the cardinal $\aleph_\omega$

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```
In[1]:= SetDirectory["l:"]; << goedel.10nov20a

        :Package Title: goedel.10nov20a              2010 November 20 at 4:30 p.m.

        It is now:  2010 Nov 22 at 13:8

        Loading Simplification Rules

        TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

        weightlimit = 40
```

summary

The cardinal $\aleph_\omega$ is the union of all smaller alephs.

the set of cardinals $\aleph_n$ where $n \in \omega$

The set of cardinals $\aleph_n$ where $\mathbf{n} \in \omega$ is **image[ℵ, ω]**.

Theorem. $\aleph_n \in$ **image[ℵ, ω]**.

```
In[2]:= SubstTest[member, APPLY[funpart[u], v], range[funpart[u]],
          {u → composite[ALEPH, id[omega]], v → nat[x]}] // Reverse

Out[2]= member[APPLY[ALEPH, nat[x]], image[ALEPH, omega]] == True

In[3]:= member[APPLY[ALEPH, nat[x_]], image[ALEPH, omega]] := True
```

Corollary. (Remove **nat** wrapper.)

```
In[4]:= SubstTest[implies, equal[x, nat[t]],
          member[APPLY[ALEPH, x], image[ALEPH, omega]], t → x] // Reverse

Out[4]= or[member[APPLY[ALEPH, x], image[ALEPH, omega]], not[member[x, omega]]] == True

In[5]:= or[member[APPLY[ALEPH, x_], image[ALEPH, omega]], not[member[x_, omega]]] := True
```

Since the successor of a natural number is another natural number, the following corollary is true.

Corollary. $\aleph_{n+1} \in$ **image[ℵ, ω]**.

```
In[6]:= SubstTest[member, APPLY[ALEPH, nat[t]], image[ALEPH, omega], t → succ[nat[x]]] // Reverse
```

```
Out[6]= member[APPLY[ALEPH, succ[nat[x]]], image[ALEPH, omega]] == True
```

```
In[7]:= member[APPLY[ALEPH, succ[nat[x_]]], image[ALEPH, omega]] := True
```

The ℵ function lists all infinite cardinals in increasing order, so each is less than the next.

Lemma. $\aleph_n \in \aleph_{n+1}$.

```
In[8]:= SubstTest[implies, member[ord[u], ord[v]],
          member[APPLY[ALEPH, ord[u]], APPLY[ALEPH, ord[v]]],
          {u → nat[x], v → succ[nat[x]]}] // Reverse
```

```
Out[8]= member[APPLY[ALEPH, nat[x]], APPLY[ALEPH, succ[nat[x]]]] == True
```

```
In[9]:= member[APPLY[ALEPH, nat[x_]], APPLY[ALEPH, succ[nat[x_]]]] := True
```

Theorem. $\aleph_n \in$ **U[image[ℵ, ω]]**.

```
In[10]:= SubstTest[implies, and[member[u, v], member[v, w]],
           member[u, U[w]], {u -> APPLY[ALEPH, nat[x]],
            v -> APPLY[ALEPH, succ[nat[x]]], w -> image[ALEPH, omega]}] // Reverse
```

```
Out[10]= member[APPLY[ALEPH, nat[x]], U[image[ALEPH, omega]]] == True
```

```
In[11]:= (% /. x → x_) /. Equal → SetDelayed
```

Theorem. Eliminate **nat** wrapper.

```
In[12]:= SubstTest[implies, equal[x, nat[t]],
           member[APPLY[ALEPH, x], U[image[ALEPH, omega]]], t → x] // Reverse
```

```
Out[12]= or[member[APPLY[ALEPH, x], U[image[ALEPH, omega]]], not[member[x, omega]]] == True
```

```
In[13]:= (% /. x → x_) /. Equal → SetDelayed
```

Theorem. (Eliminate the variable **x**.)

```
In[14]:= Map[or[subclass[image[ALEPH, omega], U[image[ALEPH, omega]]], equal[V, #]] &, SubstTest[
           class, x, member[nat[x], t], t -> image[inverse[ALEPH], U[image[ALEPH, omega]]]]]
```

```
Out[14]= subclass[image[ALEPH, omega], U[image[ALEPH, omega]]] == True
```

```
In[15]:= % /. Equal → SetDelayed
```

Corollary.

