THE $\mathbb{G}_0$-DICHOTOMY

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Abstract. In this note, we give a simple, graph-theoretic proof of the Kechris–Solecki–Todorcevic $\mathbb{G}_0$-dichotomy. We then present some consequences of the $\mathbb{G}_0$-dichotomy in Borel combinatorics, including the Luzin–Novikov and Feldman–Moore theorems. We also prove the injective version of the $\mathbb{G}_0$-dichotomy for locally countable Borel graphs and show that it fails in general (i.e., without the local countability assumption).

1. The Kechris–Solecki–Todorcevic $\mathbb{G}_0$-dichotomy

Let G be a graph on a Polish space. Recall that G is **Borel** (resp. **analytic**) if $E(G)$ is a Borel (resp. analytic) subset of the space $[V(G)]^2$. The notation $x \sim_G y$ indicates that x and y are adjacent vertices of G. A **proper coloring** of G is a function f defined on $V(G)$ such that $f(x) \neq f(y)$ whenever $x \sim_G y$. Equivalently, f is a proper coloring if for every color $\alpha \in \mathrm{im}(f)$, the set $f^{-1}(\alpha) \subseteq V(G)$ is G-independent. The **Borel chromatic number** of G, denoted by $\chi_{\mathsf{B}}(G)$, is the minimum cardinality of a Polish space C such that G admits a Borel proper coloring $f\colon V(G) \to C$. Note that $\chi_{\mathsf{B}}(G) \leqslant \aleph_0$ if and only if $V(G)$ can be covered by countably many G-independent Borel sets.

A **homomorphism** from a graph H to a graph G is a function $f\colon V(H) \to V(G)$ such that if $x \sim_H y$, then $f(x) \sim_G f(y)$, that is, f sends the edges of H to edges of G. Equivalently, f is a homomorphism if the f-preimage of every G-independent set is H-independent. If H and G are graphs on Polish spaces, then we write $H \to_{\mathsf{c}} G$ to indicate that there is a continuous homomorphism from H to G. Note that if $H \to_{\mathsf{c}} G$, then $\chi_{\mathsf{B}}(G) \geqslant \chi_{\mathsf{B}}(H)$. In this note we shall establish the following remarkable result of Kechris, Solecki, and Todorcevic, known as the $\mathbb{G}_0$**-dichotomy**:

Theorem 1.1 (Kechris–Solecki–Todorcevic [KST99])**.** *There exists a Borel graph $\mathbb{G}_0$ such that for every analytic graph G on a Polish space, precisely one of the following two alternatives holds:*

- *either $\chi_{\mathsf{B}}(G) \leqslant \aleph_0$;*
- *or $\mathbb{G}_0 \to_{\mathsf{c}} G$.*

The graph $\mathbb{G}_0$ satisfying the conclusion of Theorem 1.1 has a simple explicit construction. Let $2^{<\infty} := \bigcup_{n \in \mathbb{N}} 2^n$ be the set of all finite sequences of zeroes and ones (including the empty sequence) and let $\mathcal{C} := 2^{\mathbb{N}}$ denote the Cantor space. A set $S \subseteq 2^{<\infty}$ is **dense** if for all $t \in 2^{<\infty}$, there is $s \in S$ extending t. Given a set $S \subseteq 2^{<\infty}$, let $\mathbb{G}_S$ denote the Borel (more precisely, F_σ) graph on $\mathcal{C}$ whose edge set comprises all pairs of the form $\{s^\frown i^\frown x, s^\frown i'^\frown x\}$ with $s \in S$, $\{i, i'\} = \{0, 1\}$, and $x \in \mathcal{C}$. Here and in what follows, $^\frown$ denotes concatenation of sequences.

Theorem 1.2. *Let G be an analytic graph on a Polish space and let $S \subset 2^{<\infty}$ be a set containing exactly one sequence of each finite length* (*including the empty sequence*)*. Then either $\chi_{\mathsf{B}}(G) \leqslant \aleph_0$ or $\mathbb{G}_S \to_{\mathsf{c}} G$. If, in addition, S is dense, then these two possibilities are mutually exclusive.*

Therefore, to obtain the conclusion of Theorem 1.1, we can set $\mathbb{G}_0 := \mathbb{G}_S$ for an arbitrary dense set $S \subseteq 2^{<\infty}$ that contains exactly one sequence of each finite length. (Such a set S exists by Exercise 9.1.) In other words, Theorem 1.1 is an immediate consequence of Theorem 1.2.

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The original proof of Theorem 1.2 due to Kechris, Solecki, and Todorcevic [KST99] used methods of effective descriptive set theory. A classical proof was later discovered by Miller [Mil09]. The proof we present here follows the outline sketched by Miller in [Mil12]. A prominent feature of this argument is its reliance on graph-theoretic intuition.

2. The Borel chromatic number of $\mathbb{G}_S$

The following proposition shows that the two alternatives in Theorem 1.2 are mutually exclusive when S is dense.

Proposition 2.1. *Suppose that $S \subseteq 2^{<\infty}$ is dense. Then every Baire-measurable $\mathbb{G}_S$-independent subset of $\mathcal{C}$ is meager. In particular, $\chi_{\mathsf{B}}(\mathbb{G}_S) > \aleph_0$.*

PROOF. Let $I \subseteq \mathcal{C}$ be a Baire-measurable $\mathbb{G}_S$-independent set and suppose, toward a contradiction, that I is nonmeager. By the Baire alternative, this implies that there is a nonempty open subset $U \subseteq \mathcal{C}$ such that I is comeager in U. The topology on $\mathcal{C}$ is generated by the clopen sets of the form

$$U_s \coloneqq \{x \in \mathcal{C} \,:\, s \subset x\}, \quad \text{where } s \in 2^{<\infty}.$$

(Here "$s \subset x$" means that s is an initial segment of x.) Thus, we may assume that $U = U_s$ for some $s \in 2^{<\infty}$. Furthermore, since S is dense, we may also assume that $s \in S$.

