P[RUSSELL] can be a proper subclass of RUSSELL

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2008 May 30

```
In[1]:= << goedel.08may22a; << tools.m

:Package Title: goedel.08may22a                    2008 May 22 at 2:25 p.m.

It is now:  2008 May 30 at 9:6

Loading Simplification Rules

TOOLS.M                                        Revised 2008 May 17

weightlimit = 40
```

summary

The **Russell class** is the class of all sets that do not belong to themselves.

```
In[2]:= class[x, not[member[x, x]]]

Out[2]= RUSSELL
```

The **power class** of a class **x** is the class of all subsets of **x**. Cantor’s theorem implies that the power class of a set cannot be a subset of **x**. For proper classes, such as **RUSSELL**, this need not be the case. In fact, the power class of the Russell class is a subclass of the Russell class.

```
In[3]:= subclass[P[RUSSELL], RUSSELL]

Out[3]= True
```

An example is presented to show that, if the axiom of regularity does not hold, there can be a set that belongs to the power class of the Russell class that does not belong to the Russell class.

construction of the example

Suppose that there were a set **a** whose only members are **0** and **a** itself:

```
In[4]:= set[a, 0] = a;
```

The claim is the singleleton of **a** is a member of **RUSSELL**, but not a member of **P[RUSSELL]**.

verification of the example

Lemma.

```
In[5]:= SubstTest[member, t, set[t, 0], t → a] // Reverse

Out[5]= member[a, a] == True

In[6]:= member[a, a] := True
```

Lemma

```
In[8]:= SubstTest[member, 0, set[t, 0], t → a] // Reverse

Out[8]= member[0, a] == True

In[9]:= member[0, a] := True
```

Lemma.

```
In[10]:= SubstTest[equal, 0, set[t, 0], t → a] // Reverse

Out[10]= equal[0, a] == False

In[11]:= equal[0, a] := False
```

Lemma.

```
In[13]:= member[a, RUSSELL] // AssertTest

Out[13]= member[a, RUSSELL] == False

In[14]:= member[a, RUSSELL] := False
```

Lemma.

```
In[22]:= Map[not, SubstTest[implies, and[member[0, x], not[member[0, y]]],
            not[equal[x, y]], {x → a, y → set[a]}]] // Reverse

Out[22]= subclass[a, set[a]] == False

In[23]:= subclass[a, set[a]] := False
```

Theorem.

```
In[24]:= member[set[a], RUSSELL] // AssertTest

Out[24]= member[set[a], RUSSELL] == True

In[25]:= member[set[a], RUSSELL] := True

In[26]:= member[set[a], P[RUSSELL]]

Out[26]= False

In[28]:= Map[not[assert[#]] &, SubstTest[implies, and[member[t, RUSSELL], not[member[t, P[RUSSELL]]]],
            propersubclass[P[RUSSELL], RUSSELL], t → set[a]]] // Reverse

Out[28]= equal[0, fix[E]] == False

In[31]:= equal[0, fix[E]] := False
```

Corollary. If the set **a** exists, then **P[RUSSELL]** is a proper subclass of **RUSSELL**.

```
In[32]:= assert[propersubclass[P[RUSSELL], RUSSELL]]

Out[32]= True
```

Corollary. If the set **a** exists, then the axiom of regularity does not hold.

```
In[35]:= Map[assert, SubstTest[and, equal[V, x], subclass[x, y], {x → REGULAR, y → RUSSELL}]] // Reverse

Out[35]= AxReg == False
```