binary-fixed classes

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In[1]:= **SetDirectory["l:"]; << goedel.10jul22a; << tools.m**

```
:Package Title: goedel.10jul22a               2010 July 22 at 12:50 p.m.

It is now:  2010 Jul 23 at 12:53

Loading Simplification Rules

TOOLS.M                                      Revised 2010 February 26

weightlimit = 40
```

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summary

A class **y** is said to be **binary-closed** under **x** if **image[x, y × y] ⊂ y**. The class **x** here need not be a function. The class of all sets that are binary-closed under **x** is called **binclosed[x]**. This class is closed under arbitrary intersections and under unions of chains.

In[2]:= **fix[HULL[binclosed[x]]]**

Out[2]= binclosed[x]

In[3]:= **Uchains[binclosed[x]]**

Out[3]= binclosed[x]

A class **y** is said to be **binary-fixed** under **x** if the stronger condition **image[x, y × y] = y** holds. The class of all sets that are binary-fixed by **x** is **fix[IMAGE[x] ∘ CART ∘ DUP]**. Some rewrite rules about the classes of binary-closed and binary-fixed sets are derived in this notebook, especially for relations of the special form **(x ∘ FIRST) ⋂ (y ∘ SECOND)**.

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derivation

Observation. The class **binclosed[x]** is given by the expression:

In[4]:= **fix[composite[S, IMAGE[x], CART, DUP]]**

Out[4]= binclosed[x]

Lemma. A formula for the class of binary-fixed sets.

```
In[8]:= SubstTest[intersection, fix[composite[inverse[S], funpart[t]]],
          fix[composite[S, funpart[t]]], t → composite[IMAGE[x], CART, DUP]] // Reverse

Out[8]= intersection[binclosed[x], fix[composite[inverse[S], IMAGE[x], CART, DUP]]] ==
         fix[composite[IMAGE[x], CART, DUP]]

In[9]:= intersection[binclosed[x_], fix[composite[inverse[S], IMAGE[x_], CART, DUP]]] :=
         fix[composite[IMAGE[x], CART, DUP]]
```

Theorem. Binary-fixed sets are binary-closed.

```
In[11]:= SubstTest[subclass, intersection[u, v], u,
           {u -> binclosed[x], v -> fix[composite[inverse[S], IMAGE[x], CART, DUP]]}] // Reverse

Out[11]= subclass[fix[composite[IMAGE[x], CART, DUP]], binclosed[x]] == True

In[12]:= subclass[fix[composite[IMAGE[x_], CART, DUP]], binclosed[x_]] := True
```

invar

In this section the class **binclosed[(x ∘ FIRST) ⋂ (y ∘ SECOND)]**. is considered. The membership rule for this class is:

```
In[33]:= member[t, binclosed[intersection[composite[x, FIRST], composite[y, SECOND]]]]

Out[33]= and[member[t, V], subclass[intersection[image[x, t], image[y, t]], t]]
```

The membership rule show that this class is invariant under interchanging **x** and **y**.

Theorem. Flip symmetry.

```
In[34]:= Map[equal[#, binclosed[intersection[composite[x, SECOND], composite[y, FIRST]]]] &,
          SubstTest[binclosed, flip[t],
           t -> intersection[composite[x, FIRST], composite[y, SECOND]]]]

Out[34]= equal[binclosed[intersection[composite[x, FIRST], composite[y, SECOND]]],
           binclosed[intersection[composite[x, SECOND], composite[y, FIRST]]]] == True

In[35]:= equal[binclosed[intersection[composite[x_, FIRST], composite[y_, SECOND]]],
           binclosed[intersection[composite[x_, SECOND], composite[y_, FIRST]]]] := True
```

Theorem. A lower bound.

```
In[36]:= Map[equal[V, #] &,
          dif[invar[x], binclosed[intersection[composite[x, FIRST], composite[y, SECOND]]]] //
            complement // Normality]

Out[36]= subclass[invar[x],
           binclosed[intersection[composite[x, FIRST], composite[y, SECOND]]]] == True

In[37]:= subclass[invar[x_],
           binclosed[intersection[composite[x_, FIRST], composite[y_, SECOND]]]] := True
```

Theorem. A similar lower bound.

```
In[38]:= Map[equal[V, #] &,
          dif[invar[y], binclosed[intersection[composite[x, FIRST], composite[y, SECOND]]]] //
            complement // Normality]

Out[38]= subclass[invar[y],
           binclosed[intersection[composite[x, FIRST], composite[y, SECOND]]]] == True

In[39]:= subclass[invar[y_],
           binclosed[intersection[composite[x_, FIRST], composite[y_, SECOND]]]] := True
```

subvar

In this section the class of sets binary-fixed by **(x ∘ FIRST) ⋂ (y ∘ SECOND)** is considered.

