CONTACT TOPOLOGY AND HYDRODYNAMICS II: SOLID TORI

JOHN ETNYRE AND ROBERT GHRIST

Abstract. We prove the existence of periodic orbits for steady C^ω Euler flows on all Riemannian solid tori. By using the correspondence theorem from part I of this series, we reduce the problem to the Weinstein Conjecture for solid tori. We prove the Weinstein Conjecture on the solid torus via a combination of results due to Hofer et al. and a careful analysis of tight contact structures on solid tori.

1. Introduction and summary

Problems associated with the existence of periodic orbits in flows differ sharply from their discrete counterparts. The index-theoretic methods available for detecting periodic orbits of two-dimensional self-diffeomorphisms are not sufficient for understanding general three-dimensional flows. Indeed, the recent progress on the Seifert Conjecture by K. Kuperberg and G. Kuperberg [21, 20] has made it clear that, for flows of arbitrary regularity, there is "too much room" to have a topological forcing theory: additional constraints are required, many of which should be geometric in nature. The interesting problem is now to find sharp boundaries on the space of vector fields in dimension three which separate those fields without periodic orbits. Currently, there is great interest in the case of volume-preserving and Hamiltonain flows, since, by a classical theorem of Poincaré, almost all orbits are recurrent. The Kuperberg plug constructions do not work in this category (but see [19] for a C^1 construction).

In this series of papers, we are concerned with periodic orbits in a particular class of volume-preserving vector fields which models the motion of the simplest possible fluid and plasma flows: these are the steady, perfect, incompressible fields, or *Euler fields.* In the realm of fluid and plasma dynamics, periodic orbits play a naturally important role. For example, the existence of a hyperbolic periodic orbit in a steady Euler flow is sufficient to conclude hydrodynamic instability of the solution [13]. There are several connections between the embedding properties of periodic orbits and physical properties of fluids/plasmas, such as energy bounds, helicity, and the possibility of finite-time singularities (see, *e.g.*, [24, 25]). In part III of this series [12] we consider the knot theory of periodic orbits in Euler flows.

In part I of this series [11], we proved the existence of periodic solutions to all C^ω steady nonsingular solutions to the Euler equations for an inviscid fluid flow on a Riemannian 3-sphere. In applications, fluid flows on S^3 are not terribly significant. In [11], algebraic-topological conditions were also derived under which

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steady nonsingular Euler flows on a Riemannian cube $[0,1]^3$ with periodic boundary conditions possess a closed orbit. While this global geometry is a common domain in theoretical fluid dynamics (Fourier analysis is particularly simple here), one cannot reasonably call T^3 "physical." Perhaps the most interesting case for applications (*e.g.*, tokamaks) involves compact domains in $\mathbb{R}^3$. Under the condition that the flow is nonsingular (no fixed points), the simplest such domain is a solid torus $D^2 \times S^1$. Invariant solid tori are ubiquitous in volume-preserving flows on $\mathbb{R}^3$.

In this paper, we extend the programme of Part I to the case of a solid torus:

Theorem 2.4. *Any steady nonsingular C^ω Euler flow on $S^1 \times D^2$ leaving the boundary invariant possesses a closed flow line.*

Note that this result is independent of the geometry of the solid torus. For the proof, we rely on a dichotomy for C^ω steady nonsingular Euler flows due to Arnold which presents two scenarios: the *integrable* and the *Beltrami* fields. In part I of this series, it was shown that the crucial subclass of Beltrami fields is in fact equivalent to the class of Reeb fields associated to contact forms (see §2.1). Thus, after dealing with the integrable cases, we reduce the problem of periodic orbits for Euler fields to the problem of periodic orbits for Reeb fields, *i.e.*, the Weinstein conjecture:

Theorem 6.1. *Every Reeb field on $S^1 \times D^2$ tangent to the boundary possesses a periodic orbit.*

The work of Hofer [15] resolved the Weinstein conjecture on the 3-sphere, and on manifolds with nontrivial π_2 by considering sequences of pseudo-holomorphic curves in symplectizations. The heart of our proof for the solid torus likewise lies in Hofer's results on pseudo-holomorphic curves. A simple generalization is impossible, however, for the reason that a certain class of Reeb fields (namely those associated to *tight* structures — see §3 — requires different techniques on different manifolds. The key steps in the proof of Theorem 6.1 are a careful analysis of virtually overtwisted contact structures on the solid torus, along with an application of the recent work of Hofer et al. on finite energy foliations.

In §2 we recall the relevant definitions from hydrodynamics and discuss the relations with contact geometry. This immediately allows us to reduce Theorem 2.4 to the Weinstein conjecture. In §3 we turn to contact geometry, collecting the standard facts we need. A characterization of virtually overtwisted contact structures on the solid torus is obtained in §4. In §5 we discuss Hofer's approach to the Weinstein conjecture. In particular, we recall the use of pseudo-holomorphic curves in contact geometry. The Weinstein conjecture on solid tori is proved in §6.

2. Fluid flows on solid tori

The dynamical properties of incompressible, inviscid fluid flows are described by the Euler equations. For an overview of geometric hydrodynamics on Riemannian manifolds, the reader is referred to [4, 3]. To describe the Euler equations on a 3-manifold M we must first fix a Riemannian metric g and a volume form μ. Note that, following [4], we do not assume that μ is the volume form associated to g, though we do of course allow that possibility. The *Euler equations* governing the velocity field $u(t)$ of a perfect incompressible fluid may be written as follows:

$$\begin{aligned} \frac{(\partial \iota_u g)}{\partial t} + \iota_u \iota_{\nabla\times u}\mu &= -dP \\ \mathcal{L}_u \mu &= 0. \end{aligned} \tag{2.1}$$

Here $P(t) : M \to \mathbb{R}$ is some time-dependent (Bernoulli) function, $\mathcal{L}$ is the Lie derivative, ι denotes contraction, and $\nabla \times u$ is the vorticity, defined by the relation $\iota_{\nabla\times u}\mu = d\iota_u g$. We call u an Euler field and its flow an Euler flow if u satisfies Equation 2.1 for some function P. In this paper we concern ourselves only with steady nonsingular solutions u to Equation 2.1; that is, solutions for which all time derivatives vanish and all velocities are nonzero. For such Euler fields there exists a powerful dichotomy.

