    2.  Show your work and explain your answers and reasoning.
    3.  Calculators may be used, but pay particular attention to instruction 2.  
    **To receive credit, you must show your work.** Unexplained answers,
    and answers not supported by the work you show, will not receive
    credit.
    4.  Express your answers in simplified form.

0.  (10 -- no kidding!) Write your name and Georgia Tech ID number on every page
    now!

1.  (25) Solve the initial value problem

\[
\frac{dy}{dx}  \quad 2y = 4x \\
y(0) = 3
\]

2.  (25) Determine whether the series converges or whether it diverges.  Be sure to state
    your reasons and to verify the hypotheses of any test you use.

a.  \[ \sum_{j=0}^{\infty} \frac{(1)^j}{(j+1)3^j} \]

b.  \[ \sum_{n=0}^{\infty} \frac{n+1}{n+2} \]

c.  \[ \sum_{j=3}^{\infty} \frac{1}{j (\ln(j))^j} \]
3. (25) Find the matrices for these linear transformations
   
   a. Reflection $R$ of $\mathbb{R}^3$ across the plane $x \parallel y = 0$.
      
   $$
   \begin{bmatrix}
   1 & 1 & 1 \\
   1 & 1 & 1 \\
   1 & 1 & 1 \\
   \end{bmatrix}
   $$

   b. Orthogonal projection $P$ of $\mathbb{R}^3$ onto the image of
      
   $$
   \begin{bmatrix}
   2 & 4 & 1 \\
   1 & 1 & 1 \\
   \end{bmatrix}
   $$

   c. The composition $P$ followed by $R$.

   d. The composition $R \circ R$.

4. (25) Calculate all solutions (if any) of the systems
   
   a. 
   
   $3x + y + z = 4$
   
   $6x + 4y + 3z = 7$
   
   $3x + 7y + 5z = 4$

   $$
   \begin{bmatrix}
   3 & 1 & 1 \\
   6 & 4 & 3 \\
   3 & 7 & 5 \\
   \end{bmatrix}
   $$

   b. 
   
   $3x + y + z = 6$
   
   $9x + y + 2z = 14$
   
   $3x + 3y + z = 2$

5. (25) a. Find an orthogonal matrix $U$ and a diagonal matrix $D$ so that $U^t A U = D$,
      
      where $A = 
      
      \begin{bmatrix}
      14 & 3 \\
      3 & 6 \\
      \end{bmatrix}

   b. Use your results from part a to determine whether the equation
      
      $14x^2 - 6xy + 6y^2 = 1$ describes an ellipse or describes a hyperbola. Explain.
6. (25) For each of parts a through e there may be more than one correct answer. Circle ALL correct answers.

a. Referring to the graphs (Note: \( f(0) = 2 \), \( g(0) = 0 \))

\[
\int_{1}^{3} f(x) \, dx \quad \text{and} \quad \int_{3}^{1} f(x) \, dx
\]

\[\int_{1}^{3} g(x) \, dx \quad \text{and} \quad \int_{3}^{1} g(x) \, dx
\]

i. If \( \int_{1}^{3} f(x) \, dx \) converges, then \( \int_{3}^{1} f(x) \, dx \) converges.

ii. If \( \int_{3}^{1} f(x) \, dx \) converges, then \( \int_{1}^{3} f(x) \, dx \) converges.

iii. If \( \int_{1}^{3} g(x) \, dx \) converges, then \( \int_{3}^{1} g(x) \, dx \) converges.

iv. If \( \int_{3}^{1} g(x) \, dx \) converges, then \( \int_{1}^{3} g(x) \, dx \) converges.

b. \( \sum_{n=0}^{\infty} \left( \frac{n!}{n+1} \right) \) converges to \( \ln(2) \). The least value of \( N \) such that

\[
\left| \sum_{n=0}^{N} \left( \frac{n!}{n+1} \right) \right| < \frac{1}{100}
\]

is

i. \( N = 10 \)  \quad ii. \( N = 100 \)  \quad iii. \( N = 1000 \)  \quad iv. \( N = 10,000 \)
c. The recurrence relation \( x_{n+2} = \frac{3}{4} x_{n+1} + \frac{1}{4} x_n \) may be expressed as

\[
\begin{pmatrix}
  x_{n+2} \\
  x_{n+1}
\end{pmatrix} = \mathbf{A} \begin{pmatrix}
  x_{n+1} \\
  x_n
\end{pmatrix}
\]

where

i. \( \mathbf{A} = \begin{pmatrix} 3/4 & 1/4 \\ 0 & 1 \end{pmatrix} \)
ii. \( \mathbf{A} = \begin{pmatrix} 1/4 & 3/4 \\ 1 & 0 \end{pmatrix} \)
iii. \( \mathbf{A} = \begin{pmatrix} 3/4 & 1/4 \\ 0 & 1 \end{pmatrix} \)
iv. \( \mathbf{A} = \begin{pmatrix} 1/4 & 3/4 \\ 0 & 1 \end{pmatrix} \)

\[
\begin{aligned}
x + y + z &= 4 \\
x + y + z &= 2
\end{aligned}
\]

\[
\begin{aligned}
z &= 3 \\
z &= 3
\end{aligned}
\]

i. Infinitely many solutions.

ii. Exactly five solutions.

iii. Exactly one solution.

iv. No solutions.

v. The number of solutions depends on the phase of the moon tonight.

d. The system of equations

\[
\begin{aligned}
\begin{bmatrix} x + y \end{bmatrix} + \begin{bmatrix} z \end{bmatrix} &= 4 \\
\begin{bmatrix} x + y \end{bmatrix} + \begin{bmatrix} z \end{bmatrix} &= 2
\end{aligned}
\]

\[
\begin{aligned}
z &= 3 \\
z &= 3
\end{aligned}
\]

i. Infinitely many solutions.

ii. Exactly five solutions.

iii. Exactly one solution.

iv. No solutions.

v. The number of solutions depends on the phase of the moon tonight.

e. Circle all those that are eigenvectors of \( \mathbf{A} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \)

i. \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)
ii. \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \)
iii. \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)
iv. \( \begin{pmatrix} 2 \\ 2 \end{pmatrix} \)

v. \( \begin{pmatrix} 6 \\ 6 \end{pmatrix} \)
Answers.

1. \[ y = 2x - 1 + 4e^{2x} \]

2. a. converges absolutely by ratio test.
   b. diverges since the terms do not tend to 0
   c. converges by the integral test

3. \[ R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \quad RP = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}, \quad RR = I \]

4. a. \( x = -1, \ y = 2, \ z = -3 \)
   b. Infinitely many solutions, \( z = t, \ y = 2 - \frac{t}{2}, \ x = \frac{4}{3} - \frac{t}{6} \)

5. a. \[ U = \begin{pmatrix} \frac{1}{\sqrt{10}} & 0 \\ \frac{3}{\sqrt{10}} & 0 \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \]
   b. ellipse, both eigenvalues positive

6. a. \( I, ii, \) and \( iii \) are correct
   b. \( ii \) \( N = 100 \)
   c. \( iii \)
   d. \( iv \)
   e. All except \( ii \) are eigenvectors