domain and range of rational numbers

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```
In[1]:= SetDirectory["l:"]; << goedel.12jun03b

:Package Title: goedel.1jun03b                    2012 June 3 at 7:55 p.m.

Loading takes about seventeen minutes, half that time due to builtin pauses.

It is now:  2012 Jun 8 at 11:54

Loading Simplification Rules

TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

weightlimit = 40

Loading completed.

It is now:  2012 Jun 8 at 12:11
```

summary

In the **GOEDEL** program, rational numbers have been defined as certain functions whose graphs are straight lines through the origin in the integer plane **Z** × **Z**. The rational numbers themselves are ranges of subgroups of the direct product of the integer addition group **INTADD** with itself. Consequently, the the domain and range of any rational number are ranges of subgroups of **INTADD**.

ranges of rationals

The term **group** in the **GOEDEL** program refers to an associative composition law, and the underlying set on which this acts is its range. Accordingly the binary operation **INTADD** of integer addition is called a group, and the range of this group is the set **Z** of integers.

Lemma. If **x** is the range of a subgroup of the direct product of **INTADD** with itself, then **range[x]** is the range of a subgroup of the group **INTADD**.

```
In[2]:= SubstTest[implies, and[member[u, GROUPS], member[v, GROUPS],
          member[x, image[IMAGE[SECOND], intersection[GROUPS, P[direct[u, v]]]]]],
         member[range[x], image[IMAGE[SECOND], intersection[GROUPS, P[v]]]]],
         {u → INTADD, v → INTADD}] // Reverse

Out[2]= or[member[range[x], image[VERTSECT[INTDIV], Z]], not[member[x, image[IMAGE[SECOND],
            intersection[GROUPS, P[composite[cross[INTADD, INTADD], TWIST]]]]]]] == True
```

```
In[3]:= or[member[range[x_], image[VERTSECT[INTDIV], Z]], not[member[x_, image[IMAGE[SECOND],
          intersection[GROUPS, P[composite[cross[INTADD, INTADD], TWIST]]]]]]] := True
```

Every subgroup of integer addition is a cyclic subgroup. Thus the range of any subgroup of **INTADD** is the set of multiples of some integrer. These are just the nonempty vertical sections of the integer divisibility relation **INTDIV**.

Theorem. The range of a rational number is a nonempty vertical section of **INTDIV**.

```
In[4]:= Map[not, SubstTest[and, implies[p1, p2],
           implies[p2, p3], not[implies[p1, p3]], {p1 → member[x, RATS], p2 →
             member[x, image[IMAGE[SECOND], intersection[GROUPS, P[direct[INTADD, INTADD]]]]],
            p3 → member[range[x], image[VERTSECT[INTDIV], Z]]}]] // Reverse

Out[4]= or[member[range[x], image[VERTSECT[INTDIV], Z]], not[member[x, RATS]]] == True
```

```
In[5]:= or[member[range[x_], image[VERTSECT[INTDIV], Z]], not[member[x_, RATS]]] := True
```

Eliminating the variable **x** yields the following inclusion.

Corollary.

```
In[6]:= Map[equal[V, domain[#]] &,
         SubstTest[reify, x, case[implies[member[x, y], member[range[x], z]]],
          {y → RATS, z → image[VERTSECT[INTDIV], Z]}]]

Out[6]= subclass[image[IMAGE[SECOND], RATS], image[VERTSECT[INTDIV], Z]] == True
```

```
In[7]:= % /. Equal → SetDelayed
```

Lemma. The opposite inclusion also holds.

```
In[8]:= SubstTest[implies, subclass[u, v], subclass[image[t, u], image[t, v]],
          {t → IMAGE[SECOND], u → binhom[INTADD, INTADD], v → RATS}] // Reverse

Out[8]= subclass[image[VERTSECT[INTDIV], Z], image[IMAGE[SECOND], RATS]] == True
```

```
In[9]:= % /. Equal → SetDelayed
```

Combining the two inclusions yields an equation that can be made into a rewrite rule.

Theorem. A formula for the class of ranges of rational numbers.