Lemma.

```
In[42]:= composite[IMAGE[intersection[composite[x, FIRST], composite[y, SECOND]]],
           IMAGE[SWAP]] // FastReifNormality

Out[42]= composite[IMAGE[intersection[composite[x, FIRST], composite[y, SECOND]]],
           IMAGE[SWAP]] == IMAGE[intersection[composite[x, SECOND], composite[y, FIRST]]]

In[43]:= composite[IMAGE[intersection[composite[x_, FIRST], composite[y_, SECOND]]],
           IMAGE[SWAP]] := IMAGE[intersection[composite[x, SECOND], composite[y, FIRST]]]
```

Theorem. Flip symmetry for a class of binary-fixed sets.

```
In[44]:= Map[equal[fix[#], fix[composite[
              IMAGE[intersection[composite[x, SECOND], composite[y, FIRST]]], CART, DUP]]] &,
          Assoc[IMAGE[intersection[composite[x, FIRST], composite[y, SECOND]]],
           IMAGE[SWAP], composite[CART, DUP]]]

Out[44]= equal[fix[composite[
             IMAGE[intersection[composite[x, FIRST], composite[y, SECOND]]], CART, DUP]],
           fix[composite[IMAGE[intersection[composite[x, SECOND], composite[y, FIRST]]],
             CART, DUP]]] == True

In[45]:= equal[fix[composite[
             IMAGE[intersection[composite[x_, FIRST], composite[y_, SECOND]]], CART, DUP]],
           fix[composite[IMAGE[intersection[composite[x_, SECOND], composite[y_, FIRST]]],
             CART, DUP]]] := True
```

Lemma.

```
In[47]:= Map[empty,
          intersection[complement[subvar[x]], fix[composite[inverse[S], IMAGE[intersection[
                composite[x, FIRST], composite[y, SECOND]]], CART, DUP]]] // Normality]

Out[47]= subclass[fix[composite[inverse[S],
            IMAGE[intersection[composite[x, FIRST], composite[y, SECOND]]],
            CART, DUP]], subvar[x]] == True

In[48]:= subclass[fix[composite[inverse[S],
            IMAGE[intersection[composite[x_, FIRST], composite[y_, SECOND]]],
            CART, DUP]], subvar[x_]] := True
```

Corollary. An upper bound for a class of binary-fixed sets.

```
In[49]:= SubstTest[implies, and[subclass[u, v], subclass[v, w]],
           subclass[u, w], {u → fix[composite[
               IMAGE[intersection[composite[x, FIRST], composite[y, SECOND]]], CART, DUP]],
            v -> fix[composite[inverse[S], IMAGE[intersection[composite[x, FIRST],
                 composite[y, SECOND]]], CART, DUP]], w -> subvar[x]}] // Reverse

Out[49]= subclass[fix[composite[IMAGE[intersection[composite[x, FIRST], composite[y, SECOND]]],
            CART, DUP]], subvar[x]] == True

In[50]:= subclass[
           fix[composite[IMAGE[intersection[composite[x_, FIRST], composite[y_, SECOND]]],
            CART, DUP]], subvar[x_]] := True
```

Theorem. A similar upper bound.

```
In[51]:= Map[subclass[fix[#], subvar[y]] &,
          Assoc[IMAGE[intersection[composite[x, FIRST], composite[y, SECOND]]],
           IMAGE[SWAP], composite[CART, DUP]]]

Out[51]= subclass[fix[composite[IMAGE[intersection[composite[x, FIRST], composite[y, SECOND]]],
            CART, DUP]], subvar[y]] == True

In[52]:= subclass[
           fix[composite[IMAGE[intersection[composite[x_, FIRST], composite[y_, SECOND]]],
            CART, DUP]], subvar[y_]] := True
```

Lemma.

```
In[53]:= Map[empty,
          dif[intersection[binclosed[intersection[composite[x, FIRST], composite[y, SECOND]]],
            subvar[x], subvar[y]],
           fix[composite[IMAGE[intersection[composite[x, FIRST], composite[y, SECOND]]],
             CART, DUP]]]] // Normality]

Out[53]= subclass[
           intersection[binclosed[intersection[composite[x, FIRST], composite[y, SECOND]]],
            subvar[x], subvar[y]], fix[composite[
             IMAGE[intersection[composite[x, FIRST], composite[y, SECOND]]], CART, DUP]]] == True
```

```
In[54]:= (% /. {x → x_, y → y_}) /. Equal → SetDelayed
```

