,x_2,x_3) \mapsto \left(\tfrac{1}{2}x_1\,,\,\tfrac{1}{2}x_2\,,\,x_3 - \tfrac{1}{2}x_1x_2\right)$$

gives a contactomorphism between α_0 and the standard contact form on $\mathbb{R}^3$, $dx_3 - x_2\,dx_1$.

If $v \in \mathbb{R}^3$ is a non-zero vector, we denote the two open half rays determined by v as follows:

$$\mathbb{R}_{\pm}\cdot v = \{x \in \mathbb{R}^3\colon x = tv,\ \pm t > 0\}.$$

The *positive and negative spherical lifts* of α_0 are

$$H_+ = \left\{(x,y) \in S^*\mathbb{R}^3\colon y \in \mathbb{R}_+\cdot\alpha_0(x)\right\} \text{ and } H_- = \left\{(x,y) \in S^*\mathbb{R}^3\colon y \in \mathbb{R}_-\cdot\alpha_0(x)\right\}, \tag{3.3}$$

respectively. At all points x on a transverse link K, α_0 fails to annihilate the tangent space T_xK. Thus we have the following immediate result.

Lemma 3.5. *If K is transverse to ξ_0 then the conormal lift Λ_K of K is disjoint from $H_\pm$.* □

3.4. **Shrinking transverse links.** The following straightforward lemma reduces the study of transverse links in $\mathbb{R}^3$ with its standard contact structure to the study of transverse links lying in an arbitrary small fixed neighborhood of the origin. Let $B^d(r)$ denote the closed d-dimensional ball of radius r around the origin and let $B(r) = B^3(r)$.

Lemma 3.6. *Fix $\delta > 0$. Let $K(s)$, $s \in B^d(1)$, be a continuous family of transverse links in $(\mathbb{R}^3, \xi_0)$ such that $K(s) \subset B(\delta)$ for $s \in \partial B^d(1)$. Then there is a homotopy $K(s,t)$, $0 \le t \le 1$, with $K(s,t) = K(s)$ if $s \in \partial B^d(1)$, and such that $K(s,0) = K(s)$ and $K(s,1) \subset B(\delta)$ for each $s \in B^d(1)$. In particular, the space of transverse links in $\mathbb{R}^3$ is weakly homotopy equivalent to the space of transverse links in $B(\delta)$.*

Proof. Note that if K is a transverse link in $(\mathbb{R}^3, \alpha)$ then so is $F_c(K)$ where

$$F_c(x_1, x_2, x_3) = (\sqrt{c}\, x_1, \sqrt{c}\, x_2, c x_3)$$

for $c > 0$. Choose $\epsilon_1, \epsilon_2 > 0$ sufficiently small so that $K(s) \subset B(\epsilon_1^{-1}\delta)$ for all $s \in B^d(1)$ and $K(s) \subset B(\delta)$ for all $s \in (B^d(1) \setminus B^d(1-\epsilon_2))$. Choose any smooth $\varepsilon : B^d(1) \to [\epsilon_1, 1]$ such that

$$\varepsilon(s) = 1 \text{ if } s \in \partial B^d(1), \quad \text{and } \varepsilon(s) = \epsilon_1 \text{ if } s \in B^d(1-\epsilon_2).$$

Then $K(s,t) = F_{(1-t)+t\varepsilon(s)}(K(s))$ is a homotopy with the required properties. □

Using Lemma 3.6 in conjunction with Lemma 3.4, we can restrict our attention to a small neighborhood of the zero section in T^*S when counting holomorphic curves with boundary on the conormal lift of transverse knots.

3.5. **Almost complex structures.** We choose $S \subset \mathbb{R}^3$ as in Subsection 3.1 with the additional requirement that S is flat near the north and south poles. More precisely, for some fixed $\delta_0 > 0$ we require that

$$S \cap \left\{ y \in \mathbb{R}^3 \colon \sqrt{y_1^2 + y_2^2} \le 3\delta_0 \right\} = \left\{ (y_1, y_2, \pm 1) \in \mathbb{R}^3 \colon \sqrt{y_1^2 + y_2^2} \le 3\delta_0 \right\}. \tag{3.4}$$

For $0 < k < 3$, write

$$E_{k\delta_0} = S \cap \left\{ y \in \mathbb{R}^3 \colon \sqrt{y_1^2 + y_2^2} \le k\delta_0 \right\} = \left\{ (y_1, y_2, \pm 1) \in \mathbb{R}^3 \colon \sqrt{y_1^2 + y_2^2} \le k\delta_0 \right\},$$

and let $H_\pm(k\delta_0)$ denote the intersection $H_\pm \cap (T^*E_{k\delta_0} \times \mathbb{R})$. Note that the metric on S is flat in $E_{3\delta_0}$ and that the almost complex structure induced by the metric agrees with the standard integrable complex structure J_0 on $T^*E_{3\delta_0} \subset \mathbb{C}^2$.

Let K be a transverse link in the ball $B(\delta)$ and Λ_K its conormal lift. As usual, let $\pi_{\mathbb{C}} \colon J^1(S) \to T^*S$ denote the projection, and write $\overline{H}_\pm = \pi_{\mathbb{C}}(H_\pm)$ and $\overline{H}_\pm(k\delta_0) = \pi_{\mathbb{C}}(H_\pm(k\delta_0))$.

Lemma 3.7. *The spherical lifts $H_\pm$ of α_0 intersect the subset $\partial(T^*E_{2\delta_0} \times \mathbb{R}) \subset J^1(S)$ transversely. Moreover, $H_\pm(2\delta_0)$ is invariant under the Reeb flow and its projection $\overline{H}_\pm(2\delta_0) \subset T^*E_{2\delta_0}$ is a smooth J_0-complex subvariety.*

Proof. By Formula (3.4), the normal vector to $E_{2\delta_0}$ is $\nu = (0,0,1)$. By the definition of the contact form α_0 and Lemma 3.2,

$$\begin{aligned} H_\pm(2\delta_0) = \big\{ (q,p,z) \in T^*S \times \mathbb{R} \colon (q,p,z) = (\pm(-x_2, x_1, 1)\,,\, (x_1, x_2, 0)\,,\, x_3) \\ \text{and } x_1^2 + x_2^2 \le 4\delta_0^2 \big\}. \end{aligned} \tag{3.5}$$

This is clearly transverse to $\partial(T^*E_{2\delta_0} \times \mathbb{R})$ and invariant under the Reeb flow (which is just translation in the z-direction). Furthermore, under the identification $((u_1, u_2, \pm 1), (v_1, v_2, 0)) \mapsto (u_1 + iv_1, u_2 + iv_2) \in \mathbb{C}^2$, $\overline{H}_\pm(2\delta_0)$ corresponds to the complex line

$$(u_1 + iv_1, u_2 + iv_2) = (i\zeta, \zeta),$$

where $\zeta = x_1 + ix_2$. □

Fix $\delta \in (0, \delta_0)$ so that Lemma 3.4 holds. As pointed out in Subsection 3.4, Lemma 3.6 implies that when studying the isotopy classification of links in $\mathbb{R}^3$ which are transverse to α, it is no restriction to assume that all such links are contained in $B(\delta)$ and that all isotopies are through links inside $B(\delta)$. We thus make this assumption throughout the rest of the paper.

Lemma 3.8. *Let K be a transverse link (by our standing assumption $K \subset B(\delta)$). Then the sets $\overline{\Lambda}_K$ and $\overline{H}_\pm$ are disjoint in T^*S. In addition, $\overline{H}_\pm \setminus \overline{H}_\pm(2\delta_0)$ does not intersect any J-holomorphic disk with boundary on $\overline{\Lambda}_K$ and one positive puncture.*

Proof. Lemmas 3.5 and 3.7 imply that $\overline{H}_\pm(2\delta_0)$ and $\overline{\Lambda}_K$ are disjoint, for if $H_\pm(2\delta_0)$ intersected a Reeb flow line emanating from Λ_K, then it would intersect Λ_K itself since $H^\pm(2\delta_0)$ is foliated by Reeb flow lines.