Theorem 2.1 (Arnold [2])**.** *Let u be a C^ω nonsingular steady Euler field on a C^ω Riemannian three-manifold M. If $\partial M \neq \emptyset$ then assume u is tangent to the boundary of M. If u is not everywhere colinear with its curl, then it has a stratified integral:* i.e.*, there exists a compact stratified subset $\Sigma \subset M$ of codimension at least one which splits M into a finite collection of cells diffeomorphic to $T^2 \times \mathbb{R}$. Each $T^2 \times \{c\}$ is an invariant set for u having flow conjugate to linear flow.*

This theorem motivates the following classical definitions.

Definition 2.2. A volume-preserving vector field X on a Riemannian manifold M^3 is a *Beltrami field* if it is parallel to its curl: *i.e.*, $\nabla \times X = fX$ for some function $f : M \to \mathbb{R}$. We say that a Beltrami field is *rotational* if f is nowhere zero and that it is *irrotational* if f vanishes identically.

Furthermore, since the function f above is an integral for X, one may reduce the study of real-analytic nonsingular steady Euler fields to the following three cases:

1. rotational Beltrami fields;
2. irrotational Beltrami fields; and
3. stratified integrable fields.

This roughly outlines the plan of the proof of the main theorems.

2.1. **Reeb fields associated to Beltrami flows.** Given a rotational Beltrami vector field one may consider the 1-form $\alpha := \iota_X g$. Since $d\alpha = d\iota_X g = f\iota_X\mu$ and the kernel of α is orthogonal to X it is clear that

$$\alpha \wedge d\alpha \neq 0. \tag{2.2}$$

Thus, by definition (see §3), the plane field $\xi = \ker\,\alpha$ is a *contact structure*. Associated to α is a special vector field, the *Reeb field* X_α, defined by the conditions

$$\iota_{X_\alpha}\alpha = 1, \qquad \iota_{X_\alpha} d\alpha = 0. \tag{2.3}$$

As $\iota_X d\alpha = \iota_X d\iota_X g = \iota_X \iota_X \mu = 0$, one concludes that $X = hX_\alpha$ for some nonzero function $h : M \to \mathbb{R}$. Thus any rotational Beltrami field is a nonzero rescaling of some Reeb field (sometimes referred to as a *Reeb-like* field). This is the easy half of the following:

Theorem 2.3 (Etnyre & Ghrist [11])**.** *On a fixed 3-manifold M, the class of vector fields which are nonsingular rotational Beltrami fields for some Riemannian structure is equivalent to the class of vector fields which are nonsingular rescalings of the Reeb field of some contact form.*

If X is an irrotational Beltrami field, then the 1-form $\alpha = \iota_X g$ still defines a plane field $\xi := \ker \alpha$ transversal to X; however, in this case $d\alpha = d\iota_X g = \iota_{\nabla \times X}\mu \equiv 0$. The Frobenius Theorem implies that ξ generates a codimension-one foliation of M transverse to the boundary.

2.2. **Periodic orbits in Euler flows.** In this section we set the stage for the proof of the main theorem:

Theorem 2.4. *Any steady nonsingular C^ω Euler field on $S^1 \times D^2$ tangent to the boundary possesses a closed flowline.*

Proof: Since u is a real-analytic steady Euler field, Theorem 2.1 and the discussion above imply that one needs to consider three cases: (1) u is a Reeb vector field for a contact structure; (2) u preserves a transverse foliation; or (3) u has a stratified integral. Case (1) is precisely the Weinstein Conjecture on the solid torus, which we prove in §6. A theorem of Tischler [28] implies that, in case (2), the foliation is actually by fibers of a fibration of $S^1 \times D^2$ over S^1. Using the exact sequence for homotopy groups of a fibration one easily concludes that the fiber must be D^2. The vector field u is transverse to the fibers; thus, any fiber will provide a section to the flow, and one concludes the theorem via the Brouwer fixed point theorem.

We are left to consider an Euler field u with a stratified integral. The argument for this case essentially mirrors the argument in [11] for the integrable case on a Riemannian S^3. Specifically, the real-analytic codimension-one (or greater) set Σ from Theorem 2.1 possesses a certain [Whitney] stratification, each stratum of which is invariant under the flow of u. Thus Σ has no zero-strata. The collection of essential one-dimensional strata must be nonempty, otherwise we would have foliated $D^2 \times S^1$ by copies of T^2: impossible. These one-strata thus consist of closed 1-manifolds invariant under the flow. □

Remark 2.5. In the contact case of the above theorem, one "frequently" finds contractible periodic orbits which do not exist in the foliation case and do not necessarily exist in the integrable case. Following the outline of the above proof combined with the proof of the Weinstein conjecture on solid tori below one may extract a computable invariant of Euler fields which can be used to detect contractible periodic orbits.

3. Contact structures and characteristic foliations

Recall a *contact structure* ξ on a 3-manifold M is a plane field in TM that is maximally nonintegrable. For the remainder of this work, we may assume ξ to be transversally orientable so that $\xi = \ker \alpha$ for a nondegenerate 1-form α (since all the contact structures we encounter come with a Reeb field by Theorem 2.3). Such an α is a *contact form*. One can express the nonintegrability of ξ by the condition $\alpha \wedge d\alpha \neq 0$. Two contact structures are *contactomorphic* if there is a diffeomorphism of M that takes one of the contact structures to the other. For proofs of some of the standard results listed below see [1].

