Rainbows

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Introduction

Most of us are familiar with the sight of a rainbow after a rainstorm. The sun is at our back, and we see an arc with violet innermost, blending into blue, green, yellow and orange, with a red outer band. This is the primary rainbow, and we see it as well in the spray of a waterfall. Sometimes after a rainshower we may be fortunate enough to see another rainbow, the secondary rainbow, higher in the sky than the primary rainbow. The sequence of colors in this secondary rainbow is reversed, with the red ring innermost, and the violet band outermost.

In this project you examine how rainbows are formed and see why the sequence of colors in the two bows is as described above. First we must study the path followed by light passing from one medium to another.

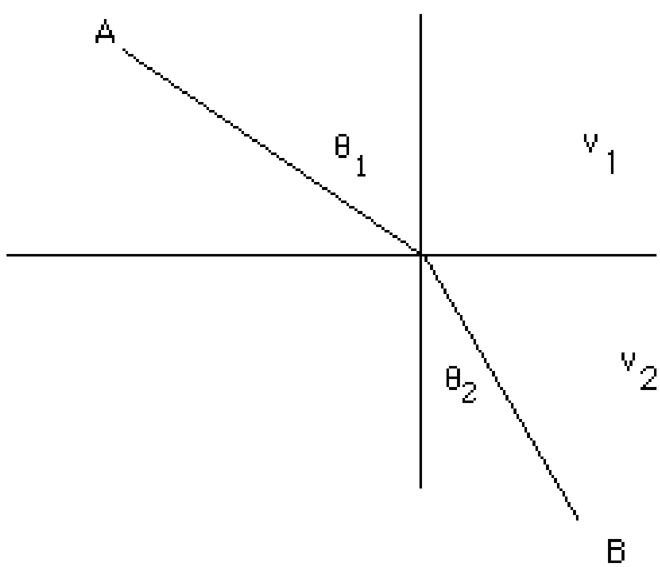

Figure 1

In traveling from A to B, the light is refracted, that is changes direction, at the interface between the two media. The angle of incidence, θ_1, and the angle of refraction, θ_2, satisfy *Snell's Law* (Willebrord Snell, 1621),

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2},$$

where v_1 is the velocity of light in the first medium, and v_2 is the velocity of light in the second. The ratio $\frac{v_{\text{water}}}{v_{\text{air}}}$ is 0.7508 for red light, and 0.7440 for violet light.

Part I. The primary rainbow

The theory of rainbows presented here was developed by Descartes. The primary rainbow is formed by rays of light which enter a drop of water (assumed to be spherical), are refracted at the boundary, reflect once internally off the back of the raindrop, and are again refracted as they leave the drop, headed toward the observer of the rainbow.

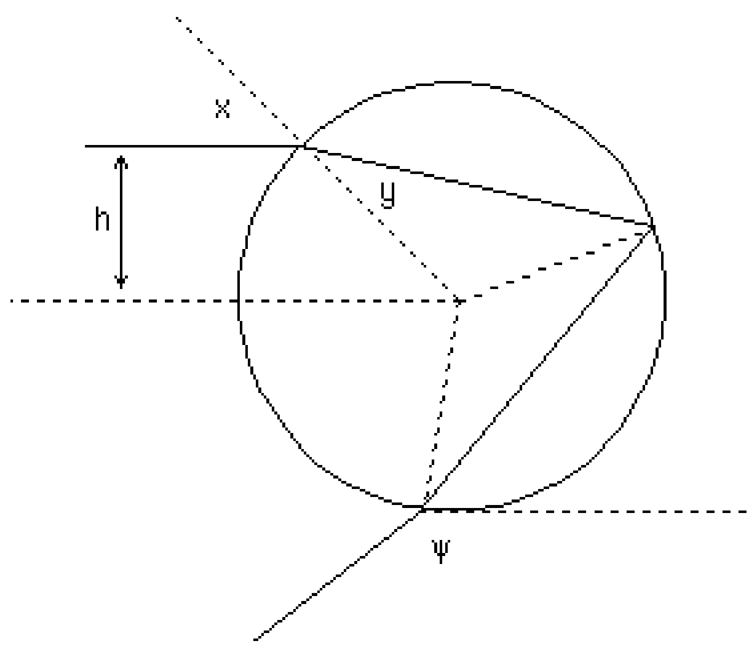

Figure 2

The distance h from the center line of the drop to the incoming ray is called the *impact parameter*, and the angle x is called the angle of incidence. The angle ψ is called the *scattering angle*. If a ray is incident with impact parameter 0, it is reflected back on its original path, and the scattering angle is π. As the impact parameter increases, the scattering angle decreases, as seen in the above figure. As h increases, the scattering angle reaches a minimum of about 138°, and then begins increasing.

