

AN INDEX FOR CLOSED ORBITS IN BELTRAMI FIELDS

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ABSTRACT. We consider the class of Beltrami fields (eigenfields of the curl operator) on three-dimensional Riemannian solid tori: such vector fields arise as steady incompressible inviscid fluids and plasmas. Using techniques from contact geometry, we construct an integer-valued index for detecting closed orbits in the flow which are topologically inessential (they have winding number zero with respect to the solid torus). This index is independent of the Riemannian structure, and is computable entirely from a C^1 approximation to the vector field on any meridional disc of the solid torus.

1. INTRODUCTION AND SUMMARY

Consider the class of *Beltrami fields* — the volume-preserving eigenfields of the curl operator. Such vector fields are the source of numerous interesting phenomena in inviscid fluids and plasmas. For example, Beltrami fields are the only steady three-dimensional Euler flows which admit chaotic Lagrangian dynamics. Beltrami fields are also common approximations to the magnetic field lines in large-scale structures within the solar corona.

Despite their importance and inherent intricacy, very little is known about the dynamics of Beltrami fields apart from numerical simulation [22, 7] and Melnikov analyses of near-integrable Beltrami fields [18, 35, 21, 5, 26] — an important but extremely small class of solutions. We consider the subtle problem of understanding how much and what kinds of dynamics Beltrami fields are forced to possess given the underlying topological features of the fluid domain.

In a series of papers [13, 14, 15], the authors develop techniques for determining forced behaviors in steady inviscid fluids via the topology of *contact structures*, the odd-dimensional analogues of symplectic structures (see, *e.g.*, [1, 11] for an introduction to contact geometry). In this paper, we give an application of these techniques to Beltrami fields on solid tori. One of the features of our topological approach is that it is independent of the Riemannian metric and is furthermore robust with respect to perturbations of

JE supported in part by NSF Grant # DMS-9705949.

RG supported in part by NSF Grant # DMS-9971629. The writing of this paper was facilitated by the hospitality of the Isaac Newton Institute at Cambridge University.

the vector field, without resorting to any hyperbolicity or nondegeneracy assumptions usually required to preserve closed orbits.

We restrict attention to Beltrami fields on Riemannian solid tori, such as would occur in the case of a force-free plasma in a containment device. In [14], it is shown using techniques from contact topology and pseudo-holomorphic curves that steady Euler fields on an invariant solid torus *always* possess a closed orbit, independent of the Riemannian structure and volume form:

Theorem: ([14]) *Any steady real-analytic solution to the Euler equations (2.2) on any invariant Riemannian solid torus possesses a closed orbit.*

Since every Beltrami field is a steady Euler field, the result holds true for all Beltrami fields. The restrictive smoothness assumption is necessary for using singularity theory arguments for the integrable Euler fields — in the setting of pure Beltrami fields, the techniques are valid up to smoothness class C^2 [23].

In this paper, we define an integer-valued index for detecting the presence of *contractible* closed orbits — those closed orbits which can be shrunk to a point within the solid torus. Thus, we are not primarily concerned with the class of Beltrami fields which possess a cross-section (and hence trivially have a closed non-contractible orbit by the Brouwer fixed point theorem). In cases where no section exists, it is very difficult to determine the existence of closed orbits. Indeed, the recent examples of fixed-point-free vector fields on a solid torus without any periodic orbits (constructible via [29] in the real-analytic case, and by [28] in the C^1 volume-preserving case), demonstrate the delicacy of the problem.

A related scenario to which our results apply is that of a Beltrami field on a long tube $D^2 \times \mathbb{R}$ which is periodic in the third variable. The problem of finding contractible closed orbits in the solid torus obtained by quotienting out the periodicity is precisely the problem of finding an orbit which is closed in the long tube.

The index we construct is a type of linking number with respect to a contact structure — the so-called *self-linking number* of a transverse knot, well-known to contact topologists. The data required to compute the index is minimal: one needs information about the Beltrami field along some (arbitrary) meridional disc in the solid torus. The principal contribution of this note is to retool the contact-topological index in a form which requires no knowledge of the contact structure per se. Since contact structures are very stable with respect to perturbation, the vector field need only be known approximately (C^1) along the disc. Hence, this index can be computed numerically with full rigor.

Sections 2 through 4 assemble the relevant ingredients. Section 5 presents the technical result on contact structures used to define the index and prove its major properties. The section §6 gives a very simple method for computing this index from a minimal amount of data: one need simply know what the vector field X approximately looks like near some finite number of points on a meridional disc of the solid torus. It is our hope that this computability may allow for utilization of this index in the analysis of experimental data.

2. BELTRAMI FIELDS ON RIEMANNIAN MANIFOLDS

Let M be an arbitrary 3-manifold with Riemannian metric g and (arbitrary) volume form μ . Given a vector field X on M , one can consider the dual 1-form $g(X, \cdot)$ to X that pairs with a vector Y to give the inner product $g(X, Y)$. In this general setting, the *curl* of a vector field X on M is the unique vector field $\nabla \times X$ satisfying

$$(2.1) \quad \mu(\nabla \times X, \cdot, \cdot) = d(g(X, \cdot)),$$

where d denotes the exterior derivative on forms. The curl operator is linear and its μ -preserving eigenfields are known as the *Beltrami fields*. In other words, X is Beltrami if and only if it is volume-preserving and $\nabla \times X = \lambda X$ for some constant λ . One can also consider the class of eigenfields with scalar fields as eigenvalues: $\nabla \times X = fX$ for $f : M \rightarrow \mathbb{R}$. Our techniques are adaptable to this more general format, but we restrict to pure eigenfields here for simplicity.

Beltrami fields arise in several contexts:

1. Beltrami fields are always steady solutions to the Euler equations for an inviscid incompressible fluid

$$(2.2) \quad \frac{\partial u}{\partial t} + \nabla_u u = -\nabla p \quad ; \quad \mathcal{L}_u \mu = 0,$$

where $\nabla_u u$ is the covariant derivative of the velocity field u along itself, and $p : M \rightarrow \mathbb{R}$, the pressure function, can be chosen to be $\frac{1}{2} \|u\|^2$. The Lie derivative $\mathcal{L}_u \mu$ of the volume form along u vanishing is equivalent to u being divergence-free.

2. Beltrami fields also yield steady solutions to the ideal MHD equations

$$(2.3) \quad \begin{aligned} \frac{\partial u}{\partial t} + \nabla_u u &= -\nabla p + (\nabla \times B) \times B \\ \frac{\partial B}{\partial t} - \nabla \times (u \times B) &= 0 \end{aligned}$$

$$\mathcal{L}_u \mu = \mathcal{L}_B \mu = 0,$$

where B denotes the magnetic field. In this context, Beltrami fields are known as force-free fields.

3. Beltrami fields are all extrema of the L^2 energy functional

$$(2.4) \quad \|u\|_2 := \frac{1}{2} \int_M \|u\|^2 d\mu$$

under the action of the volume-preserving diffeomorphism group of M . Eigenfields of curl having the smallest nonzero eigenvalue globally minimize the energy [2, 3].

Beltrami fields also play a role in the analysis of the stability of matter [30] and in the formation of dynamos [6].

The topology and dynamics of Beltrami fields are subtle: witness the complex dynamics of the well-known *ABC fields* on the Euclidean 3-torus [7]. The existence of fixed-point-free Beltrami fields on general Riemannian 3-manifolds is highly nontrivial [13], as is the presence of closed orbits within such fields [13, 14].

3. CONTACT STRUCTURES AND TOPOLOGY

Contact structures are the natural complements to Beltrami fields. Loosely put, a contact structure on an odd-dimensional manifold M is a hyperplane field which is maximally nonintegrable. More specifically, on a three-dimensional manifold, a contact structure is a smoothly-varying plane field (a choice of a two-dimensional subspace ξ_p in each tangent space $T_p M$) which cannot be stitched together into leaves of a foliation, not even at a point. Locally, every contact structure ξ is the kernel of a differential 1-form α satisfying the contact condition:

$$(3.1) \quad \alpha \wedge d\alpha \neq 0.$$

Otherwise said, $\alpha \wedge d\alpha$ is locally a volume form on M . Any 1-form satisfying (3.1) is called a contact form. If $\alpha \wedge d\alpha$ is a globally defined volume form on M , then the contact structure is said to be *cooriented*: all contact structures which arise in connection with Beltrami fields are of necessity cooriented, and we will restrict entirely to this category.

A contact form α on an oriented three-manifold M is said to be *positive* if the sign of $\alpha \wedge d\alpha$ is positive with respect to the orientation of M . Otherwise, α is said to be *negative*. The sign is a property of the contact structure and is independent of the defining 1-form.

Canonical examples of positive contact forms on \mathbb{R}^3 include $dz + x dy$, $dz + r^2 \sin r d\theta$, and $\cos r dz + r \sin r d\theta$, the latter two being given in cylindrical coordinates.

Much of the current interest in contact structures arises from fairly recent elucidations of their topological and dynamical properties (see, *e.g.*, [1, 11]). Studying contact structures by means of *characteristic foliations* is most

fruitful. Given a two-dimensional surface S embedded in M , the characteristic foliation of S , S_ξ , is the [singular] one-dimensional foliation generated by the intersections of the tangent planes of $T_p S$ with the contact planes ξ_p in $T_p M$. For all intents and purposes, S_ξ may be thought of as a vector field on S generated by ξ (by orienting the foliation). The singularities which arise on a characteristic foliation are generically saddles or spiral sources/sinks: pure centers cannot ever appear from a contact structure — the plane field twists too much for this.

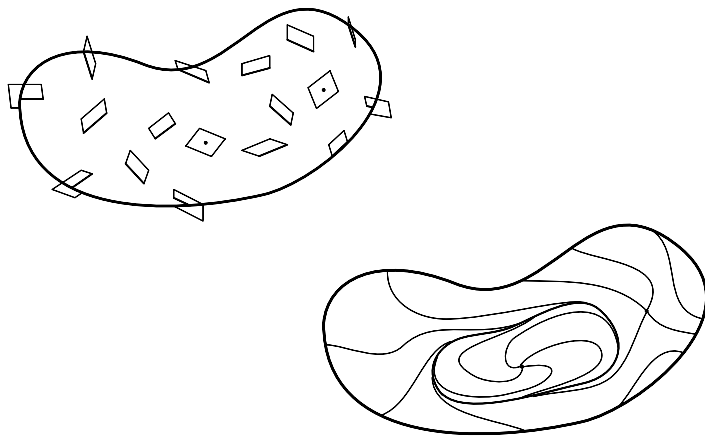


FIGURE 1. A contact structure is overtwisted if the induced characteristic foliation on some embedded disc possesses a limit cycle.

The dynamical properties of S_ξ are closely related to the topological classification of contact structures. A contact structure ξ is said to be *overtwisted* if there exists an embedded disc $D \subset M$ such that D_ξ possesses a limit cycle — a closed orbit along which nearby orbits accumulate [see Figure 1]. A contact structure is said to be *tight* if there are no such overtwisted discs anywhere in M . The contact structures for $dz + x dy$ and $dz + r^2 d\theta$ are tight [4], while that of $\cos r dz + r \sin r d\theta$ is overtwisted (*e.g.*, at the disc $\{r \leq 1, z = r^2\}$).

It is by no means apparent that the above definition is at all helpful: but in fact, the entire topological theory hangs on this dichotomy. Many major questions about contact structures are solved in the overtwisted category and unknown in the tight category (or, if known, then only recently and then by great skill and effort). For example, while overtwisted contact structures have been completely classified up to isotopy [8], the classification of tight structures appears [from what is presently known] to be delicate at best, intractable at worst [4, 19, 27, 12, 24, 25, 16, 17].

4. CONTACT DYNAMICS

The connection between contact structures and Beltrami fields is, in the present context, quite straightforward. Given any fixed-point-free Beltrami field X on M , the Beltrami condition states that

$$(4.1) \quad \mu(\lambda X, \cdot, \cdot) = d(g(X, \cdot)).$$

From this one can derive the crucial observation that the plane field orthogonal to any nonvanishing Beltrami field is indeed a contact structure as follows. Witness the 1-form $\alpha := g(X, \cdot)$ dual to X via g . The kernel of this 1-form represents the orthogonal plane field ξ to X . This form α is a contact form on M since

$$(4.2) \quad \alpha \wedge d\alpha := \lambda g(X, \cdot) \wedge \mu(X, \cdot, \cdot),$$

which, for $\lambda \neq 0$, is nowhere vanishing, as one can easily check by evaluating on local orthogonal coordinate bases of the form $(e_1 := X/\|X\|, e_2, e_3)$.

A little more is in fact true: a Beltrami field annihilates the exterior derivative of the associated contact form, since

$$(4.3) \quad (d\alpha)(X, \cdot) = \mu(X, X, \cdot) = 0.$$

Such vector fields are classical objects known as *Reeb fields*. The Reeb field of a contact form α is the unique vector field Z such that $d\alpha(Z, \cdot) = 0$ and $\alpha(Z) = 1$. We have thus observed that any Beltrami field X (nonsingular with nonzero eigenvalue) is a Reeb field for a contact form, after a possible rescaling to force $\alpha(X) = 1$. In [13], a broader version of this result was demonstrated: namely, that the class of nonsingular Beltrami fields on M (up to scaling, for any Riemannian structure and volume form) is identical to the class of Reeb fields (up to scaling, for any contact form).

