H[RUSSELL]

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In[1]:= **<< goedel56.18c; << tools.m**

```
:Package Title: goedel56.18c          2004 April 18 at 1:00 p.m.

It is now:  2004 Apr 19 at 8:34

Loading Simplification Rules

TOOLS.M                     Revised 2004 April 18

weightlimit = 40
```

summary

The hereditary core of the Russell class is its own power class and its own sum class.

discussion

The Russell class is the complement of the class of sets that belong to themselves:

In[2]:= **complement[fix[E]]**

Out[2]= RUSSELL

This class has the peculiar property that it contains its own power class, which in turn contains its power class ...

In[3]:= **subclass[P[RUSSELL], RUSSELL]**

Out[3]= True

In[4]:= **subclass[P[P[RUSSELL]], P[RUSSELL]]**

Out[4]= True

This yields an infinite decreasing chain of proper classes. What is the intersection of this chain?

iterated power classes

To study the intersection of the infinite chain **RUSSELL**, **P[RUSSELL]**, **P[P[RUSSELL]]**, ... it is useful to look at their complements:

```
In[5]:= image[E, fix[E]]

Out[5]= complement[P[RUSSELL]]

In[6]:= image[E, image[E, fix[E]]]

Out[6]= complement[P[P[RUSSELL]]]
```

The intersection of the sequence of power classes, is the union of the sequence of their complements, and these are the vertical sections of **iterate[E, fix[E]]**. It follows that this intersection is the hereditary core of the Russell class:

```
In[7]:= complement[range[iterate[E, fix[E]]]]

Out[7]= H[RUSSELL]
```

properties of H[RUSSELL]

The hereditary core of the Russell class is its own power class:

```
In[8]:= SubstTest[implies, subclass[u, v], subclass[H[u], H[v]], {u -> P[RUSSELL], v -> RUSSELL}]

Out[8]= subclass[P[H[RUSSELL]], RUSSELL] == True

In[9]:= % /. Equal -> SetDelayed

In[10]:= SubstTest[and, subclass[u, v], subclass[v, u], {u -> H[RUSSELL], v -> P[H[RUSSELL]]}]

Out[10]= True == equal[H[RUSSELL], P[H[RUSSELL]]]

In[11]:= P[H[RUSSELL]] := H[RUSSELL]
```

It is also its own sum class:

```
In[12]:= SubstTest[U, P[x], x -> H[RUSSELL]]

Out[12]= U[H[RUSSELL]] == H[RUSSELL]

In[13]:= U[H[RUSSELL]] := H[RUSSELL]
```

Note that:

```
In[14]:= subclass[H[RUSSELL], P[RUSSELL]]

Out[14]= True

In[15]:= subclass[H[RUSSELL], P[P[RUSSELL]]]

Out[15]= True
```

etc.

a natural question

The class **REGULAR** of all regular sets also has the property of being its own sum class and power class:

```
In[16]:= P[REGULAR]

Out[16]= REGULAR

In[17]:= U[REGULAR]

Out[17]= REGULAR
```

The class **REGULAR** is the union of the Zermelo−von Neumann cumulative hierarchy:

```
In[18]:= image[ZN, OMEGA]

Out[18]= REGULAR
```

It is an elementary consequence of transfinite induction that **REGULAR** is the smallest class that contains its power class. This fact was proved using **Otter**.

```
In[19]:= implies[subclass[P[x], x], subclass[REGULAR, x]]

Out[19]= True
```

This implies that the class **REGULAR** is contained in **H[RUSSELL]**:

```
In[20]:= subclass[REGULAR, H[RUSSELL]]

Out[20]= True
```

A natural question is whether these classes are equal?

```
In[21]:= equal[REGULAR, H[RUSSELL]] // AssertTest

Out[21]= equal[REGULAR, H[RUSSELL]] == subclass[H[RUSSELL], REGULAR]
```

Of course, if the axiom of regularity is assumed, the answer is trivially affirmative because then **REGULAR = V**, and all these classes are equal.