Given a surface Σ in M, the contact structure ξ induces a singular foliation Σ_ξ on Σ, generated by the line field $T\Sigma \cap \xi$, with singularities occurring at points where $T_p\Sigma = \xi_p$. This is known as the *characteristic foliation*.

Lemma 3.1 (Moser-Weinstein)**.** *Two contact structures that induce the same characteristic foliation on a surface are contactomorphic in a neighborhood of the surface.*

A contact structure is *overtwisted* if there exists an embedded disc in M whose characteristic foliation contains a limit cycle, otherwise it is called *tight.* If ξ is tight and there is a finite cover of (M, ξ) that is overtwisted then ξ is called *virtually overtwisted.*

Generically, the singularities in Σ_ξ are either of elliptic or hyperbolic type. Moreover, each singularity of Σ_ξ is assigned a sign depending on whether or not the orientations on the plane field ξ and $T\Sigma$ agree or not. Of paramount importance in detecting overtwisted discs is the Elimination Lemma:

Lemma 3.2 (Elimination Lemma [14, 6])**.** *Let Σ be a surface in a contact 3-manifold (M, ξ). Assume that p and q are singular points in Σ_ξ which are of different type (one elliptic, one hyperbolic) yet have the same sign. Finally assume that there exists a leaf γ in Σ_ξ that connects p to q. Then, given any small neighborhood U of γ in M, there exists an arbitrarily C^0-small isotopy of Σ, fixed outside of U, which removes all singularities of Σ_ξ within U.*

This lemma is most effective when used in conjunction with:

Lemma 3.3 ([10, 9])**.** *Let γ and γ' be two leaves in the characteristic foliation Σ_ξ both ending in an elliptic singularity e. By a C^0-small isotopy of Σ near e we may arrange for $\gamma \cup \gamma' \cup \{e\}$ to be a smooth curve in the new characteristic foliation.*

The above two lemmas allow us to cancel singularities of the same sign so that the leaf joining the elliptic point to the hyperbolic point and any other leaf touching the elliptic point join to form a smooth leaf after the cancelation. For example, the two branches of the unstable (or stable, depending on the orientation) manifold of a hyperbolic point terminating in an elliptic fixed point of the same sign will form a smooth leaf after the cancelation of the positive (negative respectively) singularities.

The next result tells us how to smooth corners of a surface without changing the characteristic foliation.

Lemma 3.4 (Makar-Limanov [22])**.** *Suppose Σ and Σ' are two surfaces with boundary in (M^3, ξ) that intersect transversally along their boundaries. If $\gamma := \Sigma \cap \Sigma'$ is transverse to ξ then we may isotope Σ and Σ' in an arbitrary neighborhood of γ so that $\Sigma \cup \Sigma'$ is a smooth surface and the isomorphism type of the characteristic foliation is unchanged throughout the isotopy.*

If γ is a curve embedded in a contact manifold (M, ξ) we say γ is a *transversal curve* if γ is transversal to ξ at each point of γ. If γ is nullhomologous then there is an embedded surface Σ such that $\partial\Sigma = \gamma$. Note that $\xi|_\Sigma$ is a trivial bundle so we may choose a nonzero section s of $\xi|_\Sigma$. Using this section to push off a copy γ' of γ leads naturally to the definition of the *self-linking number of* γ as the intersection number of γ' with Σ:

$$\ell(\gamma; \Sigma) := \gamma' \cdot \Sigma \tag{3.1}$$

In subsequent sections, we will consider transversal meridians of the boundary of contact solid tori. Proofs of the main theorems differ depending on the self-linking numbers of the meridians. For some of the many properties and applications of self-linking numbers in contact geometry, see [1, 5, 7].

4. Covers of tight contact tubes

Recall the definitions of tight and overtwisted from §3: overtwisted contact structures possess discs with a limit cycle in the characteristic foliation. Let ξ denote a tight contact structure on a solid torus $V = S^1 \times D^2$. We further assume that the characteristic foliation $(\partial V)_\xi$ on the boundary of V is nonsingular and contains no Reeb components (*i.e.*, annuli possessing no transversals from one boundary component to the other). In this situation we may choose a meridional curve μ on ∂V that is positively transverse to $(\partial V)_\xi$. Denote by m_ξ the self-linking number of μ relative to the disc D that μ bounds. Given that ξ is tight, one knows from the inequalities in, *e.g.*, [6], that:

$$m_\xi \leq -\chi(D) = -1. \tag{4.1}$$

We may now state:

Theorem 4.1. *Let ξ be a tight contact structure on the solid torus V for which $(\partial V)_\xi$ is a linear foliation (conjugate to a foliation by lines of constant slope). The contact structure ξ is virtually overtwisted if and only if the self-linking number of the meridian is less than -1.*

This theorem is also true in the case where the foliation is not linear; however, we provide the [much shorter] proof of the simpler case as this is all we require for the sequel. We begin with some preliminary steps. In [6] it was shown that all of the negative elliptic and positive hyperbolic singularities of a characteristic foliation on a disc may be eliminated by a C^0-small isotopy of the disc fixed near the boundary. After this elimination procedure on our meridional disc, we are left with e_+ positive elliptic and h_- negative hyperbolic singularities. One may easily check that

$$m_\xi = -(e_+ + h_-). \tag{4.2}$$

Note that since $(\partial V)_\xi$ is nonsingular we may use this foliation to define a return map Φ on μ. Recall, associated to Φ (or to any circle diffeomorphism) is a rotation number $r(\Phi) \in [0, 1)$.

