finite acyclic => wellfounded

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finite acyclic relations are wellfounded.

a lemma

If the range of a finite acyclic relation is contained in its domain, the relation is empty. Replacing $x$ by its inverse implies that a similar result holds when the domain is contained in the range.

In[2]:= SubstTest[implies, and[member[y, FINITE], equal[0, fix[trv[y]]],
    subclass[y, cart[V, V]], subclass[range[y], domain[y]]], equal[0, y], y \rightarrow inverse[x]]

Out[2]= \text{or[equal[0, domain[x]], not[equal[0, fix[trv[x]]]], not[member[domain[x], FINITE]],
    not[member[range[x], FINITE]], notsubclass[domain[x], range[x]]]} = \text{True}

In[3]:= (% /. x \rightarrow x_{\_}) /. \text{Equal} \rightarrow \text{SetDelayed}

In[4]:= Map[not, SubstTest[and, implies[p2, p5],
    implies[p2, p6], implies[and[p1, p3, p4, p5, p6, p7],
    implies[and[p3, p7], p8], not[implies[and[p1, p2, p3, p4], p8]],
    \{p1 \rightarrow equal[0, fix[trv[x]]], p2 \rightarrow member[x, FINITE], p3 \rightarrow subclass[x, cart[V, V]],
    p4 \rightarrow subclass[domain[x], range[x]], p5 \rightarrow member[domain[x], FINITE],
    p6 \rightarrow member[range[x], FINITE], p7 \rightarrow equal[0, domain[x]], p8 \rightarrow equal[0, x]\}]]

Out[4]= \text{or[equal[0, x], not[equal[0, fix[trv[x]]]], not[member[x, FINITE]],
    notsubclass[x, cart[V, V]], notsubclass[domain[x], range[x]]]} = \text{True}

In[5]:= \text{or[equal[0, x_{\_}], not[equal[0, fix[trv[x_{\_}]]]], not[member[x_{\_}, FINITE]],
    notsubclass[domain[x_{\_}], range[x_{\_}]], notsubclass[x_{\_}, cart[V, V]]} := \text{True}
subvariance corollary

Replacing \( x \) by \( \text{composite}[x, \text{id}[y]] \) in the theorem derived in the preceding section yields:

\[
\text{In}[6] := \text{SubstTest}[\text{implies},
\text{and}[\text{member}[z, \text{FINITE}], \text{equal}[0, \text{fix}[\text{trv}[z]]],
\text{subclass}[z, \text{cart}[V, V]],
\text{subclass}[\text{domain}[z], \text{range}[z]], \text{equal}[0, z], z \rightarrow \text{composite}[x, \text{id}[y]])
\]

\[
\text{Out}[6] = \text{or}[\text{equal}[0, \text{intersection}[y, \text{domain}[x]]],
\text{not}[\text{equal}[0, \text{fix}[\text{trv}[\text{composite}[x, \text{id}[y]]]])],
\text{not}[\text{member}[\text{composite}[x, \text{id}[y]], \text{FINITE}]],
\text{not}[\text{subclass}[\text{intersection}[y, \text{domain}[x]], \text{image}[x, y]]]) = \text{True}
\]

\[
\text{In}[7] := (\% / . \{x \rightarrow x_-, y \rightarrow y_\}) /. \text{Equal} \rightarrow \text{SetDelayed}
\]

The following lemma holds because any subclass of a finite relation is finite.

\[
\text{In}[8] := \text{SubstTest}[\text{implies}, \text{and}[\text{subclass}[w, x], \text{member}[x, \text{FINITE}]],
\text{member}[w, \text{FINITE}], w \rightarrow \text{composite}[x, \text{id}[y]])
\]

\[
\text{Out}[8] = \text{or}[\text{member}[\text{composite}[x, \text{id}[y]], \text{FINITE}], \text{not}[\text{member}[x, \text{FINITE}])] = \text{True}
\]

\[
\text{In}[9] := (\% / . \{x \rightarrow x_-, y \rightarrow y_\}) /. \text{Equal} \rightarrow \text{SetDelayed}
\]

In the same vein, the following is a consequence of the fact that any subclass of an acyclic relation is acyclic.

\[
\text{In}[10] := \text{SubstTest}[\text{implies}, \text{and}[\text{subclass}[w, x], \text{equal}[0, \text{fix}[\text{trv}[x]]]),
\text{equal}[0, \text{fix}[\text{trv}[w]]], w \rightarrow \text{composite}[x, \text{id}[y]])
\]

\[
\text{Out}[10] = \text{or}[\text{equal}[0, \text{fix}[\text{trv}[\text{composite}[x, \text{id}[y]]])], \text{not}[\text{equal}[0, \text{fix}[\text{trv}[x]]])] = \text{True}
\]

\[
\text{In}[11] := (\% / . \{x \rightarrow x_-, y \rightarrow y_\}) /. \text{Equal} \rightarrow \text{SetDelayed}
\]

The only class that is subvariant under \( x \) and disjoint from \( \text{domain}[x] \) is the empty set.

\[
\text{In}[12] := \text{Map}[\text{implies}[, \text{empty}[y]] \&, \text{SubstTest}[\text{and}, \text{subclass}[y, z], \text{equal}[0, z], z \rightarrow \text{image}[x, y]])
\]

\[
\text{Out}[12] = \text{or}[\text{equal}[0, y], \text{not}[\text{equal}[0, \text{intersection}[y, \text{domain}[x]]]),
\text{not}[\text{subclass}[y, \text{image}[x, y]]]) = \text{True}
\]

\[
\text{In}[13] := (\% / . \{x \rightarrow x_-, y \rightarrow y_\}) /. \text{Equal} \rightarrow \text{SetDelayed}
\]

These lemmas can be combined to obtain the following.
Normality

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\[ \text{Corollary.} \]

\[ \text{Theorem.} \]

\[ \text{Variable-free restatement:} \]

\[ \text{in infinitely divisible relations} \]

\[ \text{Corollary.} \]
In[23]:= % /. Equal -> SetDelayed

This can be restated as an equation.

In[24]:= equal[intersection[FINITE, IDEM, P[Di]], set[0]]

Out[24]= True

In[25]:= intersection[FINITE, IDEM, P[Di]] := set[0]

An idempotent irreflexive relation is infinitely divisible. Between any two points one can insert another: more precisely, if pair[u,v] belongs to such a relation, then there exists w such that pair[u,w] and pair[w,v] also belong to the relation, and u, v, w are all distinct. Since this process can be repeated over and over, it is intuitively clear that such a relation must be infinite, which is precisely what is stated by this equation.