Consider the mapping $f\colon U_s \to U_s$ defined by

$$f(s^\frown i^\frown y) \coloneqq s^\frown \bar{i}^\frown y \quad \text{for all } i \in \{0,1\} \text{ and } y \in \mathcal{C},$$

where for $i \in \{0,1\}$, we write $\bar{i} := 1 - i$. By definition, $x \sim_{\mathbb{G}_S} f(x)$ for all $x \in U_s$, and thus, since I is $\mathbb{G}_S$-independent, we have $I \cap f(I) = \varnothing$. But f is a homeomorphism of U_s, which implies that $f(I)$—and hence also $I \cap f(I)$—is comeager in U_s. This contradiction completes the proof. ∎

It is easy to see that when S contains at most one sequence of each length, the graph $\mathbb{G}_S$ is acyclic (Exercise 9.2). Therefore, taking S to be a dense set with this property, we obtain an example of an acyclic locally countable Borel graph whose Borel chromatic number is uncountable.

3. Proof of Theorem 1.2

Let G be an analytic graph on a Polish space and let $\pi\colon E^* \to [V(G)]^2$ be a continuous map from a Polish space E^* to the space $[V(G)]^2$ such that $\mathrm{im}(\pi) = E(G)$. For a finite graph H, a **copy** of H in G is a mapping φ that assigns to each vertex $u \in V(H)$ a vertex $\varphi(u) \in V(G)$ and to each edge $uv \in E(H)$ an element $\varphi(uv) \in E^*$, with the following compatibility property:

$$\pi(\varphi(uv)) = \varphi(u)\varphi(v) \quad \text{for all } uv \in E(H).$$

(We tacitly assume that the sets $V(H)$ and $E(H)$ are disjoint.) In particular, if φ is a copy of H in G, then $\varphi{\restriction}V(H)$ is a homomorphism from H to G; but φ additionally selects a specific preimage in E^* for every edge of the form $\varphi(u)\varphi(v)$. The set of all copies of H in G is denoted by $\mathsf{Hom}(H,G)$. Note that $\mathsf{Hom}(H,G)$ is a closed subset of the product space

$$V(G)^{V(H)} \times (E^*)^{E(H)}.$$

For a subset $\mathcal{H} \subseteq \mathsf{Hom}(H,G)$ and a vertex $u \in V(H)$, let

$$\mathcal{H}(u) \coloneqq \{\varphi(u) \,:\, \varphi \in \mathcal{H}\} \subseteq V(G).$$

Note that if $\mathcal{H}$ is Borel, then the set $\mathcal{H}(u)$ is analytic. Similarly, for $uv \in E(H)$, let

$$\mathcal{H}(uv) \coloneqq \{\varphi(uv) \,:\, \varphi \in \mathcal{H}\} \subseteq E^*.$$

For each finite graph H, we define a certain σ-ideal on $\mathsf{Hom}(H,G)$. Call a Borel subset $\mathcal{H} \subseteq \mathsf{Hom}(H,G)$ **tiny** if for some $u \in V(H)$, the set $\mathcal{H}(u)$ is G-independent. Call a subset (not necessarily Borel) $\mathcal{H} \subseteq \mathsf{Hom}(H,G)$ **small** if it can be covered by countably many tiny Borel sets; call $\mathcal{H}$ **large** if it is not small. By definition, small sets form a σ-ideal. Note that if $\bullet$ denotes the graph with a single vertex and no edges, then $\mathsf{Hom}(\bullet, G)$ can be identified with $V(G)$ in the obvious way, and a set $A \subseteq V(G)$ is small if and only if it can be covered by countably many G-independent Borel sets. In particular, $V(G)$ itself is small if and only if G has countable Borel chromatic number.

Let H be a finite graph and let $u \in V(H)$. Denote by $H +_u H$ the graph with vertex set $V(H) \times \{0,1\}$ and edge set given by

$$(v,i) \sim_{H+_uH} (w,j) \quad :\Longleftrightarrow \quad (i = j \text{ and } v \sim_H w) \text{ or } (i \neq j \text{ and } v = w = u).$$

In other words, $H +_u H$ is obtained from two disjoint copies of H by adding an edge between their corresponding copies of u. For $\varphi \in \mathsf{Hom}(H +_u H, G)$ and $i \in \{0,1\}$, define $\varphi^i \in \mathsf{Hom}(H, G)$ by

$$\varphi^i(v) := \varphi((v,i)) \text{ for } v \in V(H);$$
$$\varphi^i(v_1v_2) := \varphi((v_1,i)(v_2,i)) \text{ for } v_1v_2 \in E(H).$$

For $\mathcal{H} \subseteq \mathsf{Hom}(H,G)$ and $u \in V(H)$, let $\mathcal{H} +_u \mathcal{H}$ denote the set of all copies φ of $H +_u H$ in G such that φ^0, $\varphi^1 \in \mathcal{H}$. It is clear that if $\mathcal{H}$ is a Borel subset of $\mathsf{Hom}(H,G)$, then $\mathcal{H} +_u \mathcal{H}$ is a Borel subset of $\mathsf{Hom}(H +_u H, G)$. The key insight for the proof of Theorem 1.2 is given by following lemma.