```
In[10]:= SubstTest[and, subclass[u, v], subclass[v, u],
          {u → image[IMAGE[SECOND], RATS], v → image[VERTSECT[INTDIV], Z]}]

Out[10]= equal[image[IMAGE[SECOND], RATS], image[VERTSECT[INTDIV], Z]] == True
```

```
In[11]:= image[IMAGE[SECOND], RATS] := image[VERTSECT[INTDIV], Z]
```

domains of fractions

The domains of rationals, like their ranges, are also vertical sections of **INTDIV**, but with the extra wrinkle that one has to take into account that the denominator of a fraction can not be zero. The vertical straight line through the origin in the plane **Z** × **Z** is not a function, and thus not a rational number. Thus the domain of a rational number can not the range of the trivial one-element subgroup of **INTADD**.

Lemma. If **x** is the range of a subgroup of the direct product of **INTADD** with itself, then **domain[x]** is the range of a subgroup of the group **INTADD**.

```
In[12]:= SubstTest[implies, and[member[u, GROUPS], member[v, GROUPS],
            member[x, image[IMAGE[SECOND], intersection[GROUPS, P[direct[u, v]]]]]],
           member[domain[x], image[IMAGE[SECOND], intersection[GROUPS, P[u]]]]],
           {u → INTADD, v → INTADD}] // Reverse

Out[12]= or[member[domain[x], image[VERTSECT[INTDIV], Z]], not[member[x, image[IMAGE[SECOND],
              intersection[GROUPS, P[composite[cross[INTADD, INTADD], TWIST]]]]]]]] == True

In[13]:= or[member[domain[x_], image[VERTSECT[INTDIV], Z]], not[member[x_, image[IMAGE[SECOND],
              intersection[GROUPS, P[composite[cross[INTADD, INTADD], TWIST]]]]]]]] := True
```

Theorem. The domain of a rational number is a nonempty vertical section of **INTADD**.

```
In[14]:= Map[not, SubstTest[and, implies[p1, p2],
            implies[p2, p3], not[implies[p1, p3]], {p1 → member[x, RATS], p2 →
              member[x, image[IMAGE[SECOND], intersection[GROUPS, P[direct[INTADD, INTADD]]]]]],
             p3 → member[domain[x], image[VERTSECT[INTDIV], Z]]}]] // Reverse

Out[14]= or[member[domain[x], image[VERTSECT[INTDIV], Z]], not[member[x, RATS]]] == True

In[15]:= or[member[domain[x_], image[VERTSECT[INTDIV], Z]], not[member[x_, RATS]]] := True
```

Corollary. Eliminating the variable **x** yields an inclusion.

```
In[16]:= Map[equal[V, domain[#]] &,
          SubstTest[reify, x, case[implies[member[x, y], member[domain[x], z]]],
           {y → RATS, z → image[VERTSECT[INTDIV], Z]}]]

Out[16]= subclass[image[IMAGE[FIRST], RATS], image[VERTSECT[INTDIV], Z]] == True

In[17]:= % /. Equal → SetDelayed
```

Theorem. If **x** is a non-zero integer, then **inverse[inttimes[x]]** is a fraction.

```
In[18]:= SubstTest[implies, and[member[x, Z], member[y, Z], not[equal[x, id[omega]]]], member[
            composite[inverse[inttimes[x]], inttimes[y]], RATS], y → plus[set[0]]] // Reverse

Out[18]= or[equal[x, id[omega]], member[inverse[inttimes[x]], RATS], not[member[x, Z]]] == True
```

```
In[19]:= or[equal[x_, id[omega]],
           member[inverse[inttimes[x_]], RATS], not[member[x_, Z]]] := True
```

Theorem. One of the nonempty vertical sections of **INTDIV** is the singleton of the integer zero (= **id[ω]**).

```
In[20]:= SubstTest[implies, subclass[u, v],
           subclass[image[t, u], image[t, v]], {t → IMAGE[SECOND],
            u → set[cart[Z, set[id[omega]]]], v -> binhom[INTADD, INTADD]}] // Reverse

Out[20]= member[set[id[omega]], image[VERTSECT[INTDIV], Z]] == True

In[21]:= member[set[id[omega]], image[VERTSECT[INTDIV], Z]] := True
```

Lemma. (If **t** is a member of the singleton **x**, then **x = {t}**.)

```
In[22]:= SubstTest[implies, and[member[t, x], member[x, range[SINGLETON]]],
           equal[x, set[t]], t → pair[id[omega], id[omega]]] // Reverse

Out[22]= or[equal[x, cart[set[id[omega]], set[id[omega]]]],
           not[member[x, range[SINGLETON]]], not[member[pair[id[omega], id[omega]], x]]] == True

In[23]:= (% /. x → x_) /. Equal → SetDelayed
```

Theorem. There is no rational number with whose domain is **{id[ω]}**.