Proposition 4.2. *Suppose V deformation retracts to a solid torus $V' \subset V$ for which $(\partial V')_\xi$ is nonsingular. If, for any such V', the return map Ψ' for the meridian μ' of V' has an irrational rotation number and the self-linking number of μ' is strictly less than -1, then V has an overtwisted finite cover.*

Proof: Since the self-linking of μ' is strictly less than -1 there will be at least one hyperbolic point, from Equation 4.2. If there is only one such point, μ' is divided into two open intervals I_1 and I_2 by the ends of the unstable manifold of the hyperbolic point. Otherwise we can find a pair of hyperbolic points, h_1 and h_2, each of whose unstable manifold $W^u(h_j)$ divides the meridional disc D' into two subdiscs, one of which encloses a unique singular point of elliptic type. Denote by $I_j \subset \mu'$ the arc subtended by $W^u(h_j)$ which, together with $W^u(h_j)$, bounds the subdisc containing a unique elliptic point (as in Figure 1).

We claim that some iterate of Ψ' maps I_1 into I_2 (or vice-versa). Indeed since Ψ' is topologically conjugate to an irrational rigid rotation, we may argue as if it is such a rotation. Iterates of Ψ' map the clockwise endpoint of I_1 arbitrarily close to the clockwise endpoint of I_2. So either an iterate of Ψ' maps I_1 into I_2, or I_2 into I_1, or the I_j's have exactly the same length. If they all have the same length, then a small perturbation of D' near one of the hyperbolic points will change the length of one of the I_j, thus proving our claim.

Suppose $(\Psi')^n$ maps I_1 into I_2. Then the $(n+1)$-fold cover $\tilde{V}$ of V contains an overtwisted disc as illustrated in Figure 1. The cover is composed of $n+1$ copies

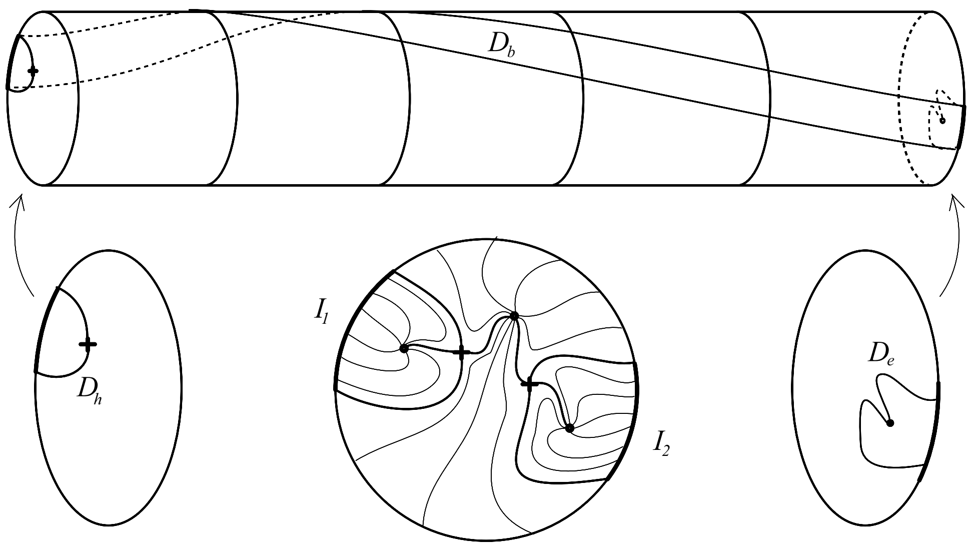

Figure 1. Above, the $(n+1)$-fold cover of V' (n fundamental domains shown, above) contains an overtwisted disc. Below, center, the characteristic foliation D'_ξ.

of V cut along its meridional disc, inside of which there are $n+1$ copies of V', labeled V'_i, cut along D'. In V'_1, say, one has a subdisc D_h of D' cut out by $W^u(h_2)$ and I_1. Let D_b be the disc consisting of all leaves in the characteristic foliation of $\partial\tilde{V}'$ emanating from I_1 in $\partial V'_1$ and terminating within I_2 in $\partial V'_n$. Finally let D_e be the subdisc of D' in V_n consisting of leaves of the characteristic foliation of D' emanating form the interval $D_b \cap I_2$ union the elliptic point to which they limit. We may now use Lemma 3.4 to smooth the corners of $D_h \cup D_b \cup D_e$ and obtain a disc D_o without changing the characteristic foliation. So ∂D_o is tangent to the characteristic foliation and contains exactly one elliptic and one hyperbolic singularity of the same sign. Using the Elimination Lemma 3.2 and Lemma 3.3 we may cancel these singularities leaving ∂D_o a closed leaf in the characteristic foliation: an overtwisted disc. □

Proposition 4.3. *There is a near-identity deformation retraction of V to V' such that $(\partial V')_\xi$ is nonsingular and irrational.*

Proof: Recall that we work under the assumption that $(\partial V)_\xi$ is a linear foliation. As such, it is a simple matter to construct a solid torus W in $(\mathbb{R}^2 \times S^1, \xi_0)$, where $\xi_0 = \ker(d\phi + r^2 d\theta)$ [in polar coordinates] so that $(\partial W)_{\xi_0}$ and $(\partial V)_\xi$ agree: simply choose the tube $\{r \leq \sqrt{-\kappa}\}$, where $\kappa < 0$ is the slope of $(\partial V)_\xi$. Using Lemma 3.1, a neighborhood of ∂V in V is contactomorphic to a neighborhood of ∂W in W — note that the contactomorphism cannot be extended over all of W since the meridians of ∂V and ∂W have different self-linking numbers. Shrinking ∂W radially yields a 1-parameter family of nearby tori with linear foliations varying nontrivially and continuously. Thus, tori with irrational foliations exist near ∂W. Pulling back the deformation retraction of W by the contactomorphism yields the desired solid torus $V' \subset V$. □

Proof: (of Theorem 4.1) Given V with $m < -1$ we may deformation retract this slightly to obtain the solid torus V' such that $(\partial V')_\xi$ is linear and irrational. The restriction of D to V' is still a meridional disc for V' with transverse boundary having self-linking number m. Proposition 4.2 then implies an overtwisted cover.