Lemmas 3.3 and 3.4 imply that any holomorphic curve with boundary on $\overline{\Lambda}_K$ will be contained in

$$\{(q,p) \in T^*S \colon |p| < 2\delta_0\}.$$

Write $x = (x_1, x_2, x_3)$. The contactomorphism of Lemma 3.2 and the properties of S, see Subsection 3.1, imply that $\overline{H}_\pm$ consists of the points $(p(x), q(x)) \in T^*S$ whose coordinates satisfy the following:

$$q(x) = b(-x_2, x_1, 1),$$

where $b > 0$ is such that $q(x) \in S$; and

$$p(x) = \left(x_1 - x_2 x_3 \sqrt{1 - a^2(x_1^2 + x_2^2)} \ , \ x_2 - x_1 x_3 \sqrt{1 - a^2(x_1^2 + x_2^2)} \ , \ a^2 x_3 (x_1^2 + x_2^2) \right),$$

where $a \geq 0$ is chosen so that $\nu(q(x)) = (-ax_2, ax_1, \sqrt{1 - a^2(x_1^2 + x_2^2)})$. It follows that

$$|p(x)|^2 = (x_1^2 + x_2^2)(1 + a^2 x_3^2).$$

Since the second factor is bigger than or equal to 1, the lemma clearly follows. □

Lemma 3.9. *There exists an almost complex structure J on T^*S which agrees with J_0 in a neighborhood of $\overline{H}_\pm$ and outside the region where $|p| \leq \delta_0$, and which is regular for $\overline{\Lambda}_K$ in the sense that 0- and 1-dimensional moduli spaces of holomorphic disks with boundary on Λ_K and one positive puncture are transversely cut out.*

Proof. Proposition 2.3(1) in [7] shows that the asserted regularity can be achieved by perturbing J in an arbitrary small neighborhood of the double points of $\overline{\Lambda}_K$. Since these double points lie neither on $\overline{H}_\pm$ nor in the region where $|p| \geq \delta_0$ the lemma follows. □

3.6. **A filtered DGA.** Following the discussion in Section 2, we now define a filtered version of the Legendrian contact homology DGA of Λ_K when K is a transverse link. See [7] for further background details on the unfiltered DGA.

Let $KC\mathcal{A}^-(K) = LC\mathcal{A}(\Lambda_K)$ be the graded free associative non-commutative unital algebra over the ring $\mathbb{Z}[H_1(\Lambda_K)][U, V]$ generated by the Reeb chords of Λ_K. Here U, V are two (formal) variables of grading 0. Other generators and coefficients have grading exactly as in the usual Legendrian contact homology DGA determined via a Maslov index. We denote the grading $|\cdot|$.

Consider a Reeb chord a, a monomial of Reeb chords $\mathbf{b} = b_1 \cdots b_m$, and a homology class $A \in H_1(\Lambda_K)$. Recall from (2.1) the moduli space $\mathcal{M}_A(a; \mathbf{b})$ of holomorphic disks with boundary on Λ_K, which has dimension $|a| - |\mathbf{b}| - |A| - 1$. For $u \in \mathcal{M}_A(a; \mathbf{b})$, let $n_U(u)$ and $n_V(u)$ denote the algebraic intersection of u and $\overline{H}_+$ and u and $\overline{H}_-$, respectively. Lemmas 3.7 and 3.8 imply that these counts are well-defined and non-negative for the J and K we consider.

Define the differential $\partial^- \colon KC\mathcal{A}^-(K) \to KC\mathcal{A}^-(K)$ by

$$\partial^- a = \sum_{\{u \in \mathcal{M}_A(a;\mathbf{b}) \ | \ |a| - |\mathbf{b}| - |A| = 1\}} (-1)^{|a|+1} \sigma(u)\, U^{n_U(u)} V^{n_V(u)}\, e^A\, \mathbf{b}, \tag{3.6}$$

where $\sigma(u) \in \{\pm 1\}$ is determined by the moduli space orientation; compare to Equation (2.3). Setting $U = V = 1$ we recover the differential used in the Legendrian contact homology DGA defined in [7].

Theorem 3.10. *The above definition gives a filtered differential: ∂^- does not decrease the exponents of U or V, and $(\partial^-)^2 = 0$.*

Proof. The unfiltered version of the differential squares to zero due to the usual transversality, gluing and compactness arguments [7]. The filtered version follows from the fact that n_U and n_V are nonnegative, the fact that gluing and compactness respects the filtration, and Lemmas 3.8 and 3.9. □

We call the above filtered DGA the *transverse knot DGA of K* and denote it $(KC\mathcal{A}^-(K), \partial^-)$. We next show that the filtered DGA of K is invariant under transverse isotopies of K up to filtered stable tame isomorphism. (For the definition of stable tame isomorphism, which extends to our situation, see [7] for example.) In particular, the homology of the filtered DGA $(KC\mathcal{A}^-(K), \partial^-)$ is a transverse link invariant.

Theorem 3.11. *The filtered DGA, $(KC\mathcal{A}^-(K), \partial^-)$, is invariant under transverse isotopies of K up to filtered stable tame isomorphism.*

Proof. By Lemma 3.6 we may assume that K_t, $0 \le t \le 1$ is an isotopy of transverse links inside $B(\delta)$ connecting two given transverse links. Then Λ_{K_t} is an isotopy of Legendrian submanifolds confined to the region in $J^1(S)$ where $|p| \le \delta$. To prove the invariance statement we generalize the invariance proof in [6]. We study parameterized moduli spaces and first note that as long as there are no disks of index -1 and no births/deaths of intersection points, the moduli spaces change by cobordisms and the filtered differential is unchanged. In order to deal with a critical instance t' we use a small perturbation of the trace of the fronts of the isotopies near the critical instance to create a Legendrian submanifold $\Lambda_{K_{t'}} \times \mathbb{R} \subset J^1(S \times \mathbb{R})$ and study a compact part of this manifold corresponding to $[-1, 1] \subset \mathbb{R}$. Straightforward modifications of Lemmas 3.4, 3.7, and 3.8 show that $\overline{H}_\pm \times \mathbb{C} \subset T^*(S \times \mathbb{R})$ (where we think of $T^*\mathbb{R}$ as $\mathbb{C}$) are disjoint from $\pi_{\mathbb{C}}(\Lambda_{K_{t'}} \times \mathbb{R})$ and is complex in regions where holomorphic disks with one positive puncture might exist. As above it then follows that $\overline{H}_\pm \times \mathbb{C}$ give filtrations on $LC\mathcal{A}^-(\Lambda_{K_{t'}} \times \mathbb{R})$ compatible with those induced on $LC\mathcal{A}^-(\Lambda_{K_{t'}})$. From this filtered differential we construct a filtered stable tame isomorphism by repeating, essentially verbatim, the construction of the stable tame isomorphism in the unfiltered case given in [6]. □

3.7. **Topological invariance of the infinity version in $\mathbb{R}^3$.** Before delving in depth into the technical details of the computation of the filtered DGA in $\mathbb{R}^3$ in Section 4, we give a geometric explanation of Theorem 1.2, which says that the infinity version of the transverse invariant in $\mathbb{R}^3$ is actually a topological link invariant. To simplify notation, we treat only the single-component knot case in this subsection. We give only a sketch of the proof here. A more detailed discussion in the algebraic setting can be found in [11], though the presentation here has the advantage that it explains the underlying geometric reason for this phenomenon.

