asymmetric part of a transitive relation

Johan G. F. Belinfante
2009 May 19

```
In[1]:= SetDirectory["l:"]; << goedel.09may18a; << tools.m

        :Package Title: goedel.09may18a                2009 May 18 at 5:55 a.m.

        It is now:  2009 May 19 at 13:29

        Loading Simplification Rules

        TOOLS.M                                        Revised 2009 May 16

        weightlimit = 40
```

summary

The asymmetric part of a transitive relation is transitive:

```
In[2]:= implies[TRANSITIVE[x], TRANSITIVE[dif[x, inverse[x]]]]

Out[2]= True
```

In this notebook a variable-free expression of this fact is derived for the special case of transitive relations that are sets. A particularly nice formulation of this uses the idempotent function **composite[DIF, id[INVERSE], inverse[FIRST]]**.

derivation

The best results are obtained if the function **IMAGE[SWAP]** which takes **x** to **inverse[x]** is replaced with its restriction **INVERSE**.

Lemma. The class of transitive relations is contained in the class of relations whose asymmetric part is transitive.

```
In[3]:= Map[subclass[TRV, intersection[P[cart[V, V]], #]] &,
          union[complement[TRV], fix[composite[
              inverse[image[inverse[DIF], TRV]], IMAGE[SWAP]]]]] // Normality] // InvertFix

Out[3]= subclass[TRV, fix[composite[INVERSE, image[inverse[DIF], TRV]]]] == True

In[4]:= % /. Equal → SetDelayed
```

Lemma. The asymmetric part of a transitive relation is transitive.

```
In[5]:= SubstTest[subclass, intersection[y, domain[funpart[x]]], image[inverse[funpart[x]], z],
          {x → composite[DIF, id[INVERSE], inverse[FIRST]], y → TRV, z → TRV}]

Out[5]= subclass[image[DIF, composite[INVERSE, id[TRV]]], TRV] == True

In[6]:= % /. Equal → SetDelayed
```

Lemma. Simplification rule.

```
In[7]:= fix[composite[complement[inverse[E]], INVERSE, id[TRV], E]] // Normality

Out[7]= fix[composite[complement[inverse[E]], INVERSE, id[TRV], E]] == Di

In[8]:= fix[composite[complement[inverse[E]], INVERSE, id[TRV], E]] := Di
```

Lemma. An irreflexive transitive relation is its own asymmetric part.

```
In[9]:= SubstTest[implies, subclass[y, fix[funpart[x]]],
          equal[image[funpart[x], y], y], {x -> composite[DIF, id[INVERSE], inverse[FIRST]],
           y → intersection[TRV, P[Di]]}] // Reverse

Out[9]= equal[image[DIF, composite[INVERSE, id[intersection[TRV, P[Di]]]]],
          intersection[TRV, P[Di]]] == True

In[10]:= image[DIF, composite[INVERSE, id[intersection[TRV, P[Di]]]]] :=
          intersection[TRV, P[Di]]
```

Lemma. Every irreflexive transitive relation is the asymmetric part of a transitive relation.

```
In[11]:= SubstTest[implies, subclass[u, v], subclass[image[t, u], image[t, v]],
           {t → composite[DIF, id[INVERSE], inverse[FIRST]],
            u → intersection[TRV, P[Di]], v → TRV}] // Reverse

Out[11]= subclass[intersection[TRV, P[Di]], image[DIF, composite[INVERSE, id[TRV]]]] == True

In[12]:= % /. Equal → SetDelayed
```

Theorem. The class of asymmetric parts of transitive relations is equal to the class of irreflexive transitive relations.

```
In[13]:= SubstTest[and, subclass[u, v], subclass[v, u],
          {u -> intersection[TRV, P[Di]], v -> image[DIF, composite[INVERSE, id[TRV]]]}]

Out[13]= equal[image[DIF, composite[INVERSE, id[TRV]]], intersection[TRV, P[Di]]] == True

In[14]:= image[DIF, composite[INVERSE, id[TRV]]] := intersection[TRV, P[Di]]
```