cartesian products of ranges of subgroups

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```
In[1]:= SetDirectory["l:"]; << goedel.12jan24b

:Package Title: goedel.12jan24b                2012 January 24 at 4:20 p.m.

Loading takes about thirteen minutes, half that time due to builtin pauses.

It is now:  2012 Jan 25 at 15:23

Loading Simplification Rules

TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

weightlimit = 40

Loading completed.

It is now:  2012 Jan 25 at 15:36
```

summary

If **u** is the range of a subgroup of a group **x** and if **v** is the range of a subgroup of a group **y**, then **u** × **v** is the range of a subgroup of the direct product of **x** and **y**.

derivation

Lemma.

```
In[15]:= Map[not, SubstTest[and, implies[p1, p2],
            implies[p1, p3], implies[and[p1, p2, p3], p4], not[implies[p1, p4]],
            {p1 -> and[member[t, GROUPS], subclass[t, x], member[x, GROUPS], equal[w, range[t]]],
             p2 -> equal[domain[t], cart[w, w]], p3 → equal[t, composite[x, id[cart[w, w]]]],
             p4 → equal[image[x, cart[w, w]], w]}]] // Reverse

Out[15]= or[equal[w, image[x, cart[w, w]]], not[equal[w, range[t]]],
           not[member[t, GROUPS]], not[member[x, GROUPS]], not[subclass[t, x]]] == True

In[16]:= (% /. {t → t_, w → w_, x → x_}) /. Equal → SetDelayed
```

Theorem. If **w** is the range of a subgroup of a group **x**, then **image[x, w × w] = w**.

```
In[17]:= Map[equal[V, domain[#]] &,
          SubstTest[reify, t, case[or[equal[w, image[x, cart[w, w]]], not[equal[w, range[t]]],
             not[member[t, z]], not[member[x, z]], not[subclass[t, x]]]], z → GROUPS]]

Out[17]= or[equal[w, image[x, cart[w, w]]],
           not[member[w, image[IMAGE[SECOND], intersection[GROUPS, P[x]]]]],
           not[member[x, GROUPS]]] == True

In[20]:= or[equal[image[x_, cart[w_, w_]], w_],
           not[member[w_, image[IMAGE[SECOND], intersection[GROUPS, P[x_]]]]],
           not[member[x_, GROUPS]]] := True
```

Theorem.

```
In[33]:= SubstTest[implies, member[w, t],
           member[range[w], image[IMAGE[SECOND], t]], t → intersection[x, P[y]]] // Reverse

Out[33]= or[member[range[w], image[IMAGE[SECOND], intersection[x, P[y]]]],
           not[member[w, x]], not[subclass[w, y]]] == True

In[34]:= or[member[range[w_], image[IMAGE[SECOND], intersection[x_, P[y_]]]],
           not[member[w_, x_]], not[subclass[w_, y_]]] := True
```

Corollary.

```
In[39]:= SubstTest[implies, and[equal[t, range[w]], member[w, s], subclass[w, z]],
           member[t, image[IMAGE[SECOND], intersection[s, P[z]]]], {t → cart[u, v],
            w -> direct[composite[x, id[cart[u, u]]], composite[y, id[cart[v, v]]]],
            s → GROUPS, z → direct[x, y]}] // Reverse

Out[39]= or[member[cart[u, v],
            image[IMAGE[SECOND], intersection[GROUPS, P[composite[cross[x, y], TWIST]]]]],
           not[equal[cart[u, v], cart[image[x, cart[u, u]], image[y, cart[v, v]]]]],
           not[member[composite[cross[composite[x, id[cart[u, u]]],
               composite[y, id[cart[v, v]]]], TWIST], GROUPS]]] == True

In[40]:= (% /. {u → u_, v → v_, x → x_, y → y_}) /. Equal → SetDelayed
```

Lemma.

```
In[44]:= Map[not, SubstTest[and, (*implies[p1,p2],implies[p1,p3],*)
            implies[and[p2, p3], p4], not[implies[p1, p4]],
            {p1 → and[member[u, image[IMAGE[SECOND], intersection[GROUPS, P[x]]]],
               member[v, image[IMAGE[SECOND], intersection[GROUPS, P[y]]]], member[x, GROUPS],
               member[y, GROUPS]], p2 → member[composite[x, id[cart[u, u]]], GROUPS],
             p3 → member[composite[y, id[cart[v, v]]], GROUPS],
             p4 → member[direct[composite[x, id[cart[u, u]]],
                composite[y, id[cart[v, v]]]], GROUPS]}]] // Reverse

Out[44]= or[member[composite[cross[composite[x, id[cart[u, u]]], composite[y, id[cart[v, v]]]],
             TWIST], GROUPS], not[member[u, image[IMAGE[SECOND], intersection[GROUPS, P[x]]]]],
           not[member[v, image[IMAGE[SECOND], intersection[GROUPS, P[y]]]]],
           not[member[x, GROUPS]], not[member[y, GROUPS]]] == True
```

```
In[45]:= (% /. {u → u_, v → v_, x → x_, y → y_}) /. Equal → SetDelayed
```

Theorem. If **u** is the range of a subgroup of a group **x** and if **v** is the range of a subgroup of a group **y**, then **u** × **v** is the range of a subgroup of the direct product of **x** and **y**.

```
In[46]:= Map[not, SubstTest[and, (*implies[p1,p4],implies[p1,p5],implies[p1,p6],*)
           implies[and[p5, p6], p7], implies[and[p4, p7], p8], not[implies[p1, p8]],
           {p1 → and[member[u, image[IMAGE[SECOND], intersection[GROUPS, P[x]]]],
              member[v, image[IMAGE[SECOND], intersection[GROUPS, P[y]]]],
              member[x, GROUPS], member[y, GROUPS]],
            p4 → member[direct[composite[x, id[cart[u, u]]], composite[y, id[cart[v, v]]]],
              GROUPS], p5 → equal[u, image[x, cart[u, u]]],
            p6 -> equal[v, image[y, cart[v, v]]], p7 → equal[cart[u, v],
              range[direct[composite[x, id[cart[u, u]]], composite[y, id[cart[v, v]]]]]],
            p8 -> member[cart[u, v], image[IMAGE[SECOND],
               intersection[GROUPS, P[composite[cross[x, y], TWIST]]]]]}]] // Reverse

Out[46]= or[member[cart[u, v],
           image[IMAGE[SECOND], intersection[GROUPS, P[composite[cross[x, y], TWIST]]]]],
          not[member[u, image[IMAGE[SECOND], intersection[GROUPS, P[x]]]]],
          not[member[v, image[IMAGE[SECOND], intersection[GROUPS, P[y]]]]],
          not[member[x, GROUPS]], not[member[y, GROUPS]]] == True

In[48]:= or[member[cart[u_, v_],
           image[IMAGE[SECOND], intersection[GROUPS, P[composite[cross[x_, y_], TWIST]]]]],
          not[member[u_, image[IMAGE[SECOND], intersection[GROUPS, P[x_]]]]],
          not[member[v_, image[IMAGE[SECOND], intersection[GROUPS, P[y_]]]]],
          not[member[x_, GROUPS]], not[member[y_, GROUPS]]] := True
```