GLB[DIV] and LUB[DIV], part 1

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```
In[1]:= SetDirectory["l:"]; << goedel90.13a; << tools.m

        :Package Title: goedel90.13a        2007 February 13 at 9:40 a.m.

        It is now:  2007 Feb 15 at 19:41

        Loading Simplification Rules

        TOOLS.M                      Revised 2007 January 7

        weightlimit = 40
```

summary

The greatest common divisor of a set **x** of natural numbers is **APPLY[GLB[DIV], x]**. Their least common multiple is **APPLY[LUB[DIV], x]**. Some elementary facts about these quantities can be obtained by instantiating existing rewrite rules about complete lattices, since the natural numbers form a complete lattice with respect to divisibility:

```
In[2]:= member[DIV, CL]

Out[2]= True
```

The bottom and top elements of the divisibility lattice are **1** and **0** respectively, since **1** divides every natural number, and every natural number divides **0**.

sethood rules

The functions **GLB[DIV]** and **LUB[DIV]** are sets:

```
In[3]:= SubstTest[member, GLB[setpart[x]], V, x → DIV] // Reverse

Out[3]= member[GLB[DIV], V] == True

In[4]:= member[GLB[DIV], V] := True

In[5]:= SubstTest[member, GLB[setpart[x]], V, x → inverse[DIV]] // Reverse

Out[5]= member[LUB[DIV], V] == True

In[6]:= member[LUB[DIV], V] := True
```

mapping rules

The greatest common divisor function maps sets of natural numbers to numbers:

```
In[7]:= SubstTest[or, member[GLB[x], map[P[fix[x]], fix[x]]],
          not[member[x, CL]], x -> DIV] // Reverse

Out[7]= member[GLB[DIV], map[P[omega], omega]] == True

In[8]:= member[GLB[DIV], map[P[omega], omega]] := True
```

A similar result holds for the least common multiple function.

```
In[9]:= SubstTest[or, member[LUB[x], map[P[fix[x]], fix[x]]],
          not[member[x, CL]], x -> DIV] // Reverse

Out[9]= member[LUB[DIV], map[P[omega], omega]] == True

In[10]:= member[LUB[DIV], map[P[omega], omega]] := True
```

Corollaries of these facts are:

```
In[12]:= SubstTest[member, APPLY[funpart[t], x], range[funpart[t]], t → GLB[DIV]] // Reverse

Out[12]= member[APPLY[GLB[DIV], x], omega] == subclass[x, omega]

In[13]:= member[APPLY[GLB[DIV], x_], omega] := subclass[x, omega]

In[14]:= SubstTest[member, APPLY[funpart[t], x], range[funpart[t]], t → LUB[DIV]] // Reverse

Out[14]= member[APPLY[LUB[DIV], x], omega] == subclass[x, omega]

In[15]:= member[APPLY[LUB[DIV], x_], omega] := subclass[x, omega]
```

pair rule

A general lemma.

```
In[16]:= equiv[member[pair[x, y], funpart[z]],
            and[member[x, domain[funpart[z]]], equal[y, APPLY[funpart[z], x]]]] // not // not

Out[16]= True

In[17]:= member[pair[x_, y_], funpart[z_]] :=
          and[equal[y, APPLY[funpart[z], x]], member[x, domain[funpart[z]]]]
```

Theorem.

```
In[18]:= SubstTest[member, pair[x, y], funpart[z], z → GLB[DIV]] // Reverse
Out[18]= member[pair[x, y], GLB[DIV]] == and[equal[y, APPLY[GLB[DIV], x]], subclass[x, omega]]

In[19]:= member[pair[x_, y_], GLB[DIV]] := and[equal[y, APPLY[GLB[DIV], x]], subclass[x, omega]]
```

Similarly:

```
In[20]:= SubstTest[member, pair[x, y], funpart[z], z → LUB[DIV]] // Reverse
Out[20]= member[pair[x, y], LUB[DIV]] == and[equal[y, APPLY[LUB[DIV], x]], subclass[x, omega]]

In[21]:= member[pair[x_, y_], LUB[DIV]] := and[equal[y, APPLY[LUB[DIV], x]], subclass[x, omega]]
```

gcd examples

The gcd of the empty set is **0**.

```
In[22]:= Map[A, SubstTest[image, funpart[t], set[0], t -> GLB[DIV]]]
Out[22]= APPLY[GLB[DIV], 0] == 0

In[23]:= APPLY[GLB[DIV], 0] := 0
```

The gcd of the set of all numbers is **1**.

```
In[24]:= Map[A, SubstTest[image, funpart[t], set[omega], t -> GLB[DIV]]]
Out[24]= APPLY[GLB[DIV], omega] == set[0]

In[25]:= APPLY[GLB[DIV], omega] := set[0]
```

lcm examples

The least common multiple of the empty set is **1**.

```
In[26]:= Map[A, SubstTest[image, funpart[t], set[0], t → LUB[DIV]]]
Out[26]= APPLY[LUB[DIV], 0] == set[0]

In[27]:= APPLY[LUB[DIV], 0] := set[0]
```

The least common multiple of the set of all natural numbers is **0**.

```
In[28]:= Map[A, SubstTest[image, funpart[t], set[omega], t → LUB[DIV]]]
Out[28]= APPLY[LUB[DIV], omega] == 0

In[29]:= APPLY[LUB[DIV], omega] := 0
```