some theorems about range and image

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In[1]:= **<< goedel54.28a; << tools.m**

```
:Package Title: goedel54.28a     2004 February 28 at 5:40 a.m.

It is now:  2004 Mar 1 at 12:31

Loading Simplification Rules

TOOLS.M                      Revised 2004 February 21

weightlimit = 40
```

summary

In this notebook some results related to Quaife's classics on **range** and **image** are derived. All of these are simple theorems that were proven several years ago using McCune's automated theorem proving program **Otter**, but which until now were not immediately recognized to be true by the **GOEDEL** program.

a corollary of Quaife's theorem (RA8)

Quaife's first corollary of his Theorem **(RA8)** is transformed as follows by the **GOEDEL** program:

In[2]:= **subclass[range[restrict[x, y, z]], intersection[range[x], z]]**

Out[2]= subclass[intersection[z, image[x, y]], range[x]]

This result can be established as follows:

In[3]:= **subclass[intersection[z, image[x, y]], range[x]] // AssertTest**

Out[3]= subclass[intersection[z, image[x, y]], range[x]] == True

In[4]:= **subclass[intersection[z_, image[x_, y_]], range[x_]] := True**

a result related to Quaife's theorem (IM2)

Lemmas.

In[5]:= **Map[or[member[u, x], #] &, SubstTest[implies,
and[member[z, w], subclass[w, t]], member[z, t], {z -> pair[u, v], t -> cart[x, y]}]]**

Out[5]= or[member[u, x], not[member[pair[u, v], w]], not[subclass[w, cart[x, y]]]] == True

```
In[6]:= or[member[u_, x_], not[member[pair[u_, v_], w_]],
          not[subclass[w_, cart[x_, y_]]]] := True

In[7]:= Map[or[member[v, y], #] &, SubstTest[implies,
          and[member[z, w], subclass[w, t]], member[z, t], {z -> pair[u, v], t -> cart[x, y]}]]

Out[7]= or[member[v, y], not[member[pair[u, v], w]], not[subclass[w, cart[x, y]]]] == True

In[8]:= or[member[v_, y_], not[member[pair[u_, v_], w_]],
          not[subclass[w_, cart[x_, y_]]]] := True
```

The following result generalizes Theorem **IM−2B**, a corollary of Quaife's theorem **(IM2)** that was proved 1998 May 3 using **Otter**.

```
In[9]:= Map[not, SubstTest[and, implies[and[p2, p3], p5], implies[and[p1, p2, p5], p6],
          not[implies[and[p1, p2, p3], p6]], {p1 → member[u, x], p2 → member[pair[u, v], w],
           p3 → subclass[w, cart[y, z]], p5 -> member[v, z], p6 → member[v, image[w, x]]}]]

Out[9]= or[member[v, image[w, x]], not[member[u, x]],
          not[member[pair[u, v], w]], not[subclass[w, cart[y, z]]]] == True

In[10]:= or[member[v_, image[w_, x_]], not[member[u_, x_]],
           not[member[pair[u_, v_], w_]], not[subclass[w_, cart[y_, z_]]]] := True
```

two results related to Quaife's theorem (IM5)

Lemma.

```
In[11]:= SubstTest[implies, subclass[u, y], subclass[image[x, u], image[x, y]], u → domain[x]]

Out[11]= or[not[subclass[domain[x], y]], subclass[range[x], image[x, y]]] == True

In[12]:= (% /. {x -> x_, y -> y_}) /. Equal -> SetDelayed
```

Theorem **IM−5B**, a corollary of Quaife's theorem **(IM5)** proved 1997 July 24 using **Otter**, resembles this lemma, but with **equal** replacing **subclass** in the conclusion:

```
In[13]:= or[not[subclass[domain[x], y]], equal[range[x], image[x, y]]] // AssertTest

Out[13]= or[equal[image[x, y], range[x]], not[subclass[domain[x], y]]] == True

In[14]:= or[equal[image[x_, y_], range[x_]], not[subclass[domain[x_], y_]]] := True
```

Theorem **IM−5C**, another corollary of Quaife's theorem **(IM5)** also proved 1997 July 24 using **Otter** is a variant of the above. Here is a slightly more general result:

```
In[15]:= Map[not, SubstTest[and, implies[p1, p2],
           implies[p2, p3], not[implies[p1, p3]], {p1 → subclass[x, cart[y, z]],
            p2 → subclass[domain[x], y], p3 → equal[image[x, y], range[x]]}]]

Out[15]= or[equal[image[x, y], range[x]], not[subclass[x, cart[y, z]]]] == True

In[16]:= or[equal[image[x_, y_], range[x_]], not[subclass[x_, cart[y_, z_]]]] := True
```

a curiosity

Theorem **IM−SU−V**, a curious result proved 1998 April 21using **Otter**, is transformed as follows by the **GOEDEL** program:

```
In[17]:= subclass[image[x, y], image[V, x]]

Out[17]= or[equal[0, intersection[y, domain[x]]], not[equal[0, x]]]
```

Its truth can be established as follows:

```
In[18]:= SubstTest[implies, equal[y, z],
          equal[image[w, y], image[w, z]], {z -> 0, w -> composite[id[x], FIRST]}]

Out[18]= or[equal[0, intersection[x, domain[y]]], not[equal[0, y]]] == True

In[19]:= or[equal[0, intersection[x_, domain[y_]]], not[equal[0, y_]]] := True
```