In the case where $m = -1$, a result of Makar-Limanov [22] states that (V, ξ) is contactomorphic to $W = \{(r, \theta, \phi) \in \mathbb{R}^2 \times S^1 | r \leq f(\theta, \phi)\}$ for some positive function $f : T^2 \to \mathbb{R}$ with the tight contact structure, $d\phi + r^2 d\theta$, on W. By lifting along the ϕ coordinate, one obtains $\tilde{W}$ an infinite cylinder in $(\mathbb{R}^2 \times \mathbb{R}^1, d\tilde{\phi} + r^2 d\theta)$. This is the standard tight contact structure on $\mathbb{R}^3$ (in polar coordiantes). Since pulling back the contact form on W (and hence on V) to its universal cover yields a tight structure, the same result holds for all finite covers. □

5. Pseduo-holomorphic curves and periodic orbits

5.1. **Reeb fields and the Weinstein conjecture.** Recall the definition of a *Reeb field*, X_α, associated to a contact form α:

$$\iota_{X_\alpha} \alpha = 1, \qquad \iota_{X_\alpha} d\alpha = 0. \tag{5.1}$$

Certain questions in Hamiltonian dynamics can be reformulated in terms of the dynamics of X_α. This relation and results of Rabinowitz [26] and Weinstein [29] concerning periodic orbits in Hamiltonian dynamics led Weinstein to pose the following:

Conjecture 5.1 (Weinstein Conjecture)**.** *For each contact form on a closed 3-manifold the corresponding Reeb vector field has a periodic orbit.*

Hofer [15] has recently made extraordinary progress on the Weinstein conjecture. Among other things he has shown the following:

Theorem 5.2. *Any contact form α associated to an overtwisted contact structure on a closed 3-manifold M possesses a closed orbit in its Reeb field which is of finite (perhaps trivial) order in $\pi_1(M)$.*

It has been observed by some experts that Hofer's theorem is still true for manifolds with boundary provided the Reeb vector field is tangent to the boundary. Unfortunately neither this result nor its proof have appeared in the literature. So for the sake of completeness we briefly sketch the proof of Theorem 5.2 noting the necessary modifications to make it valid for manifolds with boundary.

5.2. **A review of pseduo-holomorphic curves.** The main tool in the proof of Theorem 5.2 is the use of pseudo-holomorphic curves in the symplectization of the contact manifold. Given a contact 3-manifold (M, ξ) with defining form α there is an induced symplectic form

$$\omega = d(e^t \alpha) = e^t (dt \wedge \alpha + d\alpha) \tag{5.2}$$

on $W = \mathbb{R} \times M$. Choose a complex structure $J_\xi : \xi \to \xi$ on ξ so that $\alpha(v, J_\xi v) > 0$ for all $v \in \xi$, then define an almost complex structure J on W by

$$J(a, b)(h, k) = (-\alpha_b(k), J_\xi(b)\pi(k) + hX_\alpha(b)), \tag{5.3}$$

where $(h, k) \in T_{(a,b)}(\mathbb{R} \times M)$, X_α is the Reeb vector field for α, and $\pi : TM \to \xi$ is projection to ξ along X_α. Now if (S, j) is a closed Riemannian surface and Γ

is a finite subset of S then a map $u : S - \Gamma \to W$ is called *J-holomorphic* (or, *pseudo-holomorphic*, if no J is specified) if

$$du \circ j = J \circ du. \tag{5.4}$$

One may readily check that that if $\Gamma = \emptyset$ then u is constant. Given a map $u : S - \Gamma \to W$ one defines the *energy* to be

$$E(u) = \sup_{\phi \in \Sigma} \int_{S-\Gamma} u^* d(\phi\alpha), \tag{5.5}$$

where Σ is the set of all smooth maps $\phi : \mathbb{R} \to [0,1]$ satisfying $\phi' \geq 0$ and $\phi\alpha$ is the 1-form defined by $(\phi\alpha)(a,b)(h,k) = \phi(a)\alpha_b(k)$. In [15] it was shown

Theorem 5.3. *If there is a finite energy nonconstant J-holomorphic map $u : S - \Gamma \to W$, then X_α has a periodic orbit.*

This theorem is proved by examining the behavior of u near the punctures in $S - \Gamma$.

Thus to prove Theorem 5.2 we need merely find a finite energy nonconstant J-holomorphic map into W when ξ is overtwisted. To this end consider an overtwisted disc $\mathcal{D}$ in M. Orient $\mathcal{D}$ so that the unique elliptic point e in the characteristic foliation is positive (*i.e.*, ξ_e defines the orientation on $\mathcal{D}$). One then uses the following:

Theorem 5.4 (Bishop)**.** *There is a continuous map*

$$\Psi : D \times [0, \epsilon) \to W$$

so that for each $u_t = \Psi(\cdot, t)$

- *$u_t : D \to W$ is J-holomorphic.*
- *$u_t(\partial D) \subset (\mathcal{D} - \{e\}) \subset \{0\} \times M$.*
- *$u_t|_{\partial D} : \partial D \to (\mathcal{D} - \{e\})$ has winding number 1.*
- *$\Psi|_{D \times (0,\epsilon)}$ is a smooth map.*
- *$\Psi(z, 0) = e$ for all $z \in D$.*

The map Ψ is called a *Bishop filling*. Using an implicit function theorem Hofer finds a *maximal Bishop filling* $\Psi_{\max} : D \times [0, 1) \to W$. It is important to note that $\Psi_{\max}(\partial D \times [0, 1))$ cannot fill all of $\mathcal{D}$, which can be deduced from the result [15] that the map $u_t|_{\partial D} : \partial D \to \mathcal{D}$ is an embedding which is transversal to the characteristic foliation on $\mathcal{D}$. One may then argue that there is a sequence of $t_k \to 1$ and $z_k \to z_0$ so that $|\nabla \Psi_{\max}(z_k, t_k)| \to \infty$, since, if this were not the case, the sequence $\Phi_{\max}(\cdot, t_k)$ would converge to a J-holomorphic disc which would allow us to extend $\Psi_{\max}$ to a larger Bishop filling. After reparameterization, one may assume that the z_k are bounded away from the boundary of D: thus the gradients are blowing up on the interior of D.

Following [15], assume that $z_k = 0$ for all k and the norm of the gradient of $\Psi_{\max}(\cdot, t_k)$ is maximal at 0. Define a sequence of maps $v_k : D_k \to W$ where D_k is a disc of radius $R_k = |\nabla \Psi_{\max}(0, t_k)|$ by

$$v_k(z) = (a_k(z/R_k) - a_k(0), u_k(z/R_k)), \tag{5.6}$$

where $\Psi_{\max}(z, t_k) = (a_k(z), u_k(z))$. One may then check that the v_k converge to a nonconstant J-holomorphic finite energy plane $v : \mathbb{C} \to W$. Thus Theorem 5.3 implies X_α has a periodic orbit (since $\mathbb{C} = S^2 - \{\infty\}$).

The proof of Theorem 5.2 is very implicit in the sense that one has no information on the placement of the periodic orbit given the placement of the overtwisted disc. In order to extend the proof to manifolds with boundary, we will consider how the Bishop filling interacts with the boundary of the manifold: we require the following definition.

Definition 5.5. Let (W, J) be a 4-dimensional almost complex manifold. If M is a 3-dimensional submanifold then there exists a unique hyperplane field of complex tangencies in TM. By this we mean that there is a 2-dimensional subbundle η of TM such that η is J-invariant (and hence J is a complex structure on η). Choose a defining 1-form β on M such that $\eta = \ker \beta$. The *Levi form*, L, is defined to be the restriction of $d\beta(\cdot, J\cdot)$ to η. If L is identically zero we say that M is *Levi flat* (this implies that η defines a codimension-one foliation of M).

One may now easily verify:

Lemma 5.6. *Let α be a contact 1-form on M^3 whose Reeb vector field X_α is tangent to ∂M. Then the boundary of $W = \mathbb{R} \times M$ is Levi-flat with respect to the almost complex structure J in Equation 5.3; more specifically, ∂W is foliated by the complex surfaces $\mathbb{R} \times \gamma$ where $\gamma \subset \partial M$ is an orbit of X_α.*

Finally we need to recall how J-holomorphic curves intersect.

Theorem 5.7 (McDuff [23])**.** *Two closed distinct J-holomorphic curves C and C' in an almost complex 4-manifold (W, J) have only a finite number of intersection points. Each such point contributes a positive number to the algebraic intersection number $C \cdot C'$.*

We are now ready to prove:

Theorem 5.8. *If α is associated to an overtwisted contact structure on a compact 3-manifold with boundary and the Reeb vector field is tangent to the boundary, then the Reeb vector field has a closed orbit which is of finite (perhaps trivial) order in $\pi_1(M)$.*

Proof: We begin by completing M to a closed manifold M' containing M and extending α to α', a contact 1-form over M', in the standard manner. Let W' be the associated symplectization of M'. From the outline of Theorem 5.2 above, there exists a maximal Bishop family of J-holomorphic discs $\Psi : D^2 \times [0, 1) \to W'$ for some standard overtwisted disc $\mathcal{D}$ in M. Note that $\Psi(\partial D, t) \subset \mathcal{D} \subset \{0\} \times M$. We now claim that $\Psi(D, t) \subset \mathbb{R} \times M$ as well. Indeed, if this is not the case, then one of the $D_t = \Psi(D, t)$ would touch ∂W tangentially and thus, since ∂W is Levi-flat, D_t would intersect the J-holomorphic curve $C_\gamma = \mathbb{R} \times \gamma$ (where γ is the orbit of the Reeb flow on ∂M passing through the point of intersection). This contradicts Theorem 5.7 since the algebraic intersection of D_t and C_γ is zero.

Recall one obtains a finite energy plane $v : \mathbb{C} \to W'$ by rescaling the Bishop family near the points where the gradient is blowing up. But since all the $\Psi(D, t)$ lie in M, so does $v(\mathbb{C})$, implying that one has a periodic orbit of $X_{\alpha'}$ within M: a periodic orbit of X_α. $\Box$

5.3. **Finite energy foliations and surfaces of sections.** In this section we discuss some recent work of Hofer et al. [16, 18] that will be needed to complete the proof of Theorem 6.1.

Definition 5.9. Let α be a contact form on a 3-manifold M and J_α be a complex structure on $\xi = \ker(\alpha)$ as in the beginning of the previous section. A *finite energy foliation* of M is a 2-dimensional foliation $\mathcal{F}$ of $W = \mathbb{R} \times M$ which is invariant under translation along $\mathbb{R}$ and whose leaves are J-holomorphic surfaces having uniformly bounded energies.

