the chain condition in Zorn's lemma

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```
In[1]:= SetDirectory["l:"]; << goedel.08jul03a; << tools.m

:Package Title: goedel.08jul03a              2008 July 3 at 5:35 p.m.

It is now:  2008 Jul 4 at 15:2

Loading Simplification Rules

TOOLS.M                                     Revised 2008 May 17

weightlimit = 40
```

summary

The following is a special case of Zorn's lemma for restrictions of the subset relation **S**.

```
In[2]:= implies[and[axch, member[x, V], subclass[Uchains[x], image[inverse[S], x]]],
         not[subclass[x, image[inverse[PS], x]]]]

Out[2]= True
```

The general formulation of Zorn's lemma for (small) partial orders involves a different chain condition, which does not automatically simplify for restrictions of the subset relation to the condition **subclass[Uchains[x], image[inverse[S],x]]**. Instead one obtains the following more complicated condition:

```
In[3]:= and[not[empty[x]], subclass[chains[t], domain[UB[t]]]] /. t → restrict[S, x, x]

Out[3]= and[not[equal[0, x]], subclass[intersection[chains[S], P[x]],
          union[image[inverse[BIGCUP], image[inverse[S], x]], set[0]]]]
```

In this notebook it is shown that the two chain conditions are in fact logically equivalent, thereby reconciling the chain conditions for the special case of restrictions of the subset relation to the one for general partial orders. Since the focus is just on the chain condition in Zorn's lemma, and not Zorn's lemma itself, all the results in this notebook are independent of the axiom of choice.

derivation

In this section it is shown how one can reconcile the special and general cases. To relate the two versions of the chain condition, one needs to relate an inverse image of **BIGCUP** to a direct image. One can do this as follows:

```
In[4]:= SubstTest[subclass, t, image[inverse[BIGCUP], z], t → intersection[x, complement[y]]]

Out[4]= subclass[image[BIGCUP, intersection[x, complement[y]]], z] ==
         subclass[x, union[y, image[inverse[BIGCUP], z]]]

In[5]:= subclass[image[BIGCUP, intersection[x_, complement[y_]]], z_] :=
         subclass[x, union[y, image[inverse[BIGCUP], z]]]
```

To complete the derivation one final step is needed:

```
In[6]:= Map[subclass[#, image[inverse[S], x]] &, SubstTest[image, BIGCUP, union[u, v],
          {u -> intersection[chains[S], complement[set[0]], P[x]], v → set[0]}]]

Out[6]= and[not[equal[0, x]], subclass[intersection[chains[S], P[x]],
           union[image[inverse[BIGCUP], image[inverse[S], x]], set[0]]]] ==
         subclass[Uchains[x], image[inverse[S], x]]

In[7]:= and[not[equal[0, x_]], subclass[intersection[chains[S], P[x_]],
           union[image[inverse[BIGCUP], image[inverse[S], x_]], set[0]]]] :=
         subclass[Uchains[x], image[inverse[S], x]]
```

This new rewrite rule causes the chain condition in the general version of Zorn's lemma to simplify as follows for the special case of restrictions of the subset relation:

```
In[8]:= and[not[empty[x]], subclass[chains[t], domain[UB[t]]]] /. t → restrict[S, x, x]

Out[8]= subclass[Uchains[x], image[inverse[S], x]]
```