This theorem allows one to build “custom” solutions to the steady Euler equations of very high regularity. For example, [15] builds a single Beltrami field on a Riemannian \mathbb{R}^3 which possesses closed orbits of all possible knot and link types simultaneously. This can be viewed as a rigorous manifestation of the delightful results of Moffatt on knotting in the Euler equations [34].

In the present context, we will use this simple correspondence between Beltrami and Reeb fields to import technology from contact dynamics. Most specifically, we are interested in the utility of the tight/overtwisted dichotomy in describing the dynamics of Beltrami fields. One extremely important result in contact dynamics is the following theorem of Hofer [23]: *Let ξ be an overtwisted contact structure on M a compact 3-manifold without boundary. Then the Reeb field of any contact form associated to ξ possesses a closed orbit, some multiple of which bounds a disc in M .*

Loosely speaking, a closed orbit (limit cycle) in the characteristic foliation of a disc in M implies the existence of a closed Reeb orbit which bounds a disc in M . The proof relies on the delicate techniques of pseudo-holomorphic curves in symplectic manifolds. One indication of the implicit nature of the proof is that the location of the overtwisted disc has little to no correlation with the location of the implicated Reeb orbit.

It follows from well-known properties of pseudo-holomorphic curves that the proof of Hofer's theorem remains valid for a three-manifold with invariant boundary (see [14] for details). Thus, for the solid torus, it follows that an overtwisted Reeb field must possess a contractible periodic orbit (since the fundamental group of the solid torus contains no elements of finite order except the identity).

5. DEFINITION OF \mathbb{I}

From Hofer's theorem, then, one way to force a contractible orbit in a Beltrami field is by finding an overtwisted disc in the orthogonal contact structure. This is far from trivial, since it requires searching for overtwisted discs among all possible embedded discs in M : not a computationally feasible task, even if the vector field were known analytically (which, in the context of an experimentally generated flow is not generally the case).

The classification of contact structures has been successfully completed only on a selected class of three-manifolds. The classification of contact structures on the solid torus is quite recent and subtle [20, 24]. In particular, it is known that on the solid torus there are tight contact structures which are "stably overtwisted" — taking some finite covering space of the solid torus and lifting the tight contact structure downstairs yields an overtwisted contact structure on the cover [32, 14] (see Theorem 5.2). While this is a complication for contact topologists, it is a benefit to dynamicists.

Lemma 5.1. *Any Beltrami field transverse to a tight contact structure, some cover of which is overtwisted, must possess a contractible closed orbit.*

Proof: Assume that ξ is a tight contact structure, some finite cover $\tilde{\xi}$ of which is overtwisted. Then, given any Beltrami field X associated to ξ , lift this to a Beltrami field \tilde{X} on the overtwisted cover. Applying Hofer's theorem to the cover implies the existence of a contractible periodic orbit for \tilde{X} ; however, since a covering space projection takes orbits to orbits, the closed orbit upstairs (along with the disc that it bounds) *must* project to a contractible closed orbit of the original Beltrami field X . \square

Thus, our goal is to effectively determine the existence of an overtwisted or stably (with respect to coverings) overtwisted structure on a solid torus

given the least amount of information about a Beltrami field transverse to it.

To do so, we recall a common index used in contact topology (see [1, 9] for an introduction). Given a contact structure ξ on a three-manifold M , a simple closed curve (knot) is called *transverse* if its tangents are everywhere transverse to the contact planes. Assume that γ is an oriented simple closed curve in M which bounds a compact oriented surface Σ in M . Then the *self-linking number* of γ with respect to ξ and Σ is defined as follows. Choose any vector field Z on a neighborhood of Σ which has no fixed points and which is always tangent to ξ . That this is possible is a simple argument involving the classification of plane bundles. Then, flow γ for a small amount of time under Z to obtain a “push-off” curve γ_Z . The self-linking number of γ , $slk(\gamma)$ is then defined as the intersection number of γ_Z with Σ — *i.e.*, the number of transverse intersections, counted algebraically using the orientations. This integer, which can be shown to be independent of the vector field Z chosen, is an invariant of transverse curves up to isotopy through transverse curves [4]. On S^3 , the self-linking number is also independent of the surface Σ chosen so long as it bounds γ .

The following recent result allows for an application of this index to Beltrami fields.