Several useful facts concerning finite energy foliations appear in [16, 18]:

Lemma 5.10 (Hofer et al. [16, 18])**.** *Let $\mathcal{F}$ be a finite energy foliation of (M, α), then*

- *If F is a leaf of $\mathcal{F}$ invariant under some translation then $F = \mathbb{R} \times P$ where P is a periodic orbit of X_α.*
- *If a leaf F is not invariant under any translation then its projection $\hat{F}$ to M is an embedded submanifold of M transversal to X.*
- *If the projection $\hat{F}$ and $\hat{G}$ of two leaves of $\mathcal{F}$ intersect in M then F is a translate of G.*

This lemma implies that one obtains a foliation on the complement of some periodic orbits in M which is transverse to the Reeb flow.

Finite energy foliations of S^3 are in some sense generic. To make this precise, recall that a periodic orbit in a Reeb flow is *nondegenerate* if the linearized Poincaré return map associated to the orbit does not have 1 as an eigenvalue. A contact form α on a 3-manifold M is called *nondegenerate* if all the periodic orbits are nondegenerate. A result in [17] asserts that for a fixed contact form α on M, the set of positive functions f such that $f\alpha$ is nondegenerate is a Baire set in $C^\infty(M, (0, \infty))$.

Given α_0 the standard contact form on S^3, the main theorem concerning the existence of finite energy foliations is:

Theorem 5.11 ([16, 18])**.** *If α is a nondegenerate tight contact form on S^3, then there is a Baire set of admissible complex structures J on ξ for which (S^3, α, J) admits a finite energy foliation.*

6. The weinstein conjecture for solid tori

In this section we prove the Weinstein conjecture on the solid torus. Specifically,

Theorem 6.1. *Every Reeb field on $S^1 \times D^2$ tangent to the boundary possesses a periodic orbit.*

Of course, this applies to general contact 3-manifolds possessing invariant solid tori; however, the hypotheses are, for a general Reeb flow, almost as hard to verify as the existence of a closed orbit in the first place. Nevertheless, this result is useful in certain specific examples.

Lemma 6.2. *If T is a torus invariant under the flow of a Reeb field X (associated to ξ) and X has no periodic orbits on T, then the characteristic foliation T_ξ is linear (up to conjugacy).*

Proof: Since X is tangent to T the characteristic foliation is nonsingular. We may assume that the characteristic foliation is not a linear foliation by meridional curves. In the case where there does not exist a transversal meridian (*i.e.*, there is a Reeb component in T_ξ), then X must have a periodic orbit (since X must point transversely into or out of the Reeb component, hence limiting to a periodic orbit). We may thus pick a transversal meridian and consider the rotation number associated to T_ξ. If the rotation number is irrational then the characteristic foliation is conjugate to a linear foliation. If the rotation number is rational then there is a closed leaf L in the foliation. Since the flow of X preserves both ξ and T, it also preserves T_ξ. Thus T_ξ is simply a foliation by rational curves. □

Proof of Theorem 6.1: Assume there are no periodic orbits on the boundary. Suppose X is associated to the contact form α and the contact structure ξ. If ξ is overtwisted then we are done by Theorem 5.8; thus, assume that ξ is tight. Since X is tangent to the boundary, the characteristic foliation on T is nonsingular. We may assume that the foliation is linear by Lemma 6.2 , and that it is not composed of meridians by tightness. Hence, we may choose the meridian μ transverse to T_ξ. If the self-linking number $m := \ell(\mu)$ is less than -1, Theorem 4.1 implies that some finite cover of $(S^1 \times D^2, \alpha, \xi)$ is overtwisted and thus has a periodic orbit in its Reeb flow. Under the covering map, flowlines are mapped to flowlines and hence X must also have a periodic orbit.

We are left to consider the case when $m = -1$. Assume that the Reeb field does not possess a periodic orbit. Then it follows from [22] that there exists a map f from $S^1 \times D^2$ to a neighborhood V of a transversal unknot in S^3 such that $f_*(\xi) = \xi_0$ where ξ_0 is the standard tight contact structure on S^3. We may thus push α forward to V and extend it to a contact form α' on all of S^3. Thus we have a Reeb vector field $X_{\alpha'}$ associated to the tight contact structure on S^3. All the periodic orbits of $X_{\alpha'}$ must by assumption lie outside V.

Lemma 6.3. *One can perturb α' fixing V so that it is a nondegenerate contact form.*

Proof: This proof is the relative version of [17, Prop. 6.1]. Briefly, one embeds (S^3, α') into the symplectization $(\mathbb{R} \times S^3, d(e^t\alpha'))$ as $\{0\} \times S^3$, identifying the Reeb field of α' with the induced Hamiltonian field on the hypersurface. Perturbing the hypersurface $\{0\} \times S^3$ is equivalent to perturbing the Hamiltonian field. The theorem of Robinson [27, Thm. 1.B.iv] states that nondegenerate Hamiltonian fields are residual among hypersurfaces. In particular, this result holds for open manifolds in the strong C^∞ topology on the perturbations: thus, perturb the hypersurface on the complement of V, with the perturbation going to zero quickly near the boundary. This yields a nondegenerate hypersurface which, since contact forms are open in the space of 1-forms, implies a nondegenerate contact form on S^3 which agrees with α' on V. □

Given this, Theorem 5.11 yields a finite energy foliation of S^3. Let $\mathcal{F}$ denote the foliation transversal to the Reeb flow on the complement of some finite set of periodic orbits in $S^3 - V$. Since ∂V is invariant under the flow of $X_{\alpha'}$ it is transversal to $\mathcal{F}$. Thus $\mathcal{F} \cap \partial V$ is a foliation of ∂V by circles. Moreover $\mathcal{F} \cap V$ is a foliation of V by either discs or annuli; however, the presence of any annuli would clearly contradict the fact that $\mathcal{F}$ intersects ∂V transversally. Thus there is a foliation of V by discs transversal to the Reeb field. Following the proof in the case of a stratified integral, this must be a foliation by meridional discs. An application of the Brouwer fixed point theorem concludes the proof. □