Descartes' reasoning was that light exiting the drop is concentrated near the scattering angle for which the scattering angle is least sensitive to changes in the impact parameter h. That is, it is concentrated near the scattering angle for which the rate of change of scattering angle with respect to h is as small as possible, namely where

$$\frac{d\psi}{dh} = 0.$$

Of course this happens where the scattering angle ψ assumes its minimum value, and Descartes calculated scattering angles for many values of the impact parameter h, thereby determining that the reflected light is concentrated near the scattering angle of 138°. The primary rainbow is formed by light from those drops for which the angle between the incident rays and the observer's line of sight is 180 - 138 = 42°.

Exercises

1. Fermat's Principle in optics is that light travels on the path which minimizes the time of travel (See Project 2.). Use Fermat's Principle to show that in traveling from A to B (see Figure 1), light follows the path for which Snell's Law

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

holds.

2. Show that the scattering angle ψ is given by the formula

$$\psi = \pi + 2x - 4y, \text{ or}$$

$$\psi = \pi + 2x - 4\operatorname{Arcsin}\left[\frac{v_{\text{water}}}{v_{\text{air}}}\sin x\right]$$

This can be done by examining Figure 2 and either using plane geometry or by carefully calculating the ray's change of direction at each refraction or reflection.

3. Plot the scattering angle as a function of the angle of incidence, and calculate the minimum value of ψ for both red and violet light. Use this to explain why the rainbow forms in a circular arc and why the violet arc is the innermost ring and the red arc is outermost.

Part II. The secondary rainbow

The secondary rainbow is formed by rays of light which undergo two internal reflections before being refracted towards the observer. This rainbow is fainter than the primary rainbow because at each reflection a portion of the light is transmitted into the air. The path of light undergoing two internal reflections is shown in Figure 3.

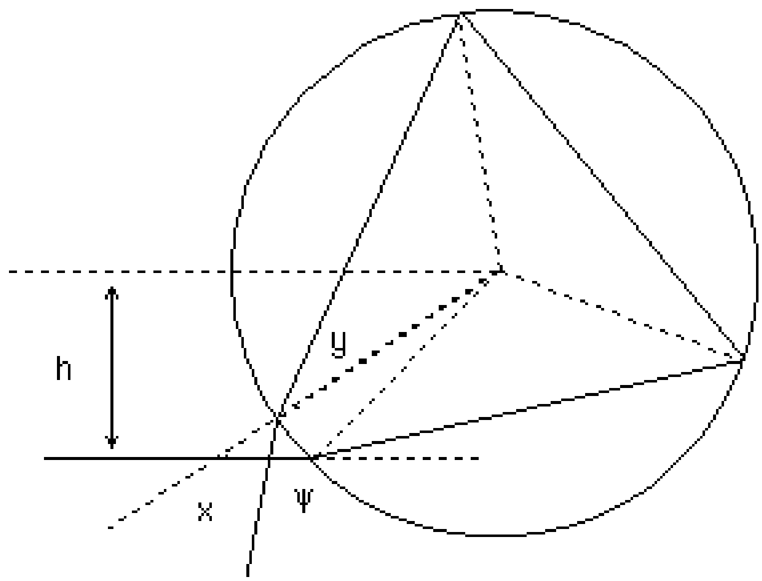

Figure 3

In this case, a ray entering a drop at impact parameter $h = 0$ is reflected twice, and leaves the drop in its original direction, with scattering angle $\psi = 0$. As h changes, the scattering angle increases to a maximum of about 130°, and then decreases. In this case, light is concentrated in the direction of this maximum scattering angle.

Exercises

4. Show that in the case of two internal reflections, the scattering angle ψ is given by the formula

$$\psi = 6y - 2x, \text{ or}$$

$$\psi = 6 \operatorname{Arcsin}\left[\frac{v_{water}}{v_{air}} \sin x\right] - 2x$$

5. Plot ψ as a function of x, and determine the maximum value of ψ for both red and violet light. Explain why the secondary rainbow appears higher in the sky than the primary rainbow, and why the sequence of bands in the secondary rainbow is reversed, with the red band innermost and violet outermost.

6. Explain why the sky is particularly dark in the region between the primary and secondary rainbows. This dark area is known as *Alexander's Dark Band*.

References

H. Moyses Nussenveig, *The Theory of the Rainbow*, Scientific American, April 1977, reprinted in Light from the Sky, W. H. Freeman and Company, San Francisco.

A. D. Andrew, G.L. Cain, S. Crum, T. D. Morley, *Calculus Projects Using Mathematica*, McGraw-Hill, 1996.