ranges of canonical projections of equivalences

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2006 October 19

```
In[1]:= SetDirectory["l:"]; << goedel86.19b; << tools.m

        :Package Title: goedel86.19b          2006 October 19 at 1:20 p.m.

        It is now:  2006 Oct 19 at 16:3

        Loading Simplification Rules

        TOOLS.M                       Revised 2006 October 12

        weightlimit = 40
```

summary

In this notebook it is shown that a set is the set of equivalence classes for some equivalence relation if and only if it is pairwise disjoint and does not hold the empty set. Since the set of equivalence classes of a (small) equivalence relation is the range of its canonical projection, one can formulate this result as a rewrite rule for the class **image[IMAGE[SECOND], image[VS, EQV]]** of all ranges of canonical projections of (small) equivalence relations.

derivation

Lemma. A formula for the class of all canonical projections of equivalences.

```
In[2]:= ImageComp[IMAGE[composite[id[E], inverse[SECOND]]],
          inverse[IMAGE[composite[id[E], inverse[SECOND]]]], FUNS] // Reverse

Out[2]= image[IMAGE[composite[id[E], inverse[SECOND]]], cliques[union[DISJOINT, Id]]] ==
         image[VS, EQV]

In[3]:= image[IMAGE[composite[id[E], inverse[SECOND]]],
          cliques[union[DISJOINT, Id]]] := image[VS, EQV]
```

Theorem. A set is the set of equivalence classes of an equivalence relation if it is pairwise disjoint and does not hold the empty set.

```
In[5]:= ImageComp[IMAGE[SECOND], IMAGE[composite[id[E], inverse[SECOND]]],
          cliques[union[DISJOINT, Id]]] // Reverse

Out[5]= image[IMAGE[SECOND], image[VS, EQV]] ==
         intersection[cliques[union[DISJOINT, Id]], P[complement[set[0]]]]
```

```
In[6]:= image[IMAGE[SECOND], image[VS, EQV]] :=
         intersection[cliques[union[DISJOINT, Id]], P[complement[set[0]]]]
```

Theorem. A formula recovering the class of equivalence relations from the class of their canonical projections.

```
In[7]:= ImageComp[IMAGE[cross[Id, inverse[E]]],
          IMAGE[composite[id[E], inverse[SECOND]]], cliques[union[DISJOINT, Id]]] // Reverse

Out[7]= image[IMAGE[cross[Id, inverse[E]]], image[VS, EQV]] == EQV

In[8]:= image[IMAGE[cross[Id, inverse[E]]], image[VS, EQV]] := EQV
```

serendipity: some general results

The empty set does not belong to the range of the canonical projection of an equivalence relation. More generally, the empty set does not belong to the range of **APPLY[VS, x]** for any set **x**.

```
In[9]:= Map[not, SubstTest[implies, subclass[u, v], subclass[image[w, u], image[w, v]],
          {u → x, v → V, w -> composite[inverse[E], IMAGE[SECOND], VS]}]]

Out[9]= member[0, range[U[image[VS, x]]]] == False

In[10]:= member[0, range[U[image[VS, x_]]]] := False
```

The following general formula was used in an earlier (and more clumsy) derivation of the results in this notebook.

```
In[11]:= ImageComp[IMAGE[SECOND], IMAGE[composite[id[E], inverse[SECOND]]], x] // Reverse

Out[11]= image[IMAGE[SECOND], image[IMAGE[composite[id[E], inverse[SECOND]]], x]] ==
          image[IMAGE[id[complement[set[0]]]], x]

In[12]:= image[IMAGE[SECOND], image[IMAGE[composite[id[E], inverse[SECOND]]], x_]] :=
          image[IMAGE[id[complement[set[0]]]], x]
```