Let K be an oriented transverse knot in $\mathbb{R}^3$. The homology $H_1(\Lambda_K) \cong \mathbb{Z}^2$ has a distinguished set of generators corresponding to the meridian and (0-framed) longitude of K, allowing us to identify $\mathbb{Z}[H_1(\Lambda_K)]$ with $R = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$. We can then rewrite Equation (3.6) as

$$\partial^- a = \sum_{\{u \in (\mathcal{M}_A(a;\mathbf{b})/\mathbb{R}) \;|\; \dim \mathcal{M} = 1\}} (-1)^{|a|+1} \sigma(u) U^{n_U(u)} V^{n_V(u)} \lambda^{\mathrm{long}(A)} \mu^{\mathrm{mer}(A)} \mathbf{b}, \tag{3.7}$$

where A is the linear combination of $\mathrm{long}(A)$ longitudes and $\mathrm{mer}(A)$ meridians.

As in [11], define the infinity DGA, $(KC\mathcal{A}^\infty(K), \partial^\infty)$, by tensoring $(KC\mathcal{A}^-(K), \partial^-)$ with $R[U^{\pm 1}, V^{\pm 1}]$ and replacing λ by $\lambda (U/V)^{-(\mathrm{sl}(K)+1)/2}$, where $\mathrm{sl}(K)$ is the self-linking number of K. We are now ready to prove Theorem 1.2 from the introduction.

Proof of Theorem 1.2. Let u denote a holomorphic disk contributing to the Legendrian contact homology of Λ_K. Just as we viewed the boundary of u in Subsection 2.3 as an element of $H_1(\Lambda_K)$ by appending capping paths at Reeb chords, we can view the entirety of u as an element $[u]$ of the relative homology $H_2(S^*\mathbb{R}^3, \Lambda_K)$ by appending capping surfaces at Reeb chords.

The exact sequence

$$\cdots \longrightarrow 0 \longrightarrow H_2(S^*\mathbb{R}^3) = \mathbb{Z} \longrightarrow H_2(S^*\mathbb{R}^3, \Lambda_K) \longrightarrow H_1(\Lambda_K) = \mathbb{Z}^2 \longrightarrow 0 \longrightarrow \cdots$$

implies that $H_2(S^*\mathbb{R}^3, \Lambda_K) \cong \mathbb{Z}^3$. Pick a basis s, l, m of $H_2(S^*\mathbb{R}^3, \Lambda_K)$ such that the following holds:

- $s = [S^2]$, the class of S^2 in $S^*\mathbb{R}^3 \cong \mathbb{R}^3 \times S^2$;
- l is the homology class of the conormal lift of a cooriented Seifert surface of K;
- m is the hemisphere of the S^2 fiber of $S^*\mathbb{R}^3$ over some point $p \in K$, bounded by the intersection of Λ_K with this fiber and containing the positive lift of ξ over p.

Note that under the boundary map $H_2(S^*\mathbb{R}^3, \Lambda_K) \longrightarrow H_1(\Lambda_K)$, the classes s, l, m map to 0, the longitude, and the meridian, respectively. Now intersecting with $\overline{H}_\pm$ defines linear maps

$$n_U, n_V : H_2(S^*\mathbb{R}^3, \Lambda_K) \to \mathbb{Z}$$

such that $n_U(s) = n_V(s) = 1$, $n_U(m) = 1$, and $n_V(m) = 0$. By the definition of self-linking number, the difference $n_U(l) - n_V(l)$ is $\mathrm{sl}(K)$; by adding the appropriate multiple of s to l, we may assume that $n_U(l) = \frac{\mathrm{sl}(K)-1}{2}$ and $n_V(l) = -\frac{\mathrm{sl}(K)+1}{2}$.

If we write $[u] = s(u)s + l(u)l + m(u)m$ for $s(u), l(u), m(u) \in \mathbb{Z}$, then $n_U[u] = s(u) + \frac{\mathrm{sl}(K)-1}{2}l(u) + m(u)$ and $n_V[u] = s(u) - \frac{\mathrm{sl}(K)+1}{2}l(u)$. From Equation (3.7), the contribution of u to the differential ∂^∞ has coefficient

$$\begin{aligned}(-1)^{|a|+1}\sigma(u)U^{n_U[u]}V^{n_V[u]}(\lambda(U/V)^{-(\mathrm{sl}(K)+1)/2})^{l(u)}\mu^{m(u)} \\ = (-1)^{|a|+1}\sigma(u)(\lambda/U)^{l(u)}(\mu U)^{m(u)}(UV)^{s(u)}.\end{aligned}$$

Now in the definition of the unfiltered Legendrian contact homology of Λ_K, which is a topological knot invariant, one could use the coefficient ring $\mathbb{Z}[H_2(S^*\mathbb{R}^3, \Lambda_K)]$ rather than $\mathbb{Z}[H_1(\Lambda_K)]$. If one writes $\tilde{\lambda}, \tilde{\mu}, \tilde{\sigma}$ for the multiplicative generators of $\mathbb{Z}[H_2(S^*\mathbb{R}^3, \Lambda_K)]$ corresponding to l, m, s, then the contribution of u to Legendrian contact homology with this enhanced coefficient ring is

$$(-1)^{|a|+1}\sigma(u)\tilde{\lambda}^{l(u)}\tilde{\mu}^{m(u)}\tilde{\sigma}^{s(u)}.$$

But this is precisely the coefficient of the contribution of u to ∂^∞, once we make the global substitutions $\tilde{\lambda} = \lambda/U$, $\tilde{\mu} = \mu U$, $\tilde{\sigma} = UV$.

It follows that this global substitution turns $(KC\mathcal{A}^\infty(K), \partial^\infty)$ into the unfiltered Legendrian contact homology DGA of Λ_K with coefficients in $\mathbb{Z}[H_2(S^*\mathbb{R}^3, \Lambda_K)]$. Since the latter is a topological invariant, the result follows. □

We remark that our choice of basis (l, m, s) for $H_2(S^*\mathbb{R}^3, \Lambda_K)$ in the proof of Theorem 1.2 is canonical, depending only on the topological type of K; as a consequence, the (stable tame) isomorphisms on Theorem 1.2 act as the identity, not just an isomorphism, on the base ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}, U^{\pm 1}, V^{\pm 1}]$. This is clear for m, which can be defined using the orientation on K rather than the contact structure, and for s. For l, suppose that there is a topological isotopy K_t between two transverse knots K_0 and K_1, with a corresponding isotopy Σ_t of Seifert surfaces, such that K_t fails to be a transverse knot at finitely many moments. At these moments, the self-linking number may jump, generically by ± 2, but $n_U([\Sigma_t])$ and $n_V([\Sigma_t])$ also each jump by ± 1. Thus the quantity $n_U([\Sigma_t]) - \frac{\mathrm{sl}(K_t)-1}{2}$ remains unchanged during the isotopy. Since l (which is $[\Sigma_t]$ plus some multiple of s) is chosen so that $n_U(l) = \frac{\mathrm{sl}(K_t)-1}{2}$, the isotopy K_t preserves l.

4. Computing the Filtered DGA of a Transverse Link

In this section we compute the filtered DGA of a transverse link to prove Theorem 1.3. We begin by describing the main strategy used in [4] for calculating knot contact homology. This description leads to a sufficient understanding of the behavior of all holomorphic disks needed for the calculation of the filtered differential.