Theorem. A connection between a class of binary-closed sets and a corresponding class of binary-fixed sets.

```
In[55]:= SubstTest[and, subclass[u, v], subclass[v, u],
          {u -> intersection[binclosed[intersection[composite[x, FIRST], composite[y, SECOND]]],
              subvar[x], subvar[y]], v -> fix[composite[
               IMAGE[intersection[composite[x, FIRST], composite[y, SECOND]]], CART, DUP]]}]

Out[55]= equal[fix[composite[
              IMAGE[intersection[composite[x, FIRST], composite[y, SECOND]]], CART, DUP]],
            intersection[binclosed[intersection[composite[x, FIRST], composite[y, SECOND]]],
             subvar[x], subvar[y]]] == True

In[56]:= intersection[binclosed[intersection[composite[x_, FIRST], composite[y_, SECOND]]],
            subvar[x_], subvar[y_]] := fix[
            composite[IMAGE[intersection[composite[x, FIRST], composite[y, SECOND]]], CART, DUP]]
```

a special case

For the special case **x = y**, the results of the preceding sections simplify.

Theorem.

```
In[57]:= binclosed[intersection[composite[x, FIRST], composite[x, SECOND]]] // Normality

Out[57]= binclosed[intersection[composite[x, FIRST], composite[x, SECOND]]] == invar[x]

In[58]:= binclosed[intersection[composite[x_, FIRST], composite[x_, SECOND]]] := invar[x]
```

Theorem.

```
In[59]:= SubstTest[intersection,
           binclosed[intersection[composite[x, FIRST], composite[y, SECOND]]],
           subvar[x], subvar[y], y → x]

Out[59]= fix[composite[IMAGE[intersection[composite[x, FIRST], composite[x, SECOND]]],
              CART, DUP]] == fix[IMAGE[x]]

In[60]:= fix[composite[IMAGE[intersection[composite[x_, FIRST], composite[x_, SECOND]]],
              CART, DUP]] := fix[IMAGE[x]]
```

cliques

Theorem. Another lower bound.

```
In[61]:= Map[equal[V, #] &,
          dif[cliques[complement[composite[inverse[y], x]]], binclosed[intersection[
               composite[x, FIRST], composite[y, SECOND]]]] // complement // Normality]

Out[61]= subclass[cliques[complement[composite[inverse[y], x]]],
           binclosed[intersection[composite[x, FIRST], composite[y, SECOND]]]] == True
```

Corollary.

```
In[62]:= Map[equal[V, #] &,
          dif[cliques[complement[composite[inverse[x], y]]], binclosed[intersection[
               composite[x, FIRST], composite[y, SECOND]]]] // complement // Normality]

Out[62]= subclass[cliques[complement[composite[inverse[x], y]]],
           binclosed[intersection[composite[x, FIRST], composite[y, SECOND]]]] == True

In[63]:= subclass[cliques[complement[composite[inverse[x_], y_]]],
           binclosed[intersection[composite[x_, FIRST], composite[y_, SECOND]]]] := True
```

For the special case **x = y** this reduces to the inclusion **P[complement[domain[x]]] ⊂ invar[x]**.

```
In[81]:= List[cliques[complement[composite[inverse[x], y]]],
           binclosed[intersection[composite[x, FIRST], composite[y, SECOND]]]] /. y → x

Out[81]= {P[complement[domain[x]]], invar[x]}
```

restrictions

Theorem.

```
In[64]:= SubstTest[implies, subclass[t, x],
           subclass[binclosed[x], binclosed[t]], t → composite[id[y], x]] // Reverse

Out[64]= subclass[binclosed[x], binclosed[composite[id[y], x]]] == True

In[65]:= subclass[binclosed[x_], binclosed[composite[id[y_], x_]]] := True
```

Theorem.

```
In[66]:= Map[equal[V, #] &, complement[dif[image[inverse[IMAGE[id[x]]], binclosed[y]],
             binclosed[composite[y, id[cart[x, x]]]]]]] // Normality]

Out[66]= subclass[image[inverse[IMAGE[id[x]]], binclosed[y]],
           binclosed[composite[y, id[cart[x, x]]]]] == True

In[67]:= subclass[image[inverse[IMAGE[id[x_]]], binclosed[y_]],
           binclosed[composite[y_, id[cart[x_, x_]]]]] := True
```

Corollary.

```
In[68]:= SubstTest[implies, subclass[u, v], subclass[image[t, u], image[t, v]],
           {t → IMAGE[id[y]], u -> image[inverse[IMAGE[id[y]]], binclosed[x]],
            v -> binclosed[composite[x, id[cart[y, y]]]]}] // Reverse

Out[68]= subclass[intersection[binclosed[x], P[y]],
           image[IMAGE[id[y]], binclosed[composite[x, id[cart[y, y]]]]]] == True

In[69]:= subclass[intersection[binclosed[x_], P[y_]],
           image[IMAGE[id[y_]], binclosed[composite[x_, id[cart[y_, y_]]]]]] := True
```