```
In[16]:= Map[not, SubstTest[implies, subclass[x, OMEGA],
            not[and[member[U[x], x], subclass[x, U[x]]]], x → image[ALEPH, omega]]] // Reverse
```

```
Out[16]= member[U[image[ALEPH, omega]], image[ALEPH, omega]] == False
```

```
In[17]:= % /. Equal → SetDelayed
```

U[image[ℵ, ω]]

The union of any set of cardinals is a cardinal: **Uclosure[fix[CARD]] = fix[CARD]**.

Theorem. The set **U[image[ℵ, ω]]** is a cardinal.

```
In[18]:= SubstTest[implies, and[member[u, P[v]], equal[Uclosure[v], v]],
           member[U[u], v], {u → image[ALEPH, omega], v → fix[CARD]}] // Reverse

Out[18]= equal[card[U[image[ALEPH, omega]]], U[image[ALEPH, omega]]] == True

In[19]:= card[U[image[ALEPH, omega]]] := U[image[ALEPH, omega]]
```

Corollary. The set **U[image[ℵ, ω]]** is an ordinal.

```
In[20]:= SubstTest[implies, and[member[t, u], subclass[u, v]], member[t, v],
           {t -> U[image[ALEPH, omega]], u → fix[CARD], v → OMEGA}] // Reverse

Out[20]= member[U[image[ALEPH, omega]], OMEGA] == True

In[21]:= member[U[image[ALEPH, omega]], OMEGA] := True
```

The derivation of following theorem uses the fact that every aleph is a limit ordinal.

Theorem. **U[image[ℵ, ω]]** $\subset \aleph_\omega$.

```
In[22]:= SubstTest[implies, subclass[u, v], subclass[U[u], U[v]],
           {u → image[ALEPH, omega], v → APPLY[ALEPH, omega]}] // Reverse

Out[22]= subclass[U[image[ALEPH, omega]], APPLY[ALEPH, omega]] == True

In[23]:= % /. Equal → SetDelayed
```

Corollary. The ordinal $\aleph_\omega$ is not less than the ordinal **U[image[ℵ, ω]]**.

```
In[24]:= Map[not, SubstTest[subclass, ord[u], ord[v],
           {u -> U[image[ALEPH, omega]], v -> APPLY[ALEPH, omega]}]]

Out[24]= member[APPLY[ALEPH, omega], U[image[ALEPH, omega]]] == False

In[25]:= member[APPLY[ALEPH, omega], U[image[ALEPH, omega]]] := False
```

Lemma. ω $\in$ **U[image[ℵ, ω]]**.

```
In[26]:= SubstTest[member, APPLY[ALEPH, nat[x]], U[image[ALEPH, omega]], x → 0] // Reverse

Out[26]= member[omega, U[image[ALEPH, omega]]] == True

In[27]:= member[omega, U[image[ALEPH, omega]]] := True
```

Theorem. The cardinal **U[image[ℵ, ω]]** is not a natural number.

```
In[28]:= Map[not, SubstTest[implies, member[ord[u], ord[v]],
            not[member[ord[v], ord[u]]], {u -> omega, v -> U[image[ALEPH, omega]]}]] // Reverse

Out[28]= member[U[image[ALEPH, omega]], omega] == False

In[29]:= member[U[image[ALEPH, omega]], omega] := False
```

It follows from this that **U[image[ℵ, ω]]** is an aleph.

```
In[30]:= member[U[image[ALEPH, omega]], range[ALEPH]]

Out[30]= True
```

monotonicity of inverse[ℵ]

The function **ℵ** is one-to-one, and its inverse is strictly monotone.

Theorem. **P[inverse[ℵ]] ⊂ monotone[E, E]**.

```
In[31]:= SubstTest[or, not[FUNCTION[t]], not[FUNCTION[inverse[t]]],
           not[subclass[t, cart[OMEGA, OMEGA]]], not[subclass[composite[t, S, inverse[t]], S]],
           subclass[t, composite[S, IMAGE[t]]], t → inverse[ALEPH]] // Reverse

Out[31]= subclass[inverse[ALEPH], composite[S, IMAGE[inverse[ALEPH]]]] == True

In[32]:= subclass[inverse[ALEPH], composite[S, IMAGE[inverse[ALEPH]]]] := True
```

Lemma.