4.1. **Scheme for calculating knot contact homology.** The calculation of knot contact homology in [4] proceeds as follows. Consider a link K braided around the unknot U with n strands and view it as a multisection of a fibration $U \times D^2$, where D^2 is the 2-disk, corresponding to a tubular neighborhood of U. As we degenerate the multisection toward U the conormal lift Λ_K approaches the conormal lift of the unknot Λ_U (with multiplicity n). More precisely, a neighborhood of $\Lambda_U \subset S^*\mathbb{R}^3$ is contactomorphic to the 1-jet space $J^1(\Lambda_U)$ of Λ_U and for K sufficiently close to U, Λ_K is a multisection of $J^1(\Lambda_U) \to \Lambda_U$ with n sheets. In other words, over any open disk $W \subset \Lambda_U$, Λ_K is given by the 1-jet extension of n functions $F_j \colon W \to \mathbb{R}$, $j = 1, \dots, n$.

It will be useful to recall that when a Legendrian submanifold Λ in $J^1(S^2)$ is given locally as the 1-jet of n functions F_i then its Reeb chords (that is double points of $\pi_{\mathbb{C}}(\Lambda)$) correspond to critical points of the difference of these local functions $F_i - F_j$.

Recall that the Legendrian contact homology algebra $KC\mathcal{A}(K)$ of Λ_K is generated by Reeb chords and that its differential is defined by a count of holomorphic disks. Close to the limit as Λ_K approaches Λ_U we have the following. The Reeb chords of Λ_K are of two types.

- **I**. Near each Reeb chord ch of Λ_U there are n^2 Reeb chords ch_{ij}, $1 \le i, j \le n$ of Λ_K, where we write ch_{ij} for the chord near ch that starts on the j^{th} local sheet of Λ_K near the start point of ch and ends at the i^{th} local sheet near the endpoint of ch.
- **II**. There are $n(n-1)$ small Reeb chords corresponding to critical points of positive local function differences of the form $F_i - F_j$. These critical points are either maxima or saddle points. We denote the former by b_{ij} and the latter by a_{ij}, where $1 \le i, j \le n$, $i \ne j$, and we use the same notational conventions for subscripts as above.

Thus Reeb chords of Λ_K are either Reeb chords of Λ_U with a small chord added or subtracted or small Reeb chords entirely inside the neighborhood of Λ_U.

One of the main technical results of [4] (see [5] for a similar result) shows that holomorphic disks admit a similar description: near the limit as Λ_K approaches Λ_U rigid holomorphic disk in T^*S with boundary on $\overline{\Lambda}_K = \pi_{\mathbb{C}}(\Lambda_K)$ and one positive puncture are of two types:

- **I**. either they lie in an arbitrarily small neighborhood of a disk with one positive puncture and boundary on Λ_U and flow trees contained in a small neighborhood of Λ_U determined by the function differences $F_i - F_j$ attached along its boundary;
- **II**. or, they lie entirely inside a small neighborhood of Λ_K and are determined by flow trees of the functional differences $F_i - F_j$.

Furthermore, any disk with its positive puncture at a chord of type **I** (resp. **II**) is of type **I** (resp. **II**). For the notion of flow trees we refer to [2, Section 2]. We do not give a complete definition of flow trees here, but merely note that they are made from pieces of flow lines of the functional differences $F_i - F_j$ and that as Λ_K collapses onto Λ_U the corresponding holomorphic disks stay in smaller and smaller neighborhoods of Λ_U.

In conclusion, in order to compute the differential $\partial \colon KC\mathcal{A}(K) \to KC\mathcal{A}(K)$ we need to understand holomorphic disks with boundary on $\overline{\Lambda}_U$ and flow trees determined by $F_i - F_j$. The latter can be understood using finite dimensional Morse theory. In order to understand the former we use the correspondence between holomorphic disks and flow trees on fronts from [2].

4.2. **The conormal lift of the unknot.** One may easily compute (or see [3, 4] and Lemma 4.1 below) the conormal lift Λ_U of the unknot U represented as the circle of radius $r < \delta$ in the x_1x_2-plane. This lift can be slightly perturbed in $J^1(S)$ so that it has two Reeb chords denoted c and e, see Figure 1.

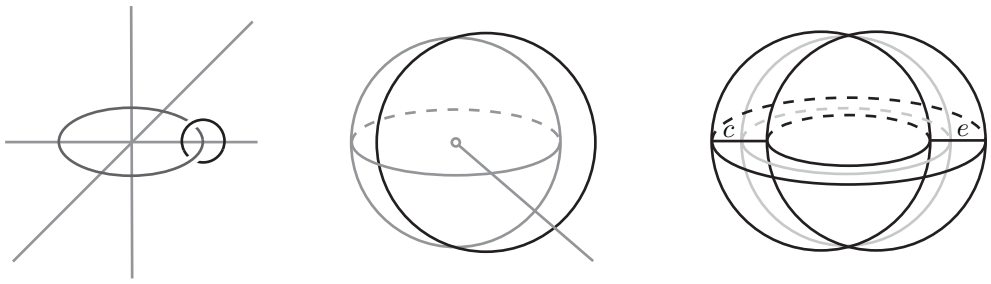

FIGURE 1. On the left is $\mathbb{R}^3$ with the unknot in the xy-plane and a normal circle at one point is shown. The middle figure shows the normal circle seen in $S^2 \times \mathbb{R} \cong \mathbb{R}^3 - \{(0,0,0)\}$ which is thought of as the (front) projection of $S^*\mathbb{R}^3 \cong J^1(S^2)$ to $S^2 \times \mathbb{R}$. On the right is the front projection of the entire conormal lift of the unknot which is obtained by rotating the circle shown in the middle figure about the line through the north and south poles. We have also slightly perturbed the picture on the right so that there are only two Reeb chords, labeled c and e in the figure.

For purposes of finding holomorphic disks via flow trees, Λ_U must be perturbed to be in general position with respect to the front projection into $S^2 \times \mathbb{R}$. Notice that in Figure 1 there is a circle in Λ_U that is mapped to the north pole and a circle mapped to the south pole. Since Λ_U is already in general position outside the fibers over the north and south pole of S we concentrate our attention there. Near the poles, the projection $\overline{\Lambda}_U$ looks as follows.

Lemma 4.1. *If U is the circle of radius $r < \delta$ in the x_1x_2-plane, then*

$$\overline{\Lambda}_U \cap T^*E_{2\delta_0} = \{(q,p)\colon q = (\xi_1, \xi_2, \pm 1),\ p = (x_1, x_2, 0)\} \approx S^1 \times (-1,1) \times \{\pm 1\},$$

where $\sqrt{x_1^2 + x_2^2} = r$ and where $-\xi_1 x_2 + \xi_2 x_1 = 0$.

Proof. This is immediate from the definition. □

In Figure 2 we see the front projection of Λ_U over the region where S is flat. The left pictures show Λ_U in a neighborhood of the circle over the north (or south) pole, as described in Lemma 4.1, and the corresponding cone in the front projection of Λ_U. The right pictures show Λ_U after small perturbation near the cone point that makes the front projection generic. Using the right representation we get the following result describing holomorphic disks of Λ_U. See Figure 5 for a description on the flow trees on the front and Figure 3 for a description of their lifts into Λ_U.

Lemma 4.2 (Ekholm–Etnyre–Ng–Sullivan 2010, [4])**.** *There are exactly six rigid holomorphic disks with boundary on $\overline{\Lambda}_U$: four (I_N, Y_N, I_S, and Y_S) with positive puncture at c, and two (E_1,E_2) with positive puncture at e and negative puncture at c. There are exactly four 1-parameter families of holomorphic disks with positive puncture at e: $\widetilde{I}_N$, $\widetilde{Y}_N$, $\widetilde{I}_S$, and $\widetilde{Y}_S$. The boundaries of these 1-parameter families are as follows:*

$$\partial\widetilde{I}_N = (E_1 \# I_N) \cup (E_2 \# I_N), \quad \partial\widetilde{Y}_N = (E_1 \# Y_N) \cup (E_2 \# Y_N),$$
$$\partial\widetilde{I}_S = (E_1 \# I_S) \cup (E_2 \# I_S), \quad \partial\widetilde{Y}_S = (E_1 \# Y_S) \cup (E_2 \# Y_S).$$

Here $E_1 \# I_N$ denotes the broken disk obtained by adjoining I_N to E_1 etc., see Figure 4.

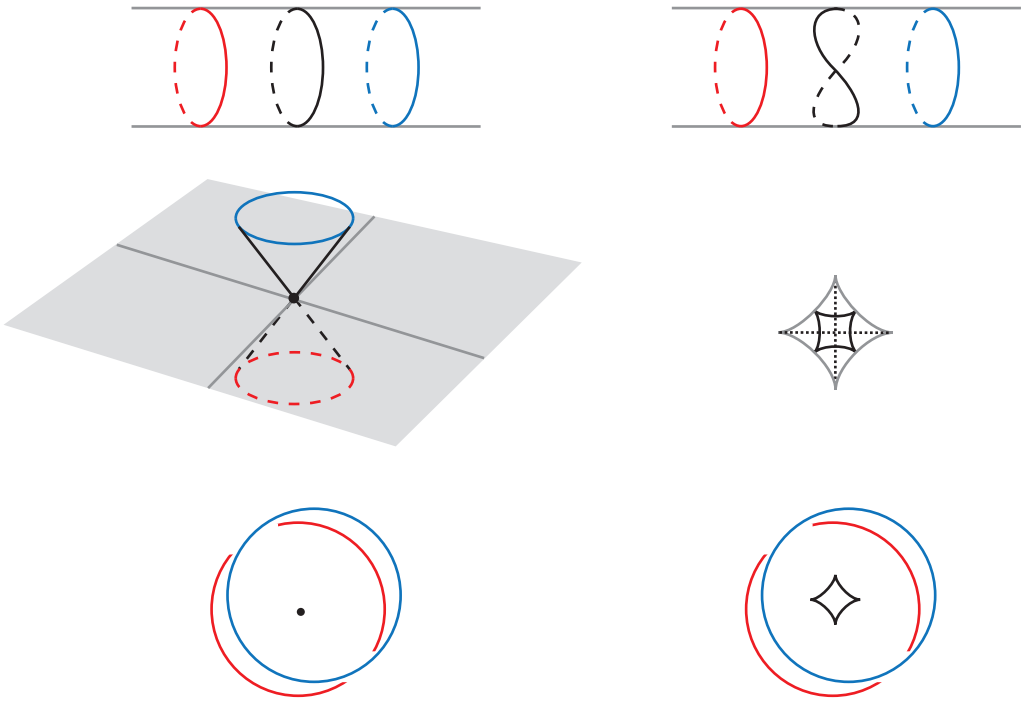

FIGURE 2. Along the top of the figure is an annular neighborhood of the circle in Λ_U that maps to the fiber above the north pole. On the middle left, we see the image of this annulus near the north pole in the front projection, a cone whose boundary is two circles. On the bottom left is the image of this annulus near the north pole in S^2 (that is, the top view of the cone where we have slightly offset the circles so that they are both visible). On the middle right, we see the top view of the cone after it has been perturbed to have a generic front projection. More specifically, the lighter outer curve is the image of the cusp curves, the dotted lines are the image of double points in the front projection and the darkest inner curve is the image of the circle that mapped to the cone point before the perturbation. On the bottom right, we see the image in S^2 of the cusp curve and the two boundary circles on Λ_U.

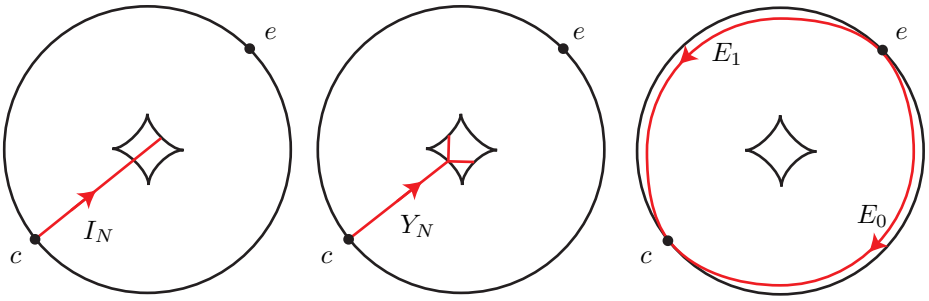

FIGURE 3. The rigid disks from Lemma 4.2 in the northern hemisphere of S^2.

Using the convention of [4]*, the disks* I_N, Y_N, I_S, Y_S *contribute* $1, \lambda, \lambda\mu, \mu$*, respectively, to* ∂c*. Similarly* E_1 *and* E_2 *contribute* $-\lambda^{-1}c$ *and* $\lambda^{-1}c$*, respectively, to* ∂e*. This gives the differential*

$$\partial c = 1 + \lambda + \lambda\mu + \mu$$
$$\partial e = (-\lambda^{-1} + \lambda^{-1})c = 0.$$

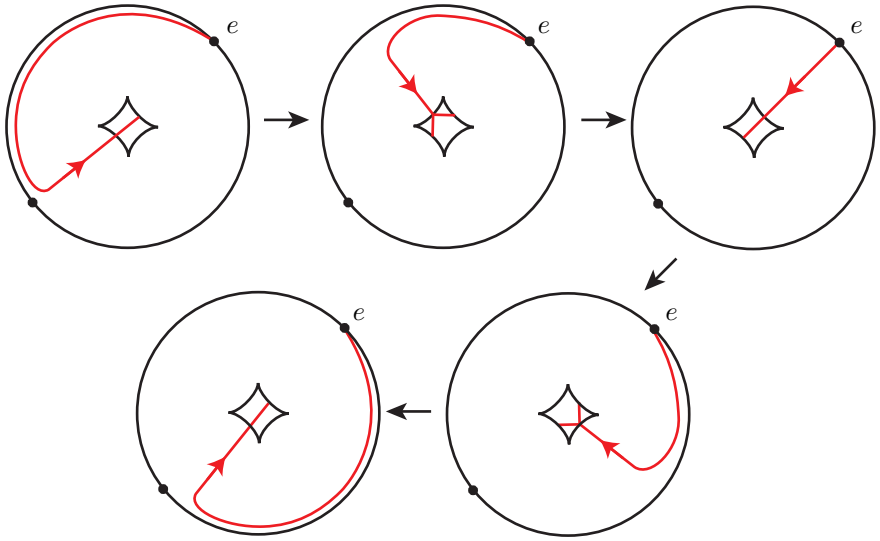

FIGURE 4. The one dimensional families of disks from Lemma 4.2 in the northern hemisphere of S^2.

□

Now that we have recalled the computation of the knot contact homology of the standard unknot in the x_1x_2-plane, we turn to computing the filtration on this knot thought of as a transverse knot.

4.3. **Intersection numbers for disks with boundary on $\overline{\Lambda}_U$.** We first note that Lemma 3.8 implies that any point in the intersection between $\overline{H}_\pm$ and a holomorphic disk with boundary on Λ_U must lie in $T^*E_{2\delta_0}$ so it will be sufficient to consider the parts of the holomorphic disks which lie in this region. As we shrink the unknot U towards the x_3 axis in $\mathbb{R}^3$, its conormal lift Λ_U approaches the 0-section in $J^1(S)$. We can also see that the perturbed front generic version of Λ_U collapses onto the 0-section. As we degenerate Λ_U onto the 0-section, the boundaries of the holomorphic disks converge to the curves on the torus Λ_U depicted in Figure 5.

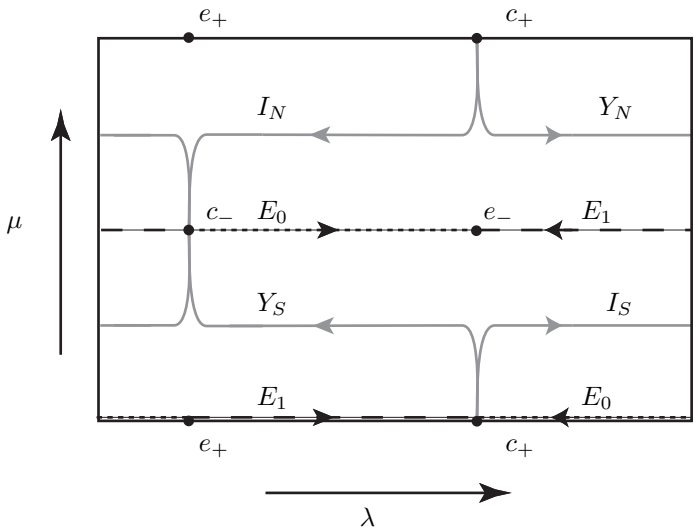

FIGURE 5. Boundaries of holomorphic disks on Λ_U.

The Legendrian torus Λ_U can be described as the 1-jet of a multifunction from S^2 to $\mathbb{R}$ and we notice that the curves in the figure are given by gradient flow lines for the functional differences of the multifunction. Furthermore, for an appropriate almost complex structure, the holomorphic disks C^1-converge to the strips corresponding to these flow lines outside any fixed neighborhood of its vertices, see [2, Lemma 5.13, Remark 5.14, and Subsection 6.4], and inside neighborhoods of its vertices they converge to other local models, see [2, Subsection 6.1]. Here the strip of a flow line consists of the line segments in the cotangent fibers between its cotangent lift. In particular, if we choose a perturbation of Λ_U so that its projection to T^*S consists of affine subspaces in a neighborhood of $(0,0,1) \in S$ then the standard complex structure for which $\overline{H}_\pm$ is a complex hypersurface is appropriate in the above sense, see [2, Subsection 4.1, 2nd bullet from the end].

Consider the flow-line disks depicted in Figure 6. These flow lines lie in the two distinct homotopy classes of the disks I_N and Y_N. (Which of these disks looks like the letter I or Y depends on the perturbation of Λ_U which makes its front generic, see also Figure 4.) The disks I_1 and I_2 correspond

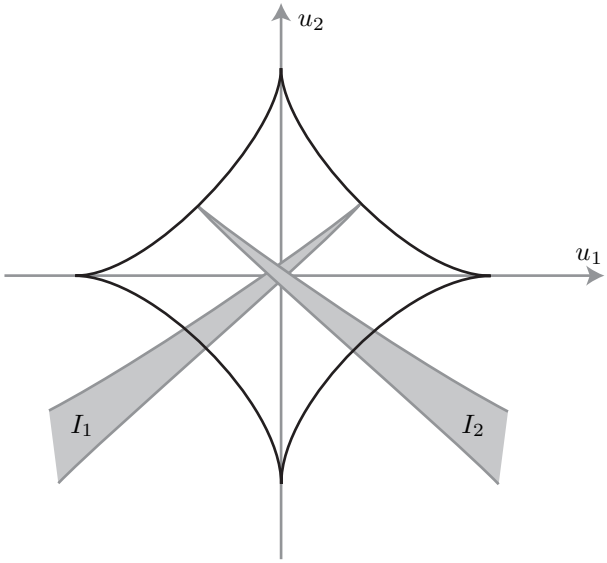

FIGURE 6. Two holomorphic disks in distinct homotopy classes. We have chosen coordinates on S^2 near the north pole so that the north pole corresponds to $(0,0)$.

to flow lines of the difference of two functions that locally describe part of Λ_U near the north pole. With notation and coordinates near the north pole of S^2 indicated in Figure 6, the gradient of the function describing the upper sheet of I_1 equals $-\partial_{u_1}$ and that of its lower sheet is ∂_{u_2}, and the flow line corresponding to I_1 in S follows the curve $C_1 = (t,t)$, $-T \le t \le \epsilon$ where $\epsilon > 0$ is arbitrarily small. Thus the flow line strip of I_1 is

$$(t,s) \mapsto ((t,t), (s(-\partial_{u_1}) + (1-s)\partial_{u_2})), \quad -T \le t \le \epsilon,\ 0 \le s \le 1,$$

near the origin. Similarly, the flow line strip of I_2 is

$$(t,s) \mapsto ((-t,t), (s\partial_{u_1} + (1-s)\partial_{u_2})), \quad -T \le t \le \epsilon,\ 0 \le s \le 1.$$

On the other hand the intersection of $\overline{H}_+$ with the fibers of T^*S over the point $(t,t) \in C_1$ is $a(t\partial_{u_1} - t\partial_{u_2})$ for some $a > 0$, see Equation (3.5). That is,

$$\overline{H}_+ \cap T^*C_1 = \{((t,t), a(t\partial_{u_1} - t\partial_{u_2}))\},$$

and we see that the strip and the hypersurface intersect once (over the point $(-\frac{1}{2a}, -\frac{1}{2a})$). Similarly, over the flow line of I_2 we have

$$\overline{H}_+ \cap T^*C_1 = \{((-t,t), a(t\partial_{u_1} + t\partial_{u_2}))\},$$

and we see that the strip and the hypersurface do not intersect: the solution of the equation which sets the fiber coordinates equal would lie over the point $(-t,t)$ where $t=\frac{1}{2a}$ but as $\epsilon>0$ is arbitrarily small this point does not lie on the flow line of I_2.

In order to relate the above calculation to Lemma 4.2, consider the disk $D_\pm$ in the fiber of T^*S bounded by the circle $((0,0,\pm 1),p)\in \overline{\Lambda}_U\cap T^*E_{2\delta_0}$ and oriented according to the induced orientation on the fiber. We notice that this circle is a lift of a longitude for U. Using the description of $\overline{H}_\pm$ in Lemma 3.7 it is easy to see that $D_\pm$ intersects $\overline{H}_\pm$ with intersection number ∓ 1. (Note here that the orientation of the base followed by the orientation of the fiber gives the orientation opposite to the complex orientation on T^*S and that the orientation induced on the normal bundle to the fiber by $\overline{H}_+$, respectively $\overline{H}_-$, agrees, respectively disagrees, with the orientation on the base.) Consulting Figure 5 and considering the algebraic topology of $\Lambda_U\subset T^*S^2$ one may easily see that the difference cycle between the disk corresponding to I_N and Y_N is homologous to D_+. Since the intersection number of these disks with $\overline{H}_+$ equals 1 or 0 (by the above calculation) it follows that the disk corresponding to I_N has intersection number 1 with $\overline{H}_+$ and the disk corresponding to Y_N has intersection number 0. Similarly, the disks corresponding to I_S and Y_S intersect $\overline{H}_-$ with intersection number 1 and 0, respectively. In addition, since $D_\mp$ does not intersect $\overline{H}_\pm$, we find that the I_S and Y_S (respectively I_N and Y_N) disks do not intersect $\overline{H}_+$ (respectively $\overline{H}_-$).

The following result follows from the computation of the knot contact homology differential for U in Lemma 4.2 and the computations of the intersection numbers between holomorphic disks and $\overline{H}_\pm$ above.

Theorem 4.3. *Let U be the transverse unknot with self-linking number -1. The filtered DGA $(KC\mathcal{A}^-(U),\partial^-)$ is filtered stable tame isomorphic to the algebra over $R[U,V]$ generated by c and e, where $|c|=1$ and $|e|=2$, and*

$$\partial^- c = U+\lambda+\lambda\mu V+\mu \quad \text{and} \quad \partial^- e = 0.$$

□

4.4. **The filtered DGA of a transverse link.** We begin by recalling the computation of the knot contact homology from [4]. Using the notation in the introduction we have the following result.

Theorem 4.4 (Ekholm–Etnyre–Ng–Sullivan 2010, [4])**.** *The differential in the Legendrian DGA associated to the conormal lift Λ_K of a framed knot K is $(KC\mathcal{A}(K),\partial)$ where $KC\mathcal{A}(K)$ is generated over $\mathbb{Z}[\mu^\pm,\lambda^\pm]$ by the a_{ij},b_{ij},c_{ij}, and e_{ij} described in Subsection 4.1 and 4.2 and the map $\partial\colon KC\mathcal{A}(K)\to KC\mathcal{A}(K)$ is determined by the following matrix equations:*

$$\begin{aligned}
\partial\mathbf{A} &= 0,\\
\partial\mathbf{B} &= -\boldsymbol{\lambda}^{-1}\cdot\mathbf{A}\cdot\boldsymbol{\lambda} \;+\; \Phi_B^L\cdot\mathbf{A}\cdot\Phi_B^R,\\
\partial\mathbf{C} &= \mathbf{A}\cdot\boldsymbol{\lambda} \;+\; \mathbf{A}\cdot\Phi_B^R,\\
\partial\mathbf{E} &= \mathbf{B}\cdot(\Phi_B^R)^{-1} \;+\; \mathbf{B}\cdot\boldsymbol{\lambda}^{-1} \;-\; \Phi_B^L\cdot\mathbf{C}\cdot\boldsymbol{\lambda}^{-1} \;+\; \boldsymbol{\lambda}^{-1}\cdot\mathbf{C}\cdot(\Phi_B^R)^{-1},
\end{aligned}$$

where $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{E},\boldsymbol{\lambda},\Phi_B^L,\Phi_B^R$ are as in Subsection 1.2, and if $\mathbf{M}$ is an $(n\times n)$-matrix, the matrix $\partial\mathbf{M}$ is defined by $(\partial\mathbf{M})_{ij}=\partial\mathbf{M}_{ij}$. □

In Subsection 4.1 above we discussed the holomorphic curves involved in the computation of the differential. In particular, one may easily conclude the following result.

Lemma 4.5 (Ekholm–Etnyre–Ng–Sullivan 2010, [4])**.** *Given any $\delta'>0$, Λ_K can be Legendrian isotoped to be close enough to Λ_U so that any holomorphic disk with boundary on $\overline{\Lambda}_K$, one positive puncture, and involving only the chords a_{ij} and b_{ij}, has its image contained within a δ'-neighborhood of $\overline{\Lambda}_U$.*

Proof. While this lemma follows directly from the results in [4], we comment that it can also be seen by observing that the Reeb chords a_{ij} and b_{ij} have small action. This action can be made arbitrarily small as we isotop K to be close to U. Now a monotonicity argument will confine the holomorphic curves to stay close to $\overline{\Lambda}_U$. □

To compute the filtration on a transverse knot we will need to explicitly describe the holomorphic disks used in the computation of the differential. More specifically we need to understand disks of Type I, discussed in Subsection 4.1. To this end we summarize the computations from [4].

Theorem 4.6 (Ekholm–Etnyre–Ng–Sullivan 2010, [4])**.** *Given any $\delta' > 0$, Λ_K can be Legendrian isotoped to be close enough to Λ_U so that any holomorphic disk with boundary on $\overline{\Lambda}_K$ has its image contained within a δ'-neighborhood of $\overline{\Lambda}_U$ and one of the disks described in Lemma 4.2.*

Specifically, the holomorphic disks that contribute to the terms in ∂c_{ij} satisfy

- *the terms in $\mathbf{A}\boldsymbol{\lambda}$ with μ coefficients are contained in neighborhoods of I_S (and $\overline{\Lambda}_U$),*
- *the terms in $\mathbf{A}\boldsymbol{\lambda}$ without μ coefficients are contained in neighborhoods of Y_N (and $\overline{\Lambda}_U$),*
- *the terms in $\mathbf{A}\Phi_B^R$ with μ coefficients are contained in neighborhoods of Y_S (and $\overline{\Lambda}_U$), and*
- *the terms in $\mathbf{A}\Phi_B^R$ without μ coefficients are contained in neighborhoods of I_N (and $\overline{\Lambda}_U$).*

The holomorphic disks that contribute to the $\Phi_B^L \cdot \mathbf{C} \cdot \boldsymbol{\lambda}^{-1} + \boldsymbol{\lambda}^{-1} \cdot \mathbf{C} \cdot (\Phi_B^R)^{-1}$ terms in ∂e_{ij} are contained in neighborhoods of either E_0 or E_1 (and $\overline{\Lambda}_U$).

The holomorphic disks that contribute to the $\mathbf{B} \cdot (\Phi_B^R)^{-1} + \mathbf{B} \cdot \boldsymbol{\lambda}^{-1}$ terms of ∂e_{ij} are contained in neighborhoods of one of the disk in the 1-parameter families $\widetilde{I}_N$, $\widetilde{Y}_N$, $\widetilde{I}_S$, and $\widetilde{Y}_S$ (and $\overline{\Lambda}_U$). That is, they are close to the union of $\overline{\Lambda}_U \cup E_1 \cup E_2$ and one of the disks I_N, Y_N, I_S, Y_S, shifted in the λ-direction some fixed distance (the same distance for all disks). More precisely,

- *disks associated to those terms in $\mathbf{B}\boldsymbol{\lambda}^{-1}$ with μ coefficients correspond to Y_S,*
- *disks associated to those terms in $\mathbf{B}\boldsymbol{\lambda}^{-1}$ without μ coefficients correspond to I_N,*
- *disks associated to those terms in $\mathbf{B}\left(\Phi_B^R\right)^{-1}$ with μ coefficients correspond to I_S, and*
- *disks associated to those terms in $\mathbf{B}\left(\Phi_B^R\right)^{-1}$ without μ coefficients correspond to Y_N.*

□

We are now in a position to prove the combinatorial expression for the filtrations given in Theorem 1.3.

Proof of Theorem 1.3. The first equation in the theorem follows for grading reasons.

The second equation in the theorem follows from Theorem 4.4 and the fact that none of the holomorphic disk contributing to the differential of a b_{ij} can intersect $\overline{H}_\pm$ by Lemma 4.5.

Since the disks E_0 and E_1 lie close to the equator, Lemma 3.8 and the third paragraph of Theorem 4.6 imply that there are no intersections of $\overline{H}_\pm$ with the disks with positive puncture at an e_{ij}-chord and one negative puncture at an c_{ij}-chord. Thus, $\Phi_B^L \cdot \mathbf{C} \cdot \boldsymbol{\lambda}^{-1} + \boldsymbol{\lambda}^{-1} \cdot \mathbf{C} \cdot (\Phi_B^R)^{-1}$ appears in both $\partial\mathbf{E}$ and $\partial^-\mathbf{E}$.

The remainder of the differentials are affected by this filtration. The effect is easily derived by the intersections of $\overline{H}_\pm$ with the disk from Lemma 4.2 which was worked out in Subsection 4.3. Some simple bookkeeping yields the desired computation. □

5. Some Examples

In this section, we present some computations of the filtered transverse knot DGA, $(KC\mathcal{A}^-, \partial^-)$. The first shows that the additional filtration structure is nontrivial; the second is an outline of a computation that shows that the filtered DGA is an effective invariant of transverse knots.

5.1. **The unknot.** We compute the transverse knot DGA for three versions of the unknot, the closure of the trivial 1-braid and the closure of the 2-braids σ_1 and σ_1^{-1}. The first two both represent the standard transverse unknot in $\mathbb{R}^3$ with self-linking number -1, while the third represents the transverse unknot with self-linking number -3. We will show that the filtered DGAs for the first two are stable tame isomorphic and are distinct from the filtered DGA for the third.

For the trivial 1-braid, Theorem 1.3 or 4.3 yields a DGA with two generators c, e and differential

$$\partial^- c = U + \lambda + \lambda\mu V + \mu, \quad \partial^- e = 0.$$

For the 2-braid σ_1, the relevant matrices are

$$\Phi^L_{\sigma_1} = \begin{pmatrix} -a_{21} & 1 \\ 1 & 0 \end{pmatrix}, \quad \Phi^R_{\sigma_1} = \begin{pmatrix} -a_{12} & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda} = \begin{pmatrix} \lambda\mu & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1+\mu & a_{12} \\ \mu a_{21} & 1+\mu \end{pmatrix}, \quad \mathbf{A}^U = \begin{pmatrix} U+\mu & Ua_{12} \\ \mu a_{21} & U+\mu \end{pmatrix}, \quad \mathbf{A}^V = \begin{pmatrix} 1+\mu V & a_{12} \\ \mu V a_{21} & 1+\mu V \end{pmatrix},$$

$$\mathbf{B}^U = \begin{pmatrix} 0 & Ub_{12} \\ \mu b_{21} & 0 \end{pmatrix}, \quad \mathbf{B}^V = \begin{pmatrix} 0 & b_{12} \\ \mu V b_{21} & 0 \end{pmatrix}.$$

We then calculate from Theorem 1.3 that $\partial^-(a_{12}) = \partial^-(a_{21}) = 0$,

$$\partial^- \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\lambda\mu}a_{12} - a_{21} \\ -\lambda\mu a_{21} - a_{12} & 0 \end{pmatrix},$$

$$\partial^- \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} \lambda\mu + \lambda\mu^2 V - \mu a_{12} & \mu + U + a_{12} \\ \mu + U + \lambda\mu^2 V a_{21} - \mu a_{21} a_{12} & 1 + \mu V + \mu a_{21} \end{pmatrix},$$

$$\partial^- \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} b_{12} + \frac{1}{\lambda\mu}(c_{12} - c_{21} + a_{21}c_{11}) & Ub_{12} - c_{22} + a_{21}c_{12} + b_{12}a_{12} + \frac{1}{\lambda\mu}(c_{11} + c_{12}a_{12}) \\ c_{22} + \frac{1}{\lambda}b_{21} - \frac{1}{\lambda\mu}c_{11} & \mu V b_{21} - c_{12} + c_{21} + c_{22}a_{12} \end{pmatrix}.$$

If we apply successively the eight tame automorphisms

$$\begin{aligned}
e_{12} &\mapsto e_{12} - b_{12}c_{12} - \tfrac{1}{\lambda\mu}c_{12}^2 \\
c_{21} &\mapsto c_{21} - \mu V b_{21} + c_{12} - c_{22}a_{12} \\
b_{12} &\mapsto b_{12} - \tfrac{1}{\mu}c_{22} + \tfrac{1}{\lambda\mu^2}c_{11} \\
b_{21} &\mapsto b_{21} + \tfrac{1}{\mu}c_{11} - \lambda c_{22} \\
e_{11} &\mapsto e_{11} + \tfrac{1}{\lambda\mu^2}c_{22}c_{11} - \tfrac{1}{\mu}e_{12} - \tfrac{1}{\lambda\mu}e_{22} + Ve_{21} \\
c_{12} &\mapsto c_{12} - \tfrac{1}{\mu}c_{11} \\
a_{21} &\mapsto a_{21} - V - \tfrac{1}{\mu} \\
a_{12} &\mapsto a_{12} + \lambda + \lambda\mu V,
\end{aligned}$$

then we obtain a DGA with the same generators but differential

$$\partial^-(c_{11}) = -\mu a_{12}, \qquad \partial^-(c_{12}) = U + \lambda + \lambda\mu V + \mu, \qquad \partial^-(c_{22}) = \mu a_{21},$$
$$\partial^-(e_{12}) = -\mu b_{12}, \qquad \partial^-(e_{21}) = \tfrac{1}{\lambda}b_{21}, \qquad \partial^-(e_{22}) = c_{21},$$
$$\partial^-(a_{12}) = \partial^-(a_{21}) = \partial^-(b_{12}) = \partial^-(b_{21}) = \partial^-(c_{21}) = \partial^-(e_{11}) = 0.$$

Destabilizing yields a DGA generated by c_{12}, e_{11} with differential $\partial^-(c_{12}) = U + \lambda + \lambda\mu V + \mu$, $\partial^-(e_{11}) = 0$, which agrees with the DGA for the trivial 1-braid above.

For the 2-braid σ_1^{-1}, we use the matrices

$$\Phi^R_{\sigma_1^{-1}} = \begin{pmatrix} 0 & 1 \\ 1 & -a_{21} \end{pmatrix}, \quad \boldsymbol{\lambda} = \begin{pmatrix} \lambda\mu^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

The expression for $\partial^- \mathbf{C}$ from Theorem 1.3 now yields in particular

$$\partial^-(c_{11}) = \lambda\mu^{-1} + \lambda V + U a_{12}.$$

To see that the filtered DGA for σ_1^{-1} differs from the filtered DGA for the trivial 1-braid, note that if we set $(U,V) = (0,0)$ in the former to obtain $(\widehat{\widehat{KC\mathcal{A}}}, \widehat{\widehat{\partial}})$, we find that $\widehat{\widehat{\partial}}(c_{11}) = \lambda\mu^{-1}$, a unit in $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$, and so the homology of $(\widehat{\widehat{KC\mathcal{A}}}, \widehat{\widehat{\partial}})$ is trivial. On the other hand, if we set $(U,V) = (0,0)$ with the trivial 1-braid, we obtain a DGA generated by c, e with $\widehat{\widehat{\partial}}(c) = \lambda + \mu$ and $\widehat{\widehat{\partial}}(e) = 0$, and it is clear that this DGA has nontrivial homology.

5.2. **The knot $m(7_6)$.** Let K_1, K_2 be the transverse knots given by the closures of the 4-braids

$$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_3^3, \qquad \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_3^3\sigma_2\sigma_3^{-1},$$

respectively. These are both transverse representatives of the mirror of the knot 7_6, with self-linking number -1. The following result demonstrates that the filtered DGA is an effective invariant of transverse knots.

Theorem 5.1. *The filtered DGAs for K_1 and K_2 are not filtered stable tame isomorphic, and thus K_1 and K_2 are not transversely isotopic.*

Proof. Consider the DGAs $(\widehat{KC\mathcal{A}}(K_i), \widehat{\partial})$ over $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ obtained from the filtered DGAs by setting $(U,V) = (0,1)$. One can count by computer the number of DGA maps (augmentations) from $(\widehat{KC\mathcal{A}}(K_i), \widehat{\partial})$ to $(\mathbb{Z}/3, 0)$ that send λ to -1 and μ to $+1$; there are 5 for K_1 and 0 for K_2. The result follows. (See [11] for more details.) □

As noted in [11], the hat version of knot Floer homology for $m(7_6)$ is 0 in the relevant bidegree $(0,0)$, and so the transverse invariants in knot Floer homology [10, 15] do not distinguish K_1 and K_2. One similarly finds that other previously developed transverse invariants (in Khovanov or Khovanov–Rozansky homology, for instance) do not distinguish K_1 and K_2. We conclude that the transverse invariant from knot contact homology is independent of previously known transverse invariants.

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