Lemma 3.1. *Let H be a finite graph and let $u \in V(H)$. Suppose that a Borel subset $\mathcal{H} \subseteq \mathsf{Hom}(H,G)$ is large. Then the set $\mathcal{H} +_u \mathcal{H} \subseteq \mathsf{Hom}(H +_u H, G)$ is also large.*

Proof. First we observe that $\mathcal{H} +_u \mathcal{H} \neq \varnothing$. Indeed, since $\mathcal{H}$ is large, hence not tiny, and Borel, the set $\mathcal{H}(u)$ is not G-independent, i.e., there exist φ_0, $\varphi_1 \in \mathcal{H}$ with $\varphi_0(u) \sim_G \varphi_1(u)$, so we can pick $e \in E^*$ with $\pi(e) = \varphi_0(u)\varphi_1(u)$. Then the map $\varphi \in \mathsf{Hom}(H +_u H, G)$ with $\varphi^0 = \varphi_0$, $\varphi^1 = \varphi_1$, and $\varphi((u,0)(u,1)) = e$ belongs to $\mathcal{H} +_u \mathcal{H}$.

Now assume, toward a contradiction, that $\mathcal{H} +_u \mathcal{H}$ is small and hence it can be expressed as

$$\mathcal{H} +_u \mathcal{H} \;=\; \bigcup_{n=0}^{\infty} \mathcal{F}_n, \tag{3.2}$$

where each $\mathcal{F}_n$ is a tiny Borel subset of $\mathsf{Hom}(H +_u H, G)$. For every $n \in \mathbb{N}$, fix some $v_n \in V(H)$ and $i_n \in \{0,1\}$ so that the set $\mathcal{F}_n((v_n, i_n))$ is G-independent. By Exercise 9.4, there are G-independent Borel sets $I_n \supseteq \mathcal{F}_n((v_n, i_n))$. Let

$$\mathcal{H}_n \;:=\; \{\varphi \in \mathcal{H} \,:\, \varphi(v_n) \in I_n\}.$$

By definition, the (Borel) sets $\mathcal{H}_n$ are tiny. Therefore, the set $\mathcal{H}' := \mathcal{H} \setminus \bigcup_{n=0}^{\infty} \mathcal{H}_n$ is large. Since $\mathcal{H}'$ is Borel, we obtain $\mathcal{H}' +_u \mathcal{H}' \neq \varnothing$. But $\mathcal{H}' +_u \mathcal{H}'$ is disjoint from $\bigcup_{n=0}^{\infty} \mathcal{F}_n$, which contradicts (3.2). ■

For the next lemma, we fix compatible complete metrics d on $V(G)$ and δ on E^*.

Lemma 3.3. *Let H be a finite graph. Suppose that a Borel subset $\mathcal{H} \subseteq \mathsf{Hom}(H,G)$ is large. Then for any $\varepsilon > 0$, there is a large Borel subset $\mathcal{H}' \subseteq \mathcal{H}$ such that:*

- *for all $u \in V(H)$, $\operatorname{diam}_d(\overline{\mathcal{H}'(u)}) < \varepsilon$;*
- *for all $(u,v) \in E(H)$, $\operatorname{diam}_\delta(\overline{\mathcal{H}'(uv)}) < \varepsilon$.*

Proof. The desired conclusion follows since $V(G)$ and E^* can be covered by countably many closed sets of diameter less than ε and small sets form a σ-ideal. ■

Now we are ready to prove Theorem 1.2. Assume that $\chi_{\mathsf{B}}(G) > \aleph_0$ (i.e., that $V(G)$ is large) and fix a set $S \subset 2^{<\infty}$ that contains a unique sequence of each finite length. Our goal is to show that $\mathbb{G}_S \to_{\mathsf{c}} G$. We shall obtain a desired continuous homomorphism from $\mathbb{G}_S$ to G as a "limit" of copies in G of certain *finite* graphs. Specifically, for each $n \in \mathbb{N}$, let T_n be the tree on $\{0,1\}^n$ whose edges are of the form $\{s^\frown i^\frown t, s^\frown i'^\frown t\}$ with $s \in S$, $\{i, i'\} = \{0,1\}$, $t \in 2^{<\infty}$, and $\mathrm{length}(s) + \mathrm{length}(t) = n-1$. For $n \in \mathbb{N}$, let s_n denote the unique element of S of length n. Then we have $T_0 = \bullet$ and $T_{n+1} = T_n +_{s_n} T_n$

for all $n \in \mathbb{N}$. Using Lemmas 3.1 and 3.3 and the fact that $V(G)$ is large, we recursively construct a sequence of large Borel subsets $\mathcal{T}_n \subseteq \mathsf{Hom}(T_n, G)$ with the following properties:

- $\mathcal{T}_{n+1} \subseteq \mathcal{T}_n +_{s_n} \mathcal{T}_n$;
- for all $u \in \{0,1\}^n$, $\mathrm{diam}_d(\overline{\mathcal{T}_n(u)}) < 2^{-n}$;
- for all $uv \in E(T_n)$, $\mathrm{diam}_\delta(\overline{\mathcal{T}_n(uv)}) < 2^{-n}$.

