maximal cliques of an equivalence relation

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In[1]:= **SetDirectory["l:"]; << goedel.11sep30a**

```
:Package Title: goedel.11sep30a            2011 September 30 at 11:20 a.m.

Loading takes about thirteen minutes, half that time due to builtin pauses.

It is now:  2011 Oct 1 at 15:48

Loading Simplification Rules

TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

weightlimit = 40

Loading completed.

It is now:  2011 Oct 1 at 16:0
```

summary

The axiom of choice is equivalent to the statement that every small relation has a maximal clique.

In[2]:= **disjoint[range[CLIQUES], subvar[inverse[PS]]]**

Out[2]= axch

In this notebook it is shown that the weaker statement that every small equivalence relation has a maximal clique is true independently of the axiom of choice. Note that the statement that a class has no maximal element is equivalent to the statement that the class is subvariant under the inverse of the proper subset relation **PS**.

In[3]:= **empty[maximal[S, x]]**

Out[3]= subclass[x, image[inverse[PS], x]]

The idea of the proof is simple. If an equivalence relation **x** is not empty, then any vertical section **image[x, {w}]** for **w ∈ fix[x]** is a maximal clique. If **x = 0**, then **0** is a maximal clique. The compound wrapper **eqv[setpart[x]]** will be used for a small equivalence relation. The variable **x** will be eliminated using a combination of **reify** and **case**.

derivation

Observation. The statement that non-empty vertical section are maximal cliques is already available in the **GOEDEL** program in the form of the following rewrite rule.

```
In[4]:=  dif[maximal[S, cliques[eqv[x]]], set[0]]

Out[4]=  image[VERTSECT[eqv[x]], fix[eqv[x]]]
```

Lemma. A temporary simplification rule.

```
In[5]:=  (subclass[fix[t], 0] // AssertTest) /. t → eqv[x]

Out[5]=  subclass[fix[eqv[x]], 0] == equal[0, eqv[x]]

In[6]:=  subclass[fix[eqv[x_]], 0] := equal[0, eqv[x]]
```

Lemma. Any non-empty small equivalence relation has a maximal clique. (The non-emptiness hypothesis is redundant.)

```
In[7]:=  SubstTest[implies, empty[u], disjoint[u, v],
           {u -> maximal[S, cliques[eqv[setpart[x]]]], v → complement[set[0]]}] // Reverse

Out[7]=  or[equal[0, eqv[setpart[x]]], not[subclass[cliques[eqv[setpart[x]]],
             image[inverse[PS], cliques[eqv[setpart[x]]]]]]] == True

In[8]:=  (% /. x → x_) /. Equal → SetDelayed
```

The redundant hypothesis is eliminated in the following theorem.

Theorem. Any small equivalence relation has a maximal clique.

```
In[9]:=  Map[not, SubstTest[and, implies[p, q],
           or[p, q], {p -> equal[0, eqv[setpart[x]]], q -> not[subclass[
               cliques[eqv[setpart[x]]], image[inverse[PS], cliques[eqv[setpart[x]]]]]]}]]

Out[9]=  subclass[cliques[eqv[setpart[x]]],
           image[inverse[PS], cliques[eqv[setpart[x]]]]] == False

In[10]:=  subclass[cliques[eqv[setpart[x_]]],
            image[inverse[PS], cliques[eqv[setpart[x_]]]]] := False
```

One can use **reify** and **case** to eliminate the variable **x**.

Theorem. A variable-free restatement of the fact that every small equivalence relation has a maximal clique.

```
In[11]:=  Map[equal[0, domain[#]] &, SubstTest[reify, x,
            case[subclass[cliques[eqv[setpart[x]]], image[t, cliques[eqv[setpart[x]]]]]],
            t -> inverse[PS]]]

Out[11]=  equal[0, intersection[image[CLIQUES, EQV], subvar[inverse[PS]]]] == True

In[12]:=  intersection[image[CLIQUES, EQV], subvar[inverse[PS]]] := 0
```