cliques of equivalence relations

Johan G. F. Belinfante
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```
In[1]:= SetDirectory["i:"]; << goedel62.01a; << tools.m

       :Package Title: goedel62.01a          2004 October 2 at 7:10 p.m.

       It is now:  2004 Oct 2 at 19:13

       Loading Simplification Rules

       TOOLS.M                    Revised 2004 September 25

       weightlimit = 40
```

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summary

A formula for **cliques[x]** is derived that yields as a corollary a certain weak factorization for equivalence relations discovered 2002 December 17 in the notebook **eqv-thin.nb**. The factorization formula holds for all equivalence relations, not just for thin ones. An elementary corollary is that an equivalence relation is uniquely determined by its cliques.

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derivation

Lemma.

```
In[2]:= composite[inverse[S], id[cliques[x]], S] // RelnNormality // Reverse

Out[2]= image[inverse[CUP], cliques[x]] == composite[inverse[S], id[cliques[x]], S]

In[3]:= image[inverse[CUP], cliques[x_]] := composite[inverse[S], id[cliques[x]], S]
```

The new formula is this:

```
In[4]:= IminComp[CUP, cross[SINGLETON, SINGLETON], cliques[x]] // Reverse

Out[4]= composite[inverse[E], id[cliques[x]], E] ==
          composite[id[fix[x]], intersection[x, inverse[x]], id[fix[x]]]
```

```
In[5]:= composite[inverse[E], id[cliques[x_]], E] :=
            composite[id[fix[x]], intersection[x, inverse[x]], id[fix[x]]]
```

Corollary: (weak factorization for any equivalence relation)

```
In[6]:= composite[inverse[E], id[cliques[eqv[x]]], E]
```

```
Out[6]= eqv[x]
```

---

variable-free formulas

The variable **x** in the factorization formula can be eliminated by reification. Note that **image[CART,Id]** is the class of all cartesian squares.

```
In[7]:= Map[VERTSECT, SubstTest[reify, x,
            composite[inverse[z], id[cliques[eqv[x]]], z], z → E]] // Reverse
```

```
Out[7]= composite[CORE[image[CART, Id]], EQUIV] == EQUIV
```

```
In[8]:= composite[CORE[image[CART, Id]], EQUIV] := EQUIV
```

Comparing the ranges, one finds that the class **EQV** of (small) equivalence relations is invariant under **CORE[image[CART,Id]]**.

```
In[9]:= ImageComp[CORE[image[CART, Id]], EQUIV, V] // Reverse
```

```
Out[9]= image[CORE[image[CART, Id]], EQV] == EQV
```

```
In[10]:= image[CORE[image[CART, Id]], EQV] := EQV
```

This result can be rewritten in various ways. A relation is reflexive and symmetric if and only if it can be written as a union of cartesian squares.

```
In[11]:= Uclosure[image[CART, Id]]
```

```
Out[11]= intersection[RFX, SYM]
```

Accordingly, the **CORE** function can also be rewritten as follows.

```
In[12]:= SubstTest[CORE, Uclosure[x], x → image[CART, Id]]
```

```
Out[12]= CORE[intersection[RFX, SYM]] == CORE[image[CART, Id]]
```

Since it is unclear how best to orient this result as a rewrite rule, we shall refrain from doing so at this time.

one-to-one property of CLIQUES

In this section a variable-free expression is derived for the fact that equivalence relations are determined by their cliques. The following function is needed for this derivation:

In[13]:= **lambda[x, composite[inverse[E], id[x], E]]**

Out[13]= composite[BIGCUP, IMAGE[CART], IMAGE[DUP]]

Note that:

In[14]:= **composite[BIGCUP, IMAGE[CART], IMAGE[DUP], CLIQUES, EQUIV]**

Out[14]= EQUIV

This can be understood as a corollary of the result in the preceding section, together with the following fact:

In[15]:= **composite[BIGCUP, IMAGE[CART], IMAGE[DUP], CLIQUES]**

Out[15]= CORE[image[CART, Id]]

The function **EQUIV** can be eliminated as follows:

In[17]:= **Assoc[CORE[image[CART, Id]], EQUIV, inverse[EQUIV]]**

Out[17]= composite[CORE[image[CART, Id]], id[EQV]] == id[EQV]

In[18]:= **composite[CORE[image[CART, Id]], id[EQV]] := id[EQV]**

It now follows that the restriction of **CLIQUES** to **EQV** is one-to-one:

In[19]:= **SubstTest[implies, subclass[composite[x, y], Id],**
    FUNCTION[composite[inverse[y], id[domain[x]]]],
   {x -> composite[BIGCUP, IMAGE[CART], IMAGE[DUP]],
    y → composite[CLIQUES, id[EQV]]}]

Out[19]= FUNCTION[composite[id[EQV], inverse[CLIQUES]]] == True

In[20]:= **FUNCTION[composite[id[EQV], inverse[CLIQUES]]] := True**

Restatement:

In[21]:= **ONEONE[composite[CLIQUES, id[EQV]]]**

Out[21]= True