The first property implies that for all $i \in \{0,1\}$,

$$\mathcal{T}_{n+1}(u^\frown i) \subseteq \mathcal{T}_n(u) \text{ for all } u \in \{0,1\}^n \quad \text{and} \quad \mathcal{T}_{n+1}(u^\frown i, v^\frown i) \subseteq \mathcal{T}_n(u,v) \text{ for all } uv \in E(T_n).$$

For $x \in \mathcal{C}$, let $f(x)$ be the unique point in $\bigcap_{n=0}^\infty \overline{\mathcal{T}_n(x\restriction n)}$, where $x \restriction n$ denotes the initial segment of x of length n (which exists due to the completeness of the metric d). The map $f\colon \mathcal{C} \to V(G)$ is clearly continuous. It remains to check that it is a homomorphism from $\mathbb{G}_S$ to G. Take any edge $xy \in E(\mathbb{G}_S)$. Then there is unique $n_0 \in \mathbb{N}$ with $x(n_0) \neq y(n_0)$, and for all $n > n_0$, we have $x\restriction n \sim_{T_n} y \restriction n$. Let $e \in E^*$ be the unique point in $\bigcap_{n=n_0+1}^\infty \overline{\mathcal{T}_n(x\restriction n, y\restriction n)}$ (which exists by the completeness of δ). Since π is continuous, we get $\pi(e) = f(x)f(y)$, i.e., $f(x) \sim_G f(y)$, as desired.

4. The Luzin–Novikov theorem

Miller [Mil09; Mil12] observed that Theorem 1.1 can be used to derive a number of important results in descriptive set theory. In this section, we give one such example: a simple combinatorial proof of the Luzin–Novikov theorem. Let X and Y be sets and let $A \subseteq X \times Y$. For $x \in X$, the **fiber** of A over x is the set $A_x := \{y \in Y : (x,y) \in A\}$.

Theorem 4.1 (Luzin–Novikov). *Let X and Y be Polish spaces and let $A \subseteq X \times Y$ be a Borel subset. Suppose that for each $x \in X$, the fiber A_x is countable. Then there exists a countable sequence of partial Borel functions $f_n\colon X \rightharpoonup Y$, $n \in \mathbb{N}$, such that for all $x \in X$ and $y \in Y$,*

$$(x,y) \in A \iff y = f_n(x) \text{ for some } n \in \mathbb{N}. \tag{4.2}$$

Proof. To construct the functions f_n, we shall apply the $\mathbb{G}_0$-dichotomy to an auxiliary graph. For each $x \in X$, let G_x be the graph with vertex set Y such that $y_1 \sim_{G_x} y_2$ if and only if $y_1 \neq y_2$ and y_1, $y_2 \in A_x$. In other words, A_x is a clique in G_x, while the vertices in $Y \setminus A_x$ are isolated in G_x. Since A_x is countable, $\chi_\mathsf{B}(G_x) \leqslant \aleph_0$. Indeed, the following is a countable Borel coloring of G_x:

$$y \mapsto \begin{cases} y & \text{if } y \in A_x; \\ * & \text{if } y \notin A_x. \end{cases}$$

(Here $*$ is an arbitrary value.) Now define a graph G with vertex set $X \times Y$ by putting edges between the pairs of vertices of the form (x, y_1) and (x, y_2), where $y_1 \sim_{G_x} y_2$. Combinatorially, G is a disjoint union of the graphs G_x taken over all $x \in X$. Since A is Borel, G is Borel as well.

Claim 4.1.1. $\chi_\mathsf{B}(G) \leqslant \aleph_0$.

$\triangleright$ Suppose not. Then, by Theorem 1.2, there is a continuous homomorphism $h\colon \mathcal{C} \to X \times Y$ from $\mathbb{G}_S$ to G, where $S \subset 2^{<\infty}$ is any dense set containing exactly one sequence of each finite length. Let $h_1\colon \mathcal{C} \to X$ and $h_2\colon \mathcal{C} \to Y$ denote the first and the second coordinates of h, respectively. Every connected component of G is contained in a set of the form $\{x\} \times Y$ for some $x \in X$, which implies that the function h_1 is constant on the connected components of $\mathbb{G}_S$. But every connected component of $\mathbb{G}_S$ is dense in $\mathcal{C}$ (Exercise 9.3), so, since h_1 is continuous, it must be constant on all of $\mathcal{C}$. If $x \in X$ is the unique value taken by h_1, then h_2 is a continuous homomorphism from $\mathbb{G}_S$ to G_x. This is impossible as $\chi_\mathsf{B}(G_x) \leqslant \aleph_0$. $\triangleleft$

Let $c\colon X \times Y \to \mathbb{N}$ be a Borel proper coloring of G. Then, for each $n \in \mathbb{N}$, we can define a Borel partial function $f_n\colon X \rightharpoonup Y$ by making $f_n(x) = y$ if and only if $y \in A_x$ and $c(x,y) = n$. Property (4.2) is immediate from the construction, so the proof is complete. ■

Corollary 4.3. *If $f\colon X \to Y$ is a countable-to-one Borel function between Polish spaces X and Y, then the set $f(X)$ is Borel.*

PROOF. Applying Theorem 4.1 to the set $\{(y, x) \in Y \times X \,:\, f(x) = y\}$ yields a countable sequence of partial Borel functions $g_n\colon Y \rightharpoonup X$, $n \in \mathbb{N}$, such that for all $x \in X$ and $y \in Y$,

$$f(x) = y \iff x = g_n(y) \text{ for some } n \in \mathbb{N}.$$

Then $f(X) = \bigcup_{n=0}^{\infty} \mathrm{dom}(g_n)$ is Borel, as desired. ■

5. Locally finite graphs

The following is a nice graph-theoretic consequence of the Luzin–Novikov Theorem 4.1:

