cardinality of omega

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```
<< goedel52.o13; << tools.m
```

```
:Package Title: goedel52.o13                          2002 May 28 at 11:15 p.m.

It is now:  2002 May 29 at 5:50

Loading Simplification Rules

TOOLS.M                   Revised 2002 May 22

weightlimit = 40
```

■ Introduction

In this notebook we derive the fact that **omega** is the first infinite cardinal. The basic idea will be to use the fact that for the cardinality of any ordinal is less than or equal to the ordinal itself, and that the cardinality of an infinite ordinal must be infinite. But there is no infinite ordinal less than **omega**, so the only possibility in this case is equality.

■ The derivation

First we have to argue that the cardinality of **omega** is a set, that is, it is in the domain of **CARD**.

```
SubstTest[implies, and[member[x, y], subclass[y, z]], member[x, z],
    {x -> omega, y -> OMEGA, z -> domain[CARD]}]
```

```
member[omega, image[Q, OMEGA]] == True
```

```
member[omega, image[Q, OMEGA]] := True
```

Next we use monotonicity to show that the cardinality of **omega** is infinite.

```
Map[not, SubstTest[implies,
  and[subclass[u, v], subclass[x, y]], subclass[image[u, x], image[v, y]],
    {u -> CARD, v -> Q, x -> singleton[omega], y -> complement[FINITE]}]]
```

```
member[card[omega], FINITE] == False
```

```
member[card[omega], FINITE] := False
```

The cardinality of **omega** does not exceed **omega** because this holds for all ordinals. That is because any ordinal is equipollent to itself.

```
SubstTest[implies, and[member[u, v], subclass[v, w]], member[u, w],
    {u -> omega, v -> OMEGA, w -> fix[composite[S, CARD]]}]
```

```
subclass[card[omega], omega] == True
```

```
subclass[card[omega], omega] := True
```

Since all ordinals less than **omega** are finite, the cardinality of **omega** cannot be less than **omega**.

```
Map[not, SubstTest[implies, and[member[u, v], subclass[v, w]], member[u, w],
    {u -> card[omega], v -> omega, w -> FINITE}]]
```

```
member[card[omega], omega] == False
```

```
member[card[omega], omega] := False
```

The only remaining possibility is equality:

```
SubstTest[member, x, intersection[y, z],
    {x -> card[omega], y -> FULL, z -> P[omega]}]
```

```
equal[omega, card[omega]] == True
```

```
card[omega] := omega
```