cardinality in arithmetic

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```
In[1]:= SetDirectory["l:"]; << goedel86.24b; << tools.m

        :Package Title: goedel86.24b          2006 October 24 at 6:50 p.m.

        It is now:  2006 Oct 26 at 8:37

        Loading Simplification Rules

        TOOLS.M                      Revised 2006 October 25

        weightlimit = 40
```

summary

This notebook is concerned with computing cardinalities of various sets encountered in the arithmetic of natural numbers and of integers.

named classes in natural arithmetic

Theorem. The function **TIMES** is countably infinite.

```
In[2]:= SubstTest[card, domain[funpart[x]], x → TIMES] // Reverse

Out[2]= card[TIMES] == omega

In[3]:= card[TIMES] := omega
```

Corollary. There are countably many binary endomorphisms of **NATADD**.

```
In[4]:= SubstTest[card, domain[funpart[x]], x → inverse[TIMES]]

Out[4]= card[binhom[NATADD, NATADD]] == omega

In[5]:= card[binhom[NATADD, NATADD]] := omega
```

Theorem. The divisibility relation for natural numbers is countably infinite.

```
In[6]:= SubstTest[implies, and[subclass[x, y], equal[omega, card[y]]],
         or[equal[card[x], omega], member[x, FINITE]], {x → DIV, y → cart[omega, omega]}]

Out[6]= equal[omega, card[DIV]] == True
```

```
In[7]:=  card[DIV] := omega
```

Lemma.

```
In[8]:=  SubstTest[implies, and[equal[omega, card[x]], equal[omega, card[y]]],
          equal[omega, card[cart[x, y]]], {x → cartsq[omega], y → cartsq[omega]}]

Out[8]=  equal[omega, card[cart[cart[omega, omega], cart[omega, omega]]]] == True

In[9]:=  card[cart[cart[omega, omega], cart[omega, omega]]] := omega
```

Theorem. The **EQUIDIFF** relation is countably infinite.

```
In[10]:=  SubstTest[implies, and[subclass[x, y], equal[omega, card[y]]],
           or[equal[card[x], omega], member[x, FINITE]],
           {x → EQUIDIFF, y → cartsq[cartsq[omega]]}]

Out[10]=  equal[omega, card[EQUIDIFF]] == True

In[11]:=  card[EQUIDIFF] := omega
```

---

named classes in integer arithmetic

Lemma. The divisibility relation for integers is not finite.

```
In[12]:=  Map[not, SubstTest[implies, and[subclass[x, y], member[y, FINITE]],
            member[x, FINITE], {x → id[Z], y → INTDIV}]]

Out[12]=  member[INTDIV, FINITE] == False

In[13]:=  member[INTDIV, FINITE] := False
```

Theorem. The relation **INTDIV** is countably infinite.

```
In[14]:=  SubstTest[implies, and[subclass[x, y], equal[omega, card[y]]],
           or[equal[card[x], omega], member[x, FINITE]], {x → INTDIV, y → cartsq[Z]}]

Out[14]=  equal[omega, card[INTDIV]] == True

In[15]:=  card[INTDIV] := omega
```

Theorem. The less-than-or-equal relation for integers is countably infinite.

```
In[16]:=  SubstTest[implies, and[subclass[x, y], equal[omega, card[y]]],
           or[equal[card[x], omega], member[x, FINITE]], {x → INTLEQ, y → cartsq[Z]}]

Out[16]=  equal[omega, card[INTLEQ]] == True

In[17]:=  card[INTLEQ] := omega
```

the set image[DIV, set[x]]

The set **image[DIV, set[x]]** is the set of natural numbers that are multiples of **x**. The **GOEDEL** program already contains some rewrite rules about the cardinality of these sets, but even better ones will be derived.

```
In[18]:= range[times[x]]

Out[18]= image[DIV, set[x]]
```

Lemma. If **x** is not a natural number, then no natural number is a multiple of **x**, and so *a fortiori* the set of multiples of **x** can not be countably infinite.

```
In[19]:= SubstTest[implies, empty[z], not[equal[omega, z]], z -> card[image[DIV, set[x]]]]

Out[19]= or[member[x, omega], not[equal[omega, card[image[DIV, set[x]]]]]] == True

In[20]:= (% /. x → x_) /. Equal → SetDelayed
```

Theorem. The set of multiples of **x** is countably infinite if and only if **x** is a non-zero natural number.