Theorem 5.1. *If G is a locally finite Borel graph on a Polish space, then $\chi_{\mathsf{B}}(G) \leqslant \aleph_0$.*

PROOF. Fix a countable base $(U_i)_{i=0}^{\infty}$ for the topology on $V(G)$. For each $x \in X$, let

$$c(x) \coloneqq \min\{i \in \mathbb{N} \,:\, x \in U_i \setminus N_G(U_i)\}.$$

Note that $c(x)$ is well-defined since x has only finitely many neighbors. It is clear that $c\colon V(G) \to \mathbb{N}$ is a proper coloring of G. It remains to argue that the function c is Borel. To this end, we apply Theorem 4.1 to the set $\{(x, y) \in V(G) \times V(G) \,:\, x \sim_G y\}$ and get a countable sequence of partial Borel functions $f_n\colon V(G) \rightharpoonup V(G)$, $n \in \mathbb{N}$, such that for all x, $y \in V(G)$,

$$x \sim_G y \iff y = f_n(x) \text{ for some } n \in \mathbb{N}.$$

Then $N_G(U_i) = \bigcup_{n=0}^{\infty} f_n^{-1}(U_i)$ is Borel, and thus c is Borel as well. ■

6. Edge-colorings and the Feldman–Moore theorem

A **proper edge-coloring** of a graph G is a function f defined on $E(G)$ such that $f(e) \neq f(h)$ whenever distinct edges e and h share an endpoint. The **Borel chromatic index** of a graph G on a Polish space, denoted by $\chi'_{\mathsf{B}}(G)$, is the minimum cardinality of a Polish space C such that G admits a Borel proper edge-coloring $f\colon E(G) \to C$. Note that in a proper edge-coloring, all the edges incident to a given vertex must receive different colors. In particular, if $\chi'_{\mathsf{B}}(G) \leqslant \aleph_0$, then G must be locally countable. Another useful and perhaps surprising consequence of the Luzin–Novikov Theorem 4.1 is a converse to this observation for Borel graphs.

Theorem 6.1 (Feldman–Moore)**.** *If G is Borel graph on a Polish space, then $\chi'_{\mathsf{B}}(G) \leqslant \aleph_0$ if and only if G is locally countable.*

PROOF. We only need to argue that $\chi'_{\mathsf{B}}(G) \leqslant \aleph_0$ for locally countable G. If G is locally countable, then Theorem 4.1 applied to the set $\{(x, y) \in V(G) \times V(G) \,:\, x \sim_G y\}$ yields a countable sequence of partial Borel functions $f_n\colon V(G) \rightharpoonup V(G)$, $n \in \mathbb{N}$, such that for all x, $y \in V(G)$,

$$x \sim_G y \iff y = f_n(x) \text{ for some } n \in \mathbb{N}.$$

For a pair of adjacent vertices x, y, let $\ell(x, y) \coloneqq \min\{n \in \mathbb{N} \,:\, y = f_n(x)\}$ and define $c\colon E(G) \to [\mathbb{N}]^{\leqslant 2}$ by $c(xy) \coloneqq \{\ell(x, y), \ell(y, x)\}$. Let H be the graph with vertex set $E(G)$ in which two distinct edges e, h of G are adjacent if and only if they share an endpoint and $c(e) = c(h)$.

Claim 6.1.1. The graph H is locally finite. In fact, every vertex in H has at most 2 neighbors.

▷ Let xy be an edge of G with $\ell(x, y) = n$ and $\ell(y, x) = m$. Suppose $xz \in E(G)$ is another edge such that $c(xz) = c(xy) = \{n, m\}$. Since $f_n(x) = y \neq z$, we have $\ell(x, z) \neq n$, so it must be that $\ell(x, z) = m$, i.e., $z = f_m(x)$. Similarly, if yz is an edge with $c(yz) = c(xy) = \{n, m\}$, then $z = f_n(y)$. Thus, the only possible neighbors of xy in H are $xf_m(x)$ and $yf_n(y)$. ◁

By Theorem 5.1, H has a Borel proper coloring $c'\colon E(G) \to \mathbb{N}$. Then the function $E(G) \to [\mathbb{N}]^{\leqslant 2} \times \mathbb{N}\colon e \mapsto (c(e), c'(e))$ is a countable Borel edge-coloring of G, as desired. ■

7. Injective homomorphisms from $\mathbb{G}_0$ for locally countable graphs

Let G be a Borel graph on a Polish space with $\chi_{\mathsf{B}}(G) > \aleph_0$. Theorem 1.1 then yields a continuous homomorphism from $\mathbb{G}_0$ to G. It turns out that for locally countable graphs G, this homomorphism can be made injective—so G has a subgraph *isomorphic* to $\mathbb{G}_0$!

Theorem 7.1 (Kechris–Solecki–Todorcevic)**.** *Let G be a locally countable Borel graph on a Polish space and let $S \subset 2^{<\infty}$ be a set containing exactly one sequence of each finite length. If $\chi_{\mathsf{B}}(G) > \aleph_0$, then there is an injective continuous homomorphism from $\mathbb{G}_S$ to G.*

In the rest of this section, we prove Theorem 7.1. Fix a locally countable Borel graph G on a Polish space. We shall use the notation and terminology from §3. In particular, we let $\pi\colon E^* \to [V(G)]^2$ be a continuous map from some Polish space E^* to $[V(G)]^2$ such that $\mathrm{im}(\pi) = E(G)$. (Since G is Borel, the map π may be assumed to be injective, but we will not make use of this fact.) We also fix compatible complete metrics d on $V(G)$ and δ on E^*.