```
In[24]:= Map[not, SubstTest[and, p1, p2, (*implies[p1,p3],implies[p1,p4],implies[p2,p5],*)
            implies[and[p3, p5], p6], implies[and[p4, p6], p7], implies[p1, not[p7]],
            {p1 → member[x, RATS], p2 → equal[domain[x], set[id[omega]]],
             p3 → FUNCTION[x], p4 → subclass[cartsq[set[id[omega]]], x],
             p5 → member[domain[x], range[SINGLETON]], p6 → member[x, range[SINGLETON]],
             p7 -> equal[cartsq[set[id[omega]]], x]}]] // Reverse

Out[24]= or[not[equal[domain[x], set[id[omega]]]], not[member[x, RATS]]] == True

In[25]:= or[not[equal[domain[x_], set[id[omega]]]], not[member[x_, RATS]]] := True
```

Corollary.

```
In[26]:= Map[not[equal[V, domain[#]]] &,
          SubstTest[reify, x, case[or[not[equal[domain[x], t]], not[member[x, y]]]],
           {t → set[id[omega]], y → RATS}]]

Out[26]= member[set[id[omega]], image[IMAGE[FIRST], RATS]] == False

In[27]:= % /. Equal → SetDelayed
```

Lemma.

```
In[28]:= SubstTest[implies, member[x, y], member[domain[x], image[IMAGE[FIRST], y]],
           {x → inverse[inttimes[u]], y → RATS}] // Reverse

Out[28]= or[member[image[INTDIV, set[u]], image[IMAGE[FIRST], RATS]],
           not[member[inverse[inttimes[u]], RATS]]] == True
```

```
In[29]:= (% /. u → u_) /. Equal → SetDelayed
```

Theorem. Every other vertical section of **INTDIV** is the domain of some rational number.

```
In[30]:= Map[not, SubstTest[and, implies[p1, p2], implies[p2, p3],
            not[implies[p1, p3]], {p1 → and[member[u, Z], not[equal[u, id[omega]]]],
             p2 -> member[inverse[inttimes[u]], RATS],
             p3 -> member[image[INTDIV, set[u]], image[IMAGE[FIRST], RATS]]}]] // Reverse

Out[30]= or[equal[u, id[omega]],
          member[image[INTDIV, set[u]], image[IMAGE[FIRST], RATS]], not[member[u, Z]]] == True

In[31]:= (% /. u → u_) /. Equal → SetDelayed
```

Lemma.

```
In[32]:= Map[equal[domain[#], V] &,
          SubstTest[reify, u, case[or[member[image[INTDIV, set[u]], image[IMAGE[FIRST], RATS]],
             not[member[u, v]]]], {v → dif[Z, set[id[omega]]]}]]

Out[32]= subclass[Z, union[
            image[inverse[VERTSECT[INTDIV]], image[IMAGE[FIRST], RATS]], set[id[omega]]]] == True

In[33]:= % /. Equal → SetDelayed
```

Corollary.

```
In[34]:= SubstTest[implies, subclass[u, v], subclass[image[t, u], image[t, v]],
           {t -> VERTSECT[INTDIV], u → Z, v -> union[image[inverse[VERTSECT[INTDIV]],
               image[IMAGE[FIRST], RATS]], set[id[omega]]]}] // Reverse

Out[34]= subclass[image[VERTSECT[INTDIV], Z],
          union[image[IMAGE[FIRST], RATS], set[set[id[omega]]]]] == True

In[35]:= % /. Equal → SetDelayed
```

Theorem. A formula for the set of domains of rational numbers.

```
In[36]:= SubstTest[and, subclass[u, v], subclass[v, u], {u -> image[IMAGE[FIRST], RATS],
          v -> dif[image[VERTSECT[INTDIV], Z], set[set[id[omega]]]]}]

Out[36]= equal[image[IMAGE[FIRST], RATS],
          intersection[complement[set[set[id[omega]]]], image[VERTSECT[INTDIV], Z]]] == True

In[37]:= image[IMAGE[FIRST], RATS] :=
          intersection[complement[set[set[id[omega]]]], image[VERTSECT[INTDIV], Z]]
```