```
In[33]:= ApComp[inverse[ALEPH], ALEPH, omega]

Out[33]= APPLY[inverse[ALEPH], APPLY[ALEPH, omega]] == omega

In[34]:= % /. Equal → SetDelayed
```

Lemma.

```
In[35]:= ApComp[ALEPH, inverse[ALEPH], U[image[ALEPH, omega]]]

Out[35]= APPLY[ALEPH, APPLY[inverse[ALEPH], U[image[ALEPH, omega]]]] == U[image[ALEPH, omega]]

In[36]:= % /. Equal → SetDelayed
```

Theorem.

```
In[37]:= Map[not,
           SubstTest[implies, member[x, omega], member[APPLY[ALEPH, x], image[ALEPH, omega]],
            x -> APPLY[inverse[ALEPH], U[image[ALEPH, omega]]]]] // Reverse

Out[37]= member[APPLY[inverse[ALEPH], U[image[ALEPH, omega]]], omega] == False

In[38]:= % /. Equal → SetDelayed
```

Theorem. **U[image[ℵ, ω]]** is not less than $\aleph_\omega$.

```
In[39]:= Map[not, SubstTest[implies, and[FUNCTION[t], subclass[t, composite[S, IMAGE[t]]],
             member[x, y], member[x, domain[t]]], member[APPLY[t, x], APPLY[t, y]],
           {t → inverse[ALEPH], x → U[image[ALEPH, omega]], y → APPLY[ALEPH, omega]}]] // Reverse

Out[39]= member[U[image[ALEPH, omega]], APPLY[ALEPH, omega]] == False

In[40]:= member[U[image[ALEPH, omega]], APPLY[ALEPH, omega]] := False
```

An application of ordinal trichotomy now yields the main theorem.

Main Theorem. **U[image[ℵ, ω]]** = $\aleph_\omega$.

```
In[41]:= SubstTest[or, member[ord[u], ord[v]], equal[ord[u], ord[v]], member[ord[v], ord[u]],
          {u -> U[image[ALEPH, omega]], v → APPLY[ALEPH, omega]}] // Reverse

Out[41]= equal[APPLY[ALEPH, omega], U[image[ALEPH, omega]]] == True

In[42]:= U[image[ALEPH, omega]] := APPLY[ALEPH, omega]
```

an iteration formula

In this section a formula for the set of alephs less than $\aleph_\omega$ is derived.

Theorem. The cardinals $\aleph_n$ where **n** $\in \omega$ are obtained iteratively by repeated application of the Hartogs operation, starting with $\aleph_0 = \omega$.

```
In[43]:= SubstTest[implies, and[equal[composite[u, w], composite[w, SUCC]],
           equal[image[w, set[0]], v]], equal[iterate[u, v], composite[w, id[omega]]],
          {u → HARTOGS, v→ set[omega], w → ALEPH}] // Reverse

Out[43]= equal[composite[ALEPH, id[omega]], iterate[HARTOGS, set[omega]]] == True

In[44]:= iterate[HARTOGS, set[omega]] := composite[ALEPH, id[omega]]
```

Corollary. The set of alephs less than $\aleph_\omega$ is the smalled set that is invariant under the Hartogs operation and holds ω.

```
In[45]:= SubstTest[range, iterate[funpart[u], set[v]], {u → HARTOGS, v→ omega}]

Out[45]= hull[invar[HARTOGS], set[omega]] == image[ALEPH, omega]

In[46]:= hull[invar[HARTOGS], set[omega]] := image[ALEPH, omega]
```

Corollary. The set of alephs less than $\aleph_\omega$ is invariant under the Hartogs operation.

```
In[47]:= SubstTest[member, hull[invar[HARTOGS], setpart[t]],
          invar[HARTOGS], t → set[omega]] // Reverse

Out[47]= subclass[image[HARTOGS, image[ALEPH, omega]], image[ALEPH, omega]] == True
```

```
In[48]:= subclass[image[HARTOGS, image[ALEPH, omega]], image[ALEPH, omega]] := True
```