1. (25) The differential equation
\[ \frac{dy}{dt} = ky + I \]
\[ y(0) = y_0 \]
may be used to model population growth with immigration. Here \( y(t) \) is the population at time \( t \), \( k \) is the intrinsic growth rate (birth rate - death rate) and \( I \) is the immigration rate.

Solve
\[ \frac{dy}{dt} = 2y + 1 \]
\[ y(0) = 10 \]

2. (25) Let \( f(x) = \sum_{n=0}^{\infty} \frac{n^3}{2^{n+1}} x^n \) and \( g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{n!}{(2n)!} x^n \).

a. Calculate the radius of convergence of the series for \( f(x) \).

b. Calculate the radius of convergence of the series for \( g(x) \).

c. Use the alternating series estimate (be sure to check hypotheses!) to find the smallest value of \( N \) guaranteed to make
\[ g(1) = \sum_{n=0}^{N} (-1)^n \frac{n!}{(2n)!} x^n \]
smaller than .0001.

3. (25) A mathematics professor's house is infested with snakes, rats, spiders, and cockroaches. Co-op students from the Georgia Institute of Pestology refuse to enter the house until they know how many of each kind of pest is present. The prof gives them these clues: "There are 750 pests with tails, and there are just as many rats as there are spiders. There are ten times as many cockroaches as all other pests."
combined, and the pests have a total of 63,000 legs.” Your mission, should you decide to accept it, is to

a. Set up a system of equations satisfied by the number of snakes (\( w \)), number of rats (\( x \)), number of spiders (\( y \)), and number of cockroaches (\( z \)) and write down its augmented matrix. Please assume snakes and rats have tails but spiders and cockroaches do not. Rats, spiders, and cockroaches have 4, 8, and 6 legs, respectively.

b. Solve the system to determine the number of each pest.

4. (25) The triangle \( T \) has vertices at \( P = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \), \( Q = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \), and \( R = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \).

a. Calculate the area of \( T \).

b. Find an equation for the plane containing \( T \).

5. Whenever \( P \) is a projection, \( H = 2P - I \) is a reflection.

a. Find the matrix \( P \) for the orthogonal projection of \( \mathbb{R}^3 \) onto the image of \( \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 2 \end{pmatrix} \).

b. With \( P \) as in part a, find the matrix for the reflection \( H = 2P - I \).

c. Find the matrix \( S \) for the reflection of \( \mathbb{R}^3 \) across the x-y plane and the matrix for \( H \) followed by \( S \).

6. (25) With \( A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \)

a. Calculate the eigenvalues and eigenvectors of \( A \).

b. Find a diagonal matrix \( D \) and a square matrix \( U \) such that \( U^{-1}AU = D \).
ANSWERS

1. \( y(t) = \frac{1}{2}(21e^{2t} - 1) \)

2. a. \( \frac{2}{3} \)  
   b. \( \infty \)  
   c. \( N = 4 \)

3. a. 

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 750 \\
0 & 1 & -1 & 0 & 0 \\
10 & 10 & 10 & -1 & 0 \\
0 & 4 & 8 & 6 & 63,000
\end{pmatrix}
\]

b. 500 snakes, 250 rats, 250 spiders, 10,000 cockroaches

4. a. \( 4\sqrt{10} \)  
   b. \( 24x - 8z = 16 \)

5. a. \( \boldsymbol{P} = \frac{1}{17} \begin{pmatrix} 13 & 4 & 6 \\ 4 & 13 & -6 \\ 6 & -6 & 8 \end{pmatrix} \)  
   b. \( \boldsymbol{H} = \frac{1}{17} \begin{pmatrix} 9 & 8 & 12 \\ 8 & 9 & -12 \\ 12 & -12 & -1 \end{pmatrix} \)

   c. \( \boldsymbol{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \), \( \boldsymbol{SH} = \frac{1}{17} \begin{pmatrix} 9 & 8 & 12 \\ 8 & 9 & -12 \\ -12 & 12 & 1 \end{pmatrix} \)

6. a. \( \lambda = 1, \begin{pmatrix} -t \\ t \end{pmatrix} \)  
   b. \( \boldsymbol{U} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \), \( \boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \)