the class of associative relations is a proper class

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```
In[1]:= << goedel52.s31; << tools.m

        :Package Title: goedel52.s31       2003 July 2 at 9:50 a.m.

        It is now:  2003 Jul 8 at 9:22

        Loading Simplification Rules

        TOOLS.M                       Revised 2003 July 6

        weightlimit = 40
```

■ summary

It is shown in this notebook that the class **ASSOCIATIVE** of all associative relations is a proper class.

■ derivation

Most of the work is in the first step. This amounts to the observation that **cart[cart[x,x],y]** is always associative.

```
In[2]:= image[inverse[cross[composite[CART, DUP], Id]],
            image[inverse[CART], ASSOCIATIVE]] // RelnNormality

Out[2]= composite[image[inverse[CART], ASSOCIATIVE], CART, DUP] == cart[V, V]
```

The result can be reformulated in the following more familiar fashion:

```
In[3]:= Map[equal[0, composite[Id, complement[#]]] &, %]

Out[3]= subclass[image[CART, cart[image[CART, Id], V]], ASSOCIATIVE] == True

In[4]:= subclass[image[CART, cart[image[CART, Id], V]], ASSOCIATIVE] := True
```

■ the sum class of ASSOCIATIVE

Lemma:

```
In[5]:= SubstTest[implies, subclass[u, v], subclass[U[u], U[v]],
            {u -> image[CART, cart[image[CART, Id], V]], v -> ASSOCIATIVE}]

Out[5]= subclass[cart[cart[V, V], V], U[ASSOCIATIVE]] == True

In[6]:= subclass[cart[cart[V, V], V], U[ASSOCIATIVE]] := True
```

The opposite inclusion also holds:

```
In[7]:= Map[equal[0, #] &, dif[ASSOCIATIVE, P[cart[cart[V, V], V]]] // Normality]

Out[7]= subclass[U[ASSOCIATIVE], cart[cart[V, V], V]] == True

In[8]:= subclass[U[ASSOCIATIVE], cart[cart[V, V], V]] := True
```

The two inclusions can be combined into an equation:

```
In[9]:= SubstTest[and, subclass[u, v], subclass[v, u],
            {u -> cart[cart[V, V], V], v -> U[ASSOCIATIVE]}]

Out[9]= True == equal[cart[cart[V, V], V], U[ASSOCIATIVE]]
```

The following rewrite rule will be permanently retained:

```
In[10]:= U[ASSOCIATIVE] := cart[cart[V, V], V]
```

■ ASSOCIATIVE is a proper class

As a corollary, one obtains the fact that **ASSOCIATIVE** is a proper class.

```
In[11]:= SubstTest[member, U[x], V, x -> ASSOCIATIVE] // Reverse

Out[11]= member[ASSOCIATIVE, V] == False

In[12]:= member[ASSOCIATIVE, V] := False
```

The following is a better rewrite rule:

```
In[13]:= member[ASSOCIATIVE, x] // AssertTest

Out[13]= member[ASSOCIATIVE, x] == False

In[14]:= member[ASSOCIATIVE, x_] := False
```