By Theorem 6.1, we can fix a Borel proper edge-coloring $c\colon E(G) \to \mathbb{N}$. Given a finite graph H and a proper edge-coloring $\xi\colon E(H) \to \mathbb{N}$, we say that a set $\mathcal{H} \subseteq \mathsf{Hom}(H,G)$ is ξ-**consistent** if

$$c(\pi(\varphi(e))) = \xi(e) \quad \text{for all } \varphi \in \mathcal{H} \text{ and } e \in E(H).$$

We say that $\mathcal{H} \subseteq \mathsf{Hom}(H,G)$ is **consistent** if it is ξ-consistent for some $\xi\colon E(H) \to \mathbb{N}$. We also say that $\mathcal{H} \subseteq \mathsf{Hom}(H,G)$ is **injective** if the sets $\overline{\mathcal{H}(u)}$, $u \in V(H)$, are pairwise disjoint. The main ingredient in the proof of Theorem 7.1 is the following modification of Lemma 3.1:

Lemma 7.2. *Let H be a connected finite graph and let $\mathcal{H} \subseteq \mathsf{Hom}(H,G)$ be a large Borel consistent injective set. Then for each $u \in V(H)$, there is a large Borel consistent injective subset $\mathcal{H}' \subseteq \mathcal{H} +_u \mathcal{H}$.*

PROOF. We already know by Lemma 3.1 that the set $\mathcal{H} +_u \mathcal{H}$ is large. For each $m \in \mathbb{N}$, let $\mathcal{H}_m$ be the set of all $\varphi \in \mathcal{H} +_u \mathcal{H}$ such that the color assigned by c to the edge joining $\varphi((u,0))$ and $\varphi((u,1))$ is m. Since small sets from a σ-ideal, there is some $m \in \mathbb{N}$ such that $\mathcal{H}_m$ is large. By construction, $\mathcal{H}_m$ is consistent, so it remains to argue that there is a large Borel subset $\mathcal{H}' \subseteq \mathcal{H}_m$ that is injective.

Claim 7.2.1. If $\varphi \in \mathcal{H}_m$, then $\varphi \upharpoonright V(H)$ is an injective function.

$\triangleright$ Suppose $\varphi \upharpoonright V(H)$ is not injective. Since $\mathcal{H}$ is injective, this implies that there must be a vertex $v \in V(H)$ with $\varphi((v,0)) = \varphi((v,1))$. Take such a vertex v whose distance to u in H is minimum and let $x := \varphi((v,0)) = \varphi((v,1))$. Let $u = v_0, v_1, \ldots, v_k = v$ be a shortest uv-path in H. Note that $k \geqslant 1$, since the vertices $\varphi((u,0))$ and $\varphi((u,1))$ are adjacent in G and hence distinct. By the choice of v, the vertices $y_0 := \varphi((v_{k-1},0))$ and $y_1 := \varphi((v_{k-1},1))$ are distinct. But x is adjacent to both y_0 and y_1, and $c(xy_0) = c(xy_1) = \xi(v_k v_{k-1})$, where $\xi\colon E(H) \to \mathbb{N}$ is an edge-coloring such that $\mathcal{H}$ is ξ-consistent. This is impossible since c is a proper edge-coloring. $\triangleleft$

Now let $\mathcal{U}$ be a countable base for the topology on $V(G)$. By the above claim, for each $\varphi \in \mathcal{H}_m$, it is possible to choose from $\mathcal{U}$ a neighborhood for every vertex of $H +_u H$ so that the closures of these neighborhoods are disjoint. Since small sets from a σ-ideal, it follows that there is a way to assign to each vertex $v \in V(H +_u H)$ a neighborhood $U_v \in \mathcal{U}$ so that the closures $\overline{U_v}$ are pairwise disjoint and the set

$$\mathcal{H}' := \{\varphi \in \mathcal{H}_m \,:\, \varphi(v) \in U_v \text{ for all } v \in V(H +_u H)\}$$

is large. This set $\mathcal{H}'$ is as desired. ∎

Now we proceed exactly as we did in §3, but with Lemma 7.2 replacing Lemma 3.1. Namely, assuming $\chi_{\mathsf{B}}(G) > \aleph_0$, we can use Lemmas 7.2 and 3.3 to recursively construct a sequence of large Borel subsets $\mathcal{T}_n \subseteq \mathsf{Hom}(T_n, G)$ with the following properties:

- $\mathcal{T}_{n+1} \subseteq \mathcal{T}_n +_{s_n} \mathcal{T}_n$;
- for all $u \in \{0,1\}^n$, $\mathrm{diam}_d(\overline{\mathcal{T}_n(u)}) < 2^{-n}$;

- for all $uv \in E(T_n)$, $\mathrm{diam}_\delta(\overline{\mathcal{T}_n(uv)}) < 2^{-n}$;
- each $\mathcal{T}_n$ is consistent and injective.

As in §3, we define a function $f\colon \mathcal{C} \to V(G)$ by letting $f(x)$ be the unique point in $\bigcap_{n=0}^\infty \overline{\mathcal{T}_n(x\restriction n)}$. Then f is a continuous homomorphism from $\mathbb{G}_S$ to G. Moreover, since each $\mathcal{T}_n$ is injective, it is straightforward to check that f is injective.

8. No injective homomorphisms from $\mathbb{G}_0$ in general

In their original paper [KST99], Kechris, Solecki, and Todorcevic asked whether there is an injective continuous homomorphism from $\mathbb{G}_S$ to G for *every* (not necessarily locally countable) Borel graph G with $\chi_\mathsf{B}(G) > \aleph_0$, and conjectured that the answer is positive. This conjecture was refuted by Lecomte [Lec07]. Here we describe an explicit counterexample. Say that a graph is K_∞-**free** is it does not contain an infinite clique. We will construct a graph G satisfying the following:

Theorem 8.1. *There exists a Borel graph G on a Polish space such that $\chi_\mathsf{B}(G) > \aleph_0$ but $\chi_\mathsf{B}(G') \leqslant \aleph_0$ for every Borel K_∞-free subgraph G' of G.*

Since $\mathbb{G}_0$ is K_∞-free (in fact, it is acyclic—see Exercise 9.2), there is no injective Borel homomorphism from $\mathbb{G}_0$ to the graph G from Theorem 8.1, even though $\chi_\mathsf{B}(G) > \aleph_0$.

For a Polish space X and a sequence $f_n\colon X \to X$, $n \in \mathbb{N}$, of Borel functions, let $G((f_n)_{n=0}^\infty)$ denote the graph **induced** by $(f_n)_{n=0}^\infty$, i.e., the graph on X given by

$$x \sim_{G((f_n)_{n=0}^\infty)} y \quad :\Longleftrightarrow \quad x \neq y \text{ and } \exists n \in \mathbb{N}\, (y = f_n(x) \text{ or } x = f_n(y)).$$

Lemma 8.2. *Let X be a Polish space. Suppose that a sequence $f_n\colon X \to X$, $n \in \mathbb{N}$, of continuous open functions satisfies the following conditions:*

(1) *the set $\{x \in X \,:\, \forall n \in \mathbb{N}\, \exists m \geqslant n\, (f_m(x) \neq x)\}$ is dense;*

(2) *for each $x \in X$, we have $\lim_{n\to\infty} f_n(x) = x$.*

Let $G := G((f_n)_{n=0}^\infty)$. Then every Baire-measurable G-independent set is meager, hence $\chi_\mathsf{B}(G) > \aleph_0$.

Proof. Let $A \subseteq X$ be a nonmeager Baire-measurable set. By the Baire alternative, A is comeager in some nonempty open set U. Choose any $x \in U$ such that for all $n \in \mathbb{N}$, there is $m \geqslant n$ with $f_m(x) \neq x$. Since $\lim_{n\to\infty} f_n(x) = x$, there is $n \in \mathbb{N}$ such that $x \neq f_n(x) \in U$, so we can pick an open subset $U_0 \subset U$ containing x such that $f_n(U_0) \subset U$ and $U_0 \cap f_n(U_0) = \varnothing$. Note that for all $y \in U_0$, we have $y \sim_G f_n(y)$. Since f_n is continuous and open and $A \cap U_0$ is nonmeager, $f_n(A \cap U_0)$ is also nonmeager. Since A is comeager in U, we have $A \cap f_n(A \cap U_0) \neq \varnothing$, i.e., for some $y \in A \cap U_0$, $f_n(y) \in A$. This implies that A is not G-independent, as desired. ∎

Lemma 8.3. *Let X be a Polish space. Suppose that a sequence $f_n\colon X \to X$, $n \in \mathbb{N}$, of Borel functions satisfies the following condition:*

(3) *for all $n \in \mathbb{N}$ and $x \in X$ with $f_n(x) \neq x$, if $m \geqslant n$, then $f_n(f_m(x)) = f_n(x)$.*

Let $G := G((f_n)_{n=0}^\infty)$. Then $\chi_\mathsf{B}(G') \leqslant \aleph_0$ for every Borel K_∞-free subgraph G' of G.

Proof. By refining the topology on X if necessary, we can make the functions $(f_n)_{n=0}^\infty$ continuous. For a subgraph $G' \subseteq G$, define a map $c_{G'}\colon X \to \mathbb{N} \cup \{\infty\}$ by

$$c_{G'}(x) := \inf\{n \in \mathbb{N} \,:\, x \sim_{G'} f_n(x)\}.$$

By construction, $c_{G'}^{-1}(\infty)$ is a G'-independent set.

Claim 8.3.1. Let G' be a subgraph of G and let $c := c_{G'}$. Suppose that $x, y \in X$ with $x \sim_{G'} y$ satisfy $c(x) = c(y) =: n$. Then n is finite and $f_n(x) = f_n(y)$.

▷ Assume that, say, $y = f_m(x)$ for some $m \in \mathbb{N}$. By the definition of $c(x)$, $m \geqslant n$. Moreover, $x \sim_{G'} f_n(x)$, which implies $f_n(x) \neq x$. Therefore, $f_n(y) = f_n(f_m(x)) = f_n(x)$. ◁

Claim 8.3.2. Let G' be an analytic subgraph of G with $\chi_{\mathsf{B}}(G') > \aleph_0$. Then there exist a Borel subset $Y \subset X$ and a point $x \in X$ such that $\chi_{\mathsf{B}}(G'[Y]) > \aleph_0$ and $x \sim_{G'} y$ for all $y \in Y$.

$\triangleright$ Curiously, the proof of this claim relies on the $\mathbb{G}_0$-dichotomy. Let $c := c_{G'}$. For each $n \in \mathbb{N} \cup \{\infty\}$, let $X_n := c^{-1}(n)$. Then $X = \bigsqcup X_n$ is a partition of X into countably many Borel sets, and hence there exists some n with $\chi_B(G'[X_n]) > \aleph_0$. Note that n is finite since X_∞ is G'-independent. By Theorem 1.2, there is a continuous homomorphism $h \colon \mathcal{C} \to X_n$ from $\mathbb{G}_S$ to $G'[X_n]$, where $S \subset 2^{<\infty}$ is any dense set containing exactly one sequence of each finite length. By Claim 8.3.1, the function f_n is constant on connected components of $G'[X_n]$. Therefore, $f_n \circ \varphi$ is constant on connected components of $\mathbb{G}_S$. Since every connected component of $\mathbb{G}_S$ is dense (Exercise 9.3) and $f_n \circ \varphi$ is continuous, $f_n \circ \varphi$ is constant on $\mathcal{C}$. In other words, there is $x \in X$ such that $\mathbb{G}_S \to_{\mathsf{c}} G'[X_n \cap f_n^{-1}(x)]$. Setting $Y := X_n \cap f_n^{-1}(x)$ completes the proof of the claim. $\triangleleft$

Suppose now that G' is a Borel subgraph of G with $\chi_{\mathsf{B}}(G') > \aleph_0$. Repeated applications of Claim 8.3.2 yield a sequence of points x_n, $n \in \mathbb{N}$, such that $x_i \sim_{G'} x_j$ for all $i < j$. In other words, G' contains an infinite complete subgraph, as desired. ■

To prove Theorem 8.1, it remains to exhibit a Polish space X and a collection of continuous open functions $f_n \colon X \to X$ that satisfy conditions (1)–(3) of Lemmas 8.2 and 8.3. A **set-theoretic binary tree** is a nonempty set $\mathbb{T} \subseteq 2^{<\infty}$ closed under taking initial segments. We identify a set-theoretic binary tree $\mathbb{T}$ with the (graph-theoretic) rooted tree in which the vertices are the sequences in $\mathbb{T}$, the empty sequence $\varnothing$ is the **root**, and the **parent** of a sequence $a_0 \ldots a_k$ is $a_0 \ldots a_{k-1}$. Note that each vertex $s \in \mathbb{T}$ has at most two **children**, namely $s^\frown 0$ and $s^\frown 1$. When $s^\frown 0$ (resp. $s^\frown 1$) is in T, we call it the **left** (resp. **right**) child of s. We say that $\mathbb{T}$ is **left-growing** if every $s \in \mathbb{T}$ has a left child in $\mathbb{T}$. Let X denote the space of all left-growing binary trees, equipped with the relative topology inherited from $2^{2^{<\infty}}$. It is easy to see that X is closed in $2^{2^{<\infty}}$, hence it is a Polish space. For a binary tree $\mathbb{T}$, let $s_n(\mathbb{T}) := \max\left(\{0,1\}^n \cap \mathbb{T}\right)$, where the maximum is taken with respect to the usual lexicographic ordering on finite binary sequences. Finally, for $n \in \mathbb{N}$, let $f_n \colon X \to X$ be given by

$$f_n(\mathbb{T}) := \{t \in \mathbb{T} \,:\, s_n(\mathbb{T})^\frown 1 \nsubseteq t\}.$$

In other words, $f_n(\mathbb{T})$ is obtained from $\mathbb{T}$ by removing the right child of $s_n(\mathbb{T})$ and all its descendants (if $s_n(\mathbb{T})$ has no right child in $\mathbb{T}$, then $f_n(\mathbb{T}) = \mathbb{T}$). It is straightforward to verify that the functions f_n are as desired (Exercise 9.6).

9. Exercises

Exercise 9.1.

(a) Show that there is a dense set $S \subset 2^{<\infty}$ that contains exactly one sequence of each length.

(b) Let X be the space of sequences of the form $(s_n)_{n=0}^\infty$, where $s_n \in \{0,1\}^n$ for all $n \in \mathbb{N}$. Then $X = \prod_{n=0}^\infty \{0,1\}^n$ carries a product topology that turns it into a compact Polish space (homeomorphic to the Cantor space). Let D be the set of all sequences $(s_n)_{n=0}^\infty \in X$ such that $\{s_n \,:\, n \in \mathbb{N}\}$ is dense. Show that D is comeager in X (hence, in particular, $D \neq \varnothing$).

Exercise 9.2. Show that if $S \subset 2^{<\infty}$ contains at most one sequence of each length, then the graph $\mathbb{G}_S$ is acyclic.

Exercise 9.3.

(a) Suppose $S \subset 2^{<\infty}$ contains at least one sequence of each length. Show that vertices x, $y \in \mathcal{C}$ are joined by a path in $\mathbb{G}_S$ is and only if the set $\{i \in \mathbb{N} \,:\, x_i \neq y_i\}$ is finite.

(b) Conclude that if $S \subset 2^{<\infty}$ contains at least one sequence of each length, then every connected component of $\mathbb{G}_S$ is dense in $\mathcal{C}$.

Exercise 9.4. Let G be an analytic graph on a Polish space. Suppose that $I \subseteq V(G)$ is an analytic G-independent set. Show that there is a Borel G-independent set $J \subseteq V(G)$ such that $J \supseteq I$.

Exercise 9.5. Let G be a locally countable Borel graph on a Polish space. Show that the function $\deg_G : V(G) \to \mathbb{N} \cup \{\infty\}$ is Borel.

Exercise 9.6. Verify that the functions $f_n : X \to X$, $n \in \mathbb{N}$, constructed in §8 are continuous and open and satisfy conditions (1)–(3) of Lemmas 8.2 and 8.3.

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