Theorem 5.2 ([14]). *Assume α is a positive contact form on a solid torus V whose Reeb field is tangent to the boundary ∂V . Choose any transverse curve γ on the boundary torus ∂V which bounds a meridional disc in V . If the self-linking number $slk(\gamma)$ of this meridian is not equal to -1 , then the pullback of α under some finite cover is an overtwisted contact form.*

It is known from the inequality of [4] that if the initial contact structure is tight, the self linking number must satisfy $slk \leq -\chi(D) = -1$. Thus, any self-linking number greater than -1 automatically implies an overtwisted structure (which is of course preserved under covers). The nontrivial result of this theorem is that for a tight structure, a self-linking number less than -1 implies an overtwisted cover. The techniques used in the proof of this theorem are a combination of perturbing characteristic foliations, manipulating singularities of characteristic foliations, and using dynamical properties of the characteristic foliation on ∂V .

Remark 5.3. It is necessary to distinguish between positive and negative contact structures. In the case of a negative contact form (one for which $\alpha \wedge d\alpha < 0$) on an invariant solid torus, the structure possesses an overtwisted cover if and only if the self-linking number of a transverse meridian is not equal to $+1$. It is an easy exercise to show that the sign of the contact form dual to a curl eigenfield is precisely the sign of the eigenvalue.

From these ingredients the following index may be defined:

Definition 5.4. Given a nonsingular Beltrami field X on an invariant Riemannian solid torus V , define the index \mathbb{I} as follows.

1. If the eigenvalue λ of X with respect to the curl operator is zero, define $\mathbb{I} := 0$.
2. Otherwise, consider the characteristic foliation $(\partial V)_\xi$ of the contact structure orthogonal to X on ∂V . If possible, choose γ any meridional curve on ∂V transverse to ∂V_ξ and define

$$(5.1) \quad \mathbb{I} := \text{Sign}(\lambda) (\text{slk}(\gamma)) + 1.$$

3. If no transverse curve γ exists, define $\mathbb{I} := \text{Sign}(\lambda)$.

Theorem 5.5. Any C^2 or smoother nonvanishing Beltrami field on an invariant Riemannian solid torus V having nonzero index \mathbb{I} possesses a contractible closed orbit.

Proof: Assume that $\lambda > 0$ as the negative case follows similarly. Since $\mathbb{I} \neq 0$, we are either in the case where there is a transverse meridian on the boundary of V with self-linking number not equal to -1 , or there is no transverse meridian. In the case where the transverse meridian exists, some finite cover of the Beltrami field has a contractible closed orbit which is preserved by the covering projection.

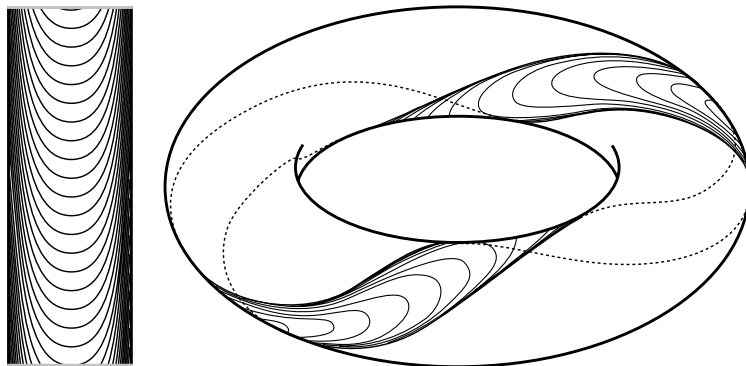


FIGURE 2. A Reeb component in a two-dimensional foliation is a foliation of an annulus by curves which limit onto the boundary components as illustrated [left, identify top and bottom]. On the boundary torus ∂V , if the characteristic foliation $(\partial V)_\xi$ possesses a Reeb component [right], then there does not exist a transverse meridional curve.

If there does not exist a closed transversal, then a basic result in foliation theory implies that either (1) the characteristic foliation $(\partial V)_\xi$ is entirely by meridional curves; or (2) the characteristic foliation $(\partial V)_\xi$ possesses a *Reeb component*, illustrated in Figure 2. In the former case, the contact structure

is clearly overtwisted: any meridional disc in general position which spans one of these meridional curves has this boundary curve as a limit cycle in its characteristic foliation. The existence of a periodic orbit then follows as earlier.

