finite full sets

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```
In[1]:= SetDirectory["l:"]; << goedel.10aug02b;<< tools.m

        :Package Title: goedel.10aug02b             2010 August 2 at 3:20 p.m.

        It is now:  2010 Aug 3 at 15:12

        Loading Simplification Rules

        TOOLS.M                                    Revised 2010 July 26

        weightlimit = 40
```

summary

A great deal is known about the class **FULL** of sets satisfying **U[x] ⊂ x**. This class holds all the ordinal numbers. Less is known about the subclass **fix[BIGCUP] ⊂ FULL**. This class of course holds all the limit ordinals, but what else?

Are there any regular finite sets satisfying **U[x] = x** other than the empty set? The **GOEDEL** program can search for finite rank examples using **ens**. No other finite rank examples are forthcoming...

```
In[2]:= Select[Range[0, 100], equal[ens[#], U[ens[#]]] &]

Out[2]= {0}
```

A theorem that explains this is derived in this notebook.

finite full sets

In this section it is shown that if a set is finite, full and regular, then it has finite rank.

Theorem. If **x** is finite and full, then **tc[x] ⊂ FINITE**.

```
In[3]:= Map[not, SubstTest[and, implies[and[p1, p2], p3], implies[and[p2, p3], p4],
           not[implies[and[p1, p2], p4]], {p1 → member[x, FINITE], p2 → full[x],
            p3 → subclass[x, FINITE], p4 → subclass[tc[x], FINITE]}]] // Reverse

Out[3]= or[not[member[x, FINITE]], not[subclass[U[x], x]], subclass[tc[x], FINITE]] == True

In[4]:= or[not[member[x_, FINITE]], not[subclass[U[x_], x_]], subclass[tc[x_], FINITE]] := True
```

Theorem. A characterization of finite rank sets.

```
In[6]:= SubstTest[member, x, image[inverse[RANK], w], w → omega] // Reverse

Out[6]= and[member[x, FINITE], member[x, REGULAR], subclass[tc[x], FINITE]] ==
         member[rank[x], omega]

In[7]:= and[member[x_, FINITE], member[x_, REGULAR], subclass[tc[x_], FINITE]] :=
         member[rank[x], omega]
```

Corollary.

```
In[8]:= or[member[rank[x], omega], not[member[x, FINITE]],
          not[member[x, REGULAR]], not[subclass[tc[x], FINITE]]] // NotNotTest

Out[8]= or[member[rank[x], omega], not[member[x, FINITE]],
          not[member[x, REGULAR]], not[subclass[tc[x], FINITE]]] == True

In[9]:= or[member[rank[x_], omega], not[member[x_, FINITE]],
          not[member[x_, REGULAR]], not[subclass[tc[x_], FINITE]]] := True
```

Theorem. Any regular, full, finite set has finite rank.

```
In[10]:= Map[not, SubstTest[and, implies[and[p1, p2], p3], implies[and[p0, p1, p3], p4],
            not[implies[and[p0, p1, p2], p4]], {p0 → member[x, REGULAR], p1 → member[x, FINITE],
             p2 → full[x], p3 → subclass[tc[x], FINITE], p4 → member[rank[x], omega]}]] // Reverse

Out[10]= or[member[rank[x], omega], not[member[x, FINITE]],
           not[member[x, REGULAR]], not[subclass[U[x], x]]] == True

In[11]:= or[member[rank[x_], omega], not[member[x_, FINITE]],
           not[member[x_, REGULAR]], not[subclass[U[x_], x_]]] := True
```

Corollary.

```
In[12]:= SubstTest[subclass, intersection[FULL, FINITE, REGULAR],
          image[inverse[RANK], t], t → omega]

Out[12]= subclass[image[RANK, intersection[FINITE, FULL]], omega] == True

In[13]:= subclass[image[RANK, intersection[FINITE, FULL]], omega] := True
```

---

finite rank sets satisfying U[x] = x

In this section the stronger condition **U[x] = x** is considered.

Lemma.

```
In[14]:= SubstTest[implies, and[member[t, omega], equal[t, U[t]]],
           equal[t, 0], t → rank[x]] // Reverse

Out[14]= or[equal[0, x], not[equal[rank[x], U[rank[x]]]], not[member[rank[x], omega]]] == True
```

```
In[15]:= (% /. x → x_) /. Equal → SetDelayed
```

Theorem.

```
In[16]:= SubstTest[implies, equal[x, y], equal[rank[x], rank[y]], y → U[x]] // Reverse

Out[16]= or[equal[rank[x], U[rank[x]]], not[equal[x, U[x]]]] == True

In[17]:= or[equal[rank[x_], U[rank[x_]]], not[equal[x_, U[x_]]]] := True
```

Corollary. The class **fix[BIGCUP]** is invariant under the function **RANK**.

```
In[18]:= Map[equal[V, #] &, SubstTest[class, x, implies[member[x, u], member[x, v]],
           {u -> intersection[REGULAR, fix[BIGCUP]], v → image[inverse[RANK], fix[BIGCUP]]}]]

Out[18]= subclass[image[RANK, fix[BIGCUP]], fix[BIGCUP]] == True

In[19]:= subclass[image[RANK, fix[BIGCUP]], fix[BIGCUP]] := True
```

Theorem. The empty set is the only finite rank set satisfying **U[x] = x**.

```
In[22]:= Map[not, SubstTest[and, implies[and[p1, p3], p4], implies[p2, p3],
            not[implies[and[p1, p2], p4]], {p1 → member[rank[x], omega],
             p2 → equal[x, U[x]], p3 → equal[rank[x], U[rank[x]]], p4 → empty[x]}]] // Reverse

Out[22]= or[equal[0, x], not[equal[x, U[x]]], not[member[rank[x], omega]]] == True

In[23]:= or[equal[0, x_], not[equal[x_, U[x_]]], not[member[rank[x_], omega]]] := True
```

A variable-free restatement of this will now be derived.

Lemma.

```
In[24]:= and[equal[x, U[x]], member[rank[x], omega], not[equal[0, x]]] // NotNotTest

Out[24]= and[equal[x, U[x]], member[rank[x], omega], not[equal[0, x]]] == False

In[25]:= (% /. x → x_) /. Equal → SetDelayed
```

Lemma.

```
In[28]:= Map[empty, SubstTest[class, x, member[x, t],
          t -> symdif[intersection[H[FINITE], REGULAR, fix[BIGCUP]], set[0]]]]

Out[28]= subclass[intersection[REGULAR, fix[BIGCUP], H[FINITE]], set[0]] == True

In[29]:= % /. Equal → SetDelayed
```

Theorem.

```
In[30]:= equal[intersection[REGULAR, fix[BIGCUP], H[FINITE]], set[0]]

Out[30]= True
```

```
In[31]:= intersection[REGULAR, fix[BIGCUP], H[FINITE]] := set[0]
```

Lemma.

```
In[41]:= SubstTest[implies, and[subclass[u, v], subclass[v, w]],
           subclass[u, w], {u -> intersection[FINITE, REGULAR, fix[BIGCUP]],
            v -> intersection[FINITE, REGULAR, FULL], w → H[FINITE]}] // Reverse

Out[41]= subclass[intersection[FINITE, REGULAR, fix[BIGCUP]], H[FINITE]] == True

In[42]:= (% /. Equal → SetDelayed)
```

Lemma.

```
In[44]:= SubstTest[subclass, t, intersection[u, v],
           {t -> intersection[FINITE, REGULAR, fix[BIGCUP]], u → H[FINITE],
            v -> intersection[REGULAR, fix[BIGCUP]]}] // Reverse

Out[44]= subclass[intersection[FINITE, REGULAR, fix[BIGCUP]], