Theorem FIN−DJ3

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```
In[1]:= << goedel53.23a; << tools.m

        :Package Title: goedel53.23a      2004 January 23 at 8:40 p.m.

        It is now:  2004 Jan 26 at 15:34

        Loading Simplification Rules

        TOOLS.M                      Revised 2004 January 3

        weightlimit = 40
```

summary

In this notebook, Theorem **FIN−DJ3**, proved 2000 March 16 using **Otter**, is rederived. This theorem says that no finite set can belong to a class that is subvariant under the proper subset relation **PS**.

a comment about lemma FIN−DJ1

By definition, a set is finite if it does not belong to any set that is subvariant under **PS**. Lemma **FIN−DJ1**, which says that no finite set can belong to a set **x** that is subvariant under **PS**, follows immediately from this definition:

```
In[2]:= SubstTest[implies, and[member[x, y], subclass[y, z]], member[x, z],
            {y -> subvar[PS], z -> P[complement[FINITE]]}]

Out[2]= or[equal[0, intersection[FINITE, x]],
          not[member[x, V]], not[subclass[x, image[PS, x]]]] == True
```

The more general result that will be derived below is similar, but leaves out the hypothesis that **x** be a set. The derivation of the more general result will make no use of this lemma.

two lemmas

The derivation of Theorem **FIN−DJ3** makes use of Lemma **FIN−PS3**, obtained by introducing variables into Theorem **FIN−PS2** , just as was done in **Otter**'s proof:

```
In[3]:= Map[implies[#, equal[0, x]] &,
          SubstTest[member, x, intersection[y, z], {y -> subvar[PS], z -> FINITE}]] // Reverse

Out[3]= or[equal[0, x], not[member[x, FINITE]], not[subclass[x, image[PS, x]]]] == True

In[4]:= or[equal[0, x_], not[member[x_, FINITE]], not[subclass[x_, image[PS, x_]]]] := True
```

A second lemma is needed:

```
In[5]:= SubstTest[implies, and[subclass[u, v], member[v, FINITE]], member[u, FINITE],
            {u -> intersection[x, P[y]], v -> P[y]}]

Out[5]= or[member[intersection[x, P[y]], FINITE], not[member[y, FINITE]]] == True

In[6]:= or[member[intersection[x_, P[y_]], FINITE], not[member[y_, FINITE]]] := True
```

derivation of Theorem FIN−DJ3

Theorem **FIN−DJ3** now follows:

```
In[7]:= Map[equal[V, class[y, not[and[member[y, x], #]]]] &,
         SubstTest[and, implies[p1, p4], implies[p2, p3], implies[and[p3, p4], p5],
                not[implies[and[p1, p2], p5]], {p1 -> member[y, FINITE],
           p2 -> subclass[x, image[PS, x]], p3 -> subvariant[PS, intersection[x, P[y]]],
                p4 -> member[intersection[x, P[y]], FINITE],
           p5 -> equal[0, intersection[x, P[y]]]}]]

Out[7]= or[equal[0, intersection[FINITE, x]], not[subclass[x, image[PS, x]]]] == True

In[8]:= or[equal[0, intersection[FINITE, x_]], not[subclass[x_, image[PS, x_]]]] := True
```

Restatement:

```
In[9]:= implies[subvariant[PS, x], disjoint[FINITE, x]]

Out[9]= True
```

comments on a more general result

The same argument that was used to derive Theorem **FIN−DJ3** in the preceding section can be used to derive a more general result, obtained by simply omitting the unneeded hypothesis **member[y,x]** :

```
In[10]:= Map[equal[V, class[y, not[#]]] &,
          SubstTest[and, implies[p2, p5], implies[p3, p4], implies[and[p4, p5], p6],
                 not[implies[and[p2, p3], p6]], {p2 -> member[y, FINITE],
            p3 -> subclass[x, image[PS, x]], p4 -> subvariant[PS, intersection[x, P[y]]],
                 p5 -> member[intersection[x, P[y]], FINITE],
            p6 -> equal[0, intersection[x, P[y]]]}]]

Out[10]= or[equal[0, intersection[FINITE, image[S, x]]], not[subclass[x, image[PS, x]]]] == True
```

This more general result says that no finite set can contain a member of a class that is subvariant under **PS**. This is true because any subset of a finite set is finite. To show this, two lemmas are required:

```
In[11]:= SubstTest[implies, subclass[x, y], subclass[image[z, x], image[z, y]],
            {y -> complement[FINITE], z -> S}]

Out[11]= or[equal[0, intersection[FINITE, image[S, x]]],
           not[equal[0, intersection[FINITE, x]]]] == True

In[12]:= (% /. x -> x_) /. Equal -> SetDelayed
```

The following goes in the opposite direction:

```
In[13]:= SubstTest[implies, and[subclass[x, y], subclass[y, z]], subclass[x, z],
           {y -> image[S, x], z -> complement[FINITE]}]

Out[13]= or[equal[0, intersection[FINITE, x]],
           not[equal[0, intersection[FINITE, image[S, x]]]]] == True

In[14]:= (% /. x -> x_) /. Equal -> SetDelayed
```

These two lemmas can be combined:

```
In[15]:= equiv[equal[0, intersection[FINITE, image[S, x]]], equal[0, intersection[FINITE, x]]]

Out[15]= True
```

This justifies the following general rewrite rule that has the effect of rewriting the more general theorem to Theorem **FIN−DJ3**.

```
In[16]:= equal[0, intersection[FINITE, image[S, x_]]] := equal[0, intersection[FINITE, x]]
```