Remark 6.4. We note that an alternate approach to the final step in the proof of Theorem 6.1 exists. Instead of using finite-energy foliations, one can use the (currently developing) *contact homology* of Hofer and Eliashberg [8]. This is a homology theory that is defined in terms of a contact 1-form (actually a Reeb vector field) but depends only on the underlying contact structure. The chain groups for this homology are generated by periodic orbits in the Reeb flow and the boundary operator is defined using pseudo-holomorphic curves in the symplectization that limit to the periodic orbits in various ways. It seems the discussion in §5.2 is sufficient to allow one to define this contact homology for manifolds with boundary (if the implicated Reeb fields are all tangent to the boundary). Then, using standard models for all the universally tight contact structures, one can compute that the contact homology is non-trivial and thus there must be periodic orbits in any Reeb field associated to a universally tight contact form on $S^1 \times D^2$.

References

[1] B. Aebisher. *Symplectic Geometry.* Number 124 in Progress in Math. Birkhaüser, Berlin, 1994.

[2] V. I. Arnold. Sur la géométrie differentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier*, 16:316–361, 1966.

[3] V. I. Arnold and B. Khesin. Topological methods in hydrodynamics. *Annu. Rev. Fuid Mech.*, 24:145–166, 1992.

[4] V. I. Arnold and B. Khesin. *Topological Methods in Hydrodynamics.* Springer-Verlag, Berlin, Heidelberg, New York, 1998.

[5] D. Bennequin. Entrelacements et équations de Pfaff. *Asterisque*, 107-108:87–161, 1983.

[6] Y. Eliashberg. Contact 3-manifolds twenty years since J. Martinet's work. *Ann. Inst. Fourier, Grenoble*, 42(1-2):165–192, 1992.

[7] Y. Eliashberg. Invariants in contact topology. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, number Extra Vol. II, pages 327–338 (electronic), 1998.

[8] Y. Eliashberg and M. Fraser. Classification of topologically trivial Legendrian knots. In *Geometry, topology, and dynamics (Montreal, PQ, 1995)*, pages 17–51. Amer. Math. Soc., Providence, RI, 1998.

[9] Y. Eliashberg and W. Thurston. *Confoliations*, volume 13 of *University Lecture Series.* American Mathematical Society, 1998.

[10] J. Etnyre. *Symplectic Constructions on 4-Manifolds.* PhD thesis, University of Texas, Austin, 1996.

[11] J. Etnyre and R. Ghrist. Contact topology and hydrodynamics I: existence and the Seifert conjecture. Preprint, 1998.

[12] J. Etnyre and R. Ghrist. Contact topology and hydrodynamics III: knotted flowlines. Preprint, 1999.

[13] S. Friedlander and M. Vishik. Instability criteria for steady flows of a perfect fluid. *Chaos*, 3:455–460, 1992.

[14] E. Giroux. Convexité en topologie de contact. *Comment. Math. Helv.*, 66(4):637–677, 1991.

[15] H. Hofer. Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three. *Invent. Math.*, 114:515–563, 1993.

[16] H. Hofer. Holomorphic curves and dynamics in dimension three. To appear in PCMI/IAS Lecture Notes, 1998.

[17] H. Hofer, K. Wysocki, and E. Zehnder. The dynamics on three-dimensional strictly convex energy surfaces. *Ann. of Math. (2)*, 148(1):197–289, 1998.

[18] H. Hofer, K. Wysocki, and E. Zehnder. Finite energy foliations of tight three-spheres. In preparation, 1999.

[19] G. Kuperberg. A volume-preserving counterexample to the Seifert conjecture. *Comment. Math. Helv.*, 71(1):70–97, 1996.

[20] G. Kuperberg and K. Kuperberg. Generalized counterexamples to the Seifert conjecture. *Ann. of Math.*, 144:239–268, 1994.

[21] K. Kuperberg. A smooth counterexample to the Seifert conjecture. *Ann. of Math.*, 140:723–732, 1994.

[22] S. Makar-Limanov. Tight contact structures on solid tori. *Trans. Am. Math. Soc.*, 350:1013–1044, 1998.

[23] D. McDuff. The local behavior of holomorphic curves in almost complex 4-manifolds. *J. Diff. Geom.*, 34:143–164, 1991.

[24] H. Moffatt. Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology: part I. *J. Fluid Mech.*, 159:359–378, 1985.

[25] H. Moffatt, editor. *Topological aspects of the dynamics of fluids and plasmas. Proceedings of the Program of the Institute for Theoretical Physics on Topological Fluid Dynamics held August–December 1991 and the NATO Advanced Research Workshop held at the University of California, Santa Barbara, California, November 1–5, 1991*, number 218 in NATO Advanced Science Institutes Series E: Applied Sciences. Kluwer Academic Press, 1994.

[26] P. Rabinowitz. Periodic solutions of Hamiltonian systems on a prescribed energy surface. *J. Diff. Equ.*, 33:336–352, 1979.

[27] J. Robbins. A structural stability theorem. *Ann. Math.*, 94:447–493, 1971.

[28] D. Tischler. On fibering certain foliated manifolds over S^1. *Topology*, 9:153–154, 1970.

[29] A. Weinstein. On the hypothesis of Rabinowitz's periodic orbit theorem. *J. Diff. Equ.*, 33:353–358, 1979.

Department of Mathematics, Stanford University, Palo Alto, CA, 94305

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332