```
In[21]:= equiv[equal[omega, card[image[DIV, set[x]]]],
            and[member[x, omega], not[equal[0, x]]]] // not // not

Out[21]= True

In[22]:= equal[omega, card[image[DIV, set[x_]]]] := and[member[x, omega], not[equal[0, x]]]
```

Corollary. Variable-free restatement of this fact: the set of multiples of **x** has cardinality **omega** if and only if **x** is a non-zero natural number.

```
In[23]:= image[inverse[VERTSECT[DIV]], image[Q, set[omega]]] // Normality

Out[23]= image[inverse[VERTSECT[DIV]], image[Q, set[omega]]] ==
          intersection[omega, complement[set[0]]]

In[24]:= image[inverse[VERTSECT[DIV]], image[Q, set[omega]]] :=
          intersection[omega, complement[set[0]]]
```

a finiteness rule for image[DIV, set[x]]

Lemma. If **x** is not a natural number, then no natural number is a multiple of **x**, and so *a fortiori* the set of multiples of **x** is finite.

```
In[25]:= SubstTest[implies, empty[z], member[z, FINITE], z → image[DIV, set[x]]]

Out[25]= or[member[x, omega], member[image[DIV, set[x]], FINITE]] == True

In[26]:= (% /. x → x_) /. Equal → SetDelayed
```

Lemma. If there are only a finite number of multiples of some natural number **x**, then **x** is zero.

```
In[27]:= SubstTest[implies, and[FUNCTION[u], member[v, FINITE]],
          member[image[u, v], FINITE], {u → inverse[times[x]], v → image[DIV, set[x]]}]

Out[27]= or[equal[0, x], not[member[x, omega]], not[member[image[DIV, set[x]], FINITE]]] == True

In[28]:= (% /. x → x_) /. Equal → SetDelayed
```

Theorem. The set of multiples of **x** is finite if any only if **x** is either **0** or not a natural number.

```
In[29]:= equiv[member[image[DIV, set[x]], FINITE],
            or[equal[0, x], not[member[x, omega]]]] // not // not

Out[29]= True

In[30]:= member[image[DIV, set[x_]], FINITE] := or[equal[0, x], not[member[x, omega]]]
```

Corollary. Variable-free restatement of the finiteness rule.

```
In[31]:= image[inverse[VERTSECT[DIV]], FINITE] // Normality

Out[31]= image[inverse[VERTSECT[DIV]], FINITE] == union[complement[omega], set[0]]

In[32]:= image[inverse[VERTSECT[DIV]], FINITE] := union[complement[omega], set[0]]
```

---

counting divisors

Counting the number of divisors of **x** is more interesting than counting multiples of **x**. Here are some known rules.

```
In[33]:= equal[0, card[image[inverse[DIV], set[x]]]]

Out[33]= not[member[x, omega]]

In[34]:= equal[set[0], card[image[inverse[DIV], set[x]]]]

Out[34]= equal[x, set[0]]

In[35]:= equal[succ[set[0]], card[image[inverse[DIV], set[x]]]]

Out[35]= member[x, PRIMES]
```

Theorem. The number of divisors of **x** is finite except when **x** = **0**.

```
In[36]:= SubstTest[member, U[y], FINITE, y → image[inverse[DIV], set[x]]] // Reverse

Out[36]= member[image[inverse[DIV], set[x]], FINITE] == not[equal[0, x]]

In[37]:= member[image[inverse[DIV], set[x_]], FINITE] := not[equal[0, x]]
```

Corollary. Variable-free restatement of this theorem.

```
In[38]:= SubstTest[class, x, member[image[u, set[x]], v],
           {u → inverse[DIV], v → FINITE}] // Reverse

Out[38]= image[inverse[VERTSECT[inverse[DIV]]], FINITE] == complement[set[0]]

In[39]:= image[inverse[VERTSECT[inverse[DIV]]], FINITE] := complement[set[0]]
```

Lemma.

```
In[40]:= SubstTest[implies, equal[omega, card[z]],
          not[member[z, FINITE]], z -> image[inverse[DIV], set[x]]]

Out[40]= or[equal[0, x], not[equal[omega, card[image[inverse[DIV], set[x]]]]]] == True

In[41]:= (% /. x → x_) /. Equal → SetDelayed
```

Theorem.

```
In[42]:= equiv[equal[omega, card[image[inverse[DIV], set[x]]]], equal[0, x]]

Out[42]= True

In[43]:= equal[omega, card[image[inverse[DIV], set[x_]]]] := equal[0, x]
```