In the latter case, where a Reeb component exists in $(\partial V)_\xi$, the following argument eliminates this possibility as a Beltrami field. Consider the Reeb field Z associated to the contact form α dual to X . Since Z preserves the contact structure ξ and the boundary tours ∂V , it must likewise preserve the characteristic foliation $(\partial V)_\xi$. Thus, if $(\partial V)_\xi$ contains a closed curve, then, since Z is everywhere transverse to $(\partial V)_\xi$, the entire boundary torus is swept out by forward images of this curve under the flow of Z , and $(\partial V)_\xi$ is a foliation by closed curves. Thus, a Reeb component (which always possesses both closed and open curves as in the illustration) cannot arise as the characteristic foliation on an invariant solid torus. \square

6. COMPUTATION OF \mathbb{I}

Given a contact structure ξ on a 3-manifold M , the determination of whether it is a tight or overtwisted structure is a difficult question in general. By the Darboux Theorem (see, *e.g.*, [1, 33]), every contact structure in dimension three is locally equivalent to the kernel of $dz + x dy$, which is a tight structure [4]. Thus, on the one hand, the property of being overtwisted is a decidedly global feature. However, the process of *Lutz twisting* [31] allows one to change a tight structure into an overtwisted structure by means of a C^0 alteration on an arbitrarily small open set in M . Thus, given a Beltrami field X on M , determining whether the contact structure orthogonal to X is overtwisted is computationally intractable. Determining whether the universal cover is overtwisted is no less difficult.

However, to compute the index \mathbb{I} , one does not need information about the vector field on the entire three-dimensional regime, but rather on some (arbitrary) two-dimensional meridional disc. We outline a method for easily computing \mathbb{I} from a C^1 approximation to X along a two-dimensional slice of the solid torus.

Choose a meridional disc D with boundary curve γ . Orient γ so that the contact form $\alpha := g(X, \cdot)$ evaluates to a positive number on the tangents to γ : *i.e.*, γ points in roughly the same direction as X . This orientation on γ induces in the usual way an orientation on the disc D . Denote by X_D the projection of the vector field X on to D by orthogonal projection onto the tangent planes (orthogonal with respect to the metric g).

Proposition 6.1. *If the vector field X_D is generic (possesses a finite number of nondegenerate rest points), the index \mathbb{I} of the Beltrami field X can be*

computed by

$$(6.1) \quad \mathbb{I} = \text{Sign}(\lambda) \left(1 + \sum_{p: X_D(p)=0} \sigma(p) \text{Ind}(X_D; p) \right)$$

where for every rest point p of X_D , $\sigma(p)$ is defined to be the sign (+/−) of the dot product of $X(p)$ with the positive normal vector to D at p , and the term $\text{Ind}(X_D; p)$ denotes the standard Euler-Poincaré index of the planar vector field X_D at p .

Proof: Assume that the characteristic foliation D_ξ is generic in the above sense: there are a finite number of singular points p at which the contact structure ξ is tangent to D , and the characteristic foliation about these points appears locally as a source/sink or a saddle. Following [4, 9], there is a standard formula for computing the self-linking number of the transverse curve $\gamma = \partial D$:

$$(6.2) \quad \text{slk}(\gamma) = \sum_{p: D_\xi(p)=0} \text{sign}(T_p D, \xi_p) \text{Ind}(D_\xi; p)$$

Here, the sign of $(T_p D, \xi_p)$ is +1 when the orientations on the contact plane ξ and the orientations on the tangent plane to disc D at p agree. Otherwise the sign is −1. However, we only know the Beltrami field X and the disc D — not the characteristic foliation. Determining the indices of the rest points of the characteristic foliation D_ξ is accomplished via the projected field X_D as follows. Since the contact structure for X is the plane field ξ orthogonal to X , the characteristic foliation at every point $q \in D$ is given by

$$D_\xi(q) := \xi_q \cap T_q D = (X_D(q))^\perp,$$

the line field orthogonal to X_D at q . The proposition is proved by noting (1) fixed points of X_D occur exactly at fixed points of D_ξ ; (2) the Euler-Poincaré index of D_ξ at a fixed point p is unchanged by looking at the orthogonal vector field, as illustrated in Figure 3. \square

Computationally, this is extremely simple as D can be chosen almost arbitrarily (one presumably chooses a D which is “nice” in coordinates) and information about X is required only on D itself. The local index calculation is the most delicate portion of the computation: the location of the orthogonal point p is easy and the sign $\sigma(p)$ merely measures whether X agrees with the oriented normal to D , which is trivial to determine.

An example of a characteristic foliation D_ξ and the resulting self-linking number is given in Figure 4.

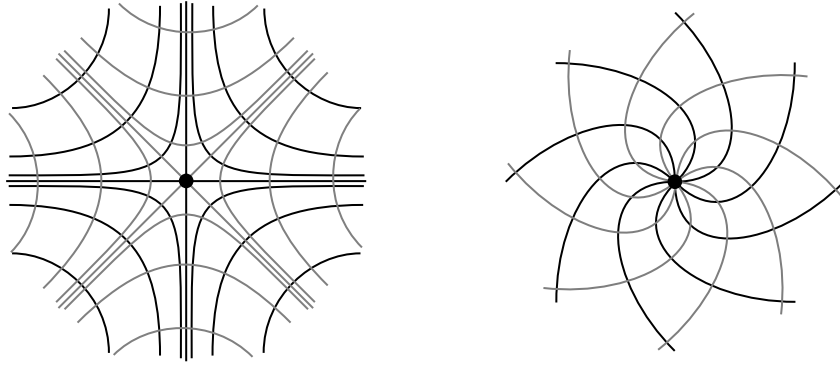


FIGURE 3. Taking the orthogonal line field X_D (grey) to the characteristic foliation D_ξ (black) leaves the index invariant.

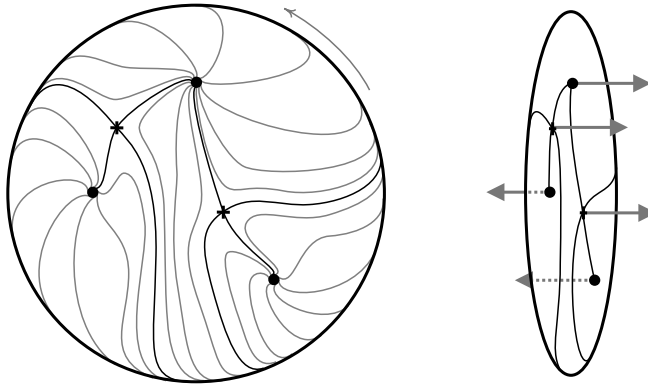


FIGURE 4. An example of an slk computation given the characteristic foliation on an oriented spanning disc D . [left] D_ξ ; [right] the directions of the field X , positive to the left and negative to the right. The self-linking number is $slk = -3$. Thus the Beltrami field X has index $\mathbb{I} = -2$.

7. CONCLUSIONS

The index \mathbb{I} is an unusual object in that one inputs information about the vector field which is strictly *two*-dimensional, yet one obtains data about the dynamics which is fully *three*-dimensional. In Figure 4, the *only* information about the Beltrami field known is that (1) it is orthogonal to the disc D at five points; (2) locally near those five points the projected field is either source/sink or saddle type; and (3) at those five points the field points out or in as illustrated. From this data, it is inevitable that *somewhere* in the flow there exists a contractible closed flowline. This is a corollary of the contact-topological methods used — the moral of the story is that the Beltrami

condition hides within it certain constraints on the dynamics which couple the dynamics of the vector field to the topology of the orthogonal plane field.

A deficiency of our theory is that it is not sharp. It is certainly possible for a Beltrami field to have the value $\mathbb{I} = 0$ and yet still have a contractible periodic orbit: indeed, any tight contact structure which has been Lutz twisted along a contractible closed curve necessarily has trivial index as well as a contractible orbit. It appears certain (due to this mechanism of arbitrarily small Lutz twists) that no completely sharp computable index can be defined. What \mathbb{I} does, however, is detect if there are contractible orbits forced by the presence of non-localized overtwisted discs, and in this regime it is efficacious.

It would be interesting to find a sharp lower bound on the number of periodic orbits present in the case of $\mathbb{I} \neq 0$ (cf. the recent body of theory surrounding the *contact homology* of Eliashberg, Givental, and Hofer [10]).

REFERENCES

- [1] B. Aebischer et al. *Symplectic Geometry*. Number 124 in Progress in Math. Birkhäuser, Berlin, 1994.
- [2] V. I. Arnold. The asymptotic Hopf invariant and its applications. *Sel. Math. Sov.*, 5:327–345, 1986.
- [3] V. I. Arnold and B. Khesin. *Topological Methods in Hydrodynamics*. Springer-Verlag, Berlin, Heidelberg, New York, 1998.
- [4] D. Bennequin. Entrelacements et équations de Pfaff. *Asterisque*, 107-108:87–161, 1983.
- [5] C. Chicone. A geometric approach to regular perturbation theory with an application to hydrodynamics. *TAMS*, 347(12):4559–4598, 1996.
- [6] S. Childriss and A. Gilbert. *Stretch, Twist, Fold: The Fast Dynamo*. Springer-Verlag, 1995.
- [7] T. Dombre, U. Frisch, J. Greene, M. Hénon, A. Mehr, and A. Soward. Chaotic streamlines in the ABC flows. *J. Fluid Mech.*, 167:353–391, 1986.
- [8] Y. Eliashberg. Classification of overtwisted contact structures on 3-manifolds. *Invent. Math.*, 98:623–637, 1989.
- [9] Y. Eliashberg. Legendrian and transversal knots in tight contact 3-manifolds. In *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, pages 171–193. Publish or Perish, Houston, TX, 1993.
- [10] Y. Eliashberg, A. Givental, and H. Hofer. An introduction to symplectic field theory. [ArXiv preprint math.SG/0010059](#).
- [11] Y. Eliashberg and W. Thurston. *Confoliations*, volume 13 of *University Lecture Series*. American Mathematical Society, 1998.
- [12] J. Etnyre. Tight contact structures on lens spaces. *Comm. Cont. Math.*, 2(4):559–577, 2000.
- [13] J. Etnyre and R. Ghrist. Contact topology and hydrodynamics I: Beltrami fields and the Seifert conjecture. *Nonlinearity*, 13:441–458, 2000.
- [14] J. Etnyre and R. Ghrist. Contact topology and hydrodynamics II: solid tori. [ArXiv preprint math.SG/9907112](#), 1999.
- [15] J. Etnyre and R. Ghrist. Contact topology and hydrodynamics III: knotted flowlines. To appear in *Trans. Amer. Math. Soc.* Preprint, 1999.

- [16] J. Etnyre and K. Honda. On the non-existence of tight contact structures. To appear, *Ann. Math.*
- [17] J. Etnyre and K. Honda. Tight contact structures with no symplectic fillings. [ArXiv preprint math.GT/0010044](#).
- [18] J.-L. Gautero. Chaos lagrangien pour une classe d'écoulements de beltrami. *C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Univers Sci. Terre*, 301(15):1095–1098, 1985.
- [19] E. Giroux. Une structure de contact, même tendue est plus ou moins tordue. *Ann. Scient. Ec. Norm. Sup.*, 27:697–705, 1994.
- [20] E. Giroux. Structures de contact en dimension trois et bifurcations de feuilletages de surfaces. *Invent. Math.*, 141(3):615–689, 2000.
- [21] A. Goriely and M. Tabor. The singularity analysis for nearly integrable systems: homoclinic intersections and local multivaluedness. *Phys. D*, 85(1-2):93–125, 1995.
- [22] M. Hénon. Sur la topologie des lignes de courant dans un cas particulier. *C. R. Acad. Sci. Paris*, 262:312–314, 1976.
- [23] H. Hofer. Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three. *Invent. Math.*, 114:515–563, 1993.
- [24] K. Honda. On the classification of tight contact structures I. *Geom. & Top.*, 4(11):309–368, 2000.
- [25] K. Honda. On the classification of tight contact structures II. preprint, 1999.
- [26] D.-B. Huang, X.-H. Zhao, and H.-H. Dai. Invariant tori and chaotic streamlines in the ABC flow. *Phys. Lett. A*, 237(3):136–140, 1998.
- [27] Y. Kanda. The classification of tight contact structures on the 3-torus. *Comm. in Anal. and Geom.*, 5:413–438, 1997.
- [28] G. Kuperberg. A volume-preserving counterexample to the Seifert conjecture. *Comment. Math. Helv.*, 71(1):71–97, 1996.
- [29] G. Kuperberg and K. Kuperberg. Generalized counterexamples to the Seifert conjecture. *Ann. of Math.*, 144:239–268, 1994.
- [30] M. Loss and H.-T. Yau. Stability of Coulomb systems with magnetic fields III: zero-energy bound states of the Pauli operator. *Comm. Math. Phys.*, 104:283–290, 1986.
- [31] R. Lutz. Structures de contact sur les fibrés principaux en cercles de dimension trois. *Ann. Inst. Fourier*, 3:1–15, 1977.
- [32] S. Makar-Limanov. Tight contact structures on solid tori. *Trans. Am. Math. Soc.*, 350:1013–1044, 1998.
- [33] D. McDuff and D. Salamon. *Introduction to Symplectic Topology*. Oxford University Press, New York, 1995.
- [34] H. Moffatt. Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology: part I. *J. Fluid Mech.*, 159:359–378, 1985.
- [35] X.-H. Zhao, K.-H. Kwek, J.-B. Li, and K.-L. Huang. Chaotic and resonant streamlines in the ABC flow. *SIAM J. Appl. Math.*, 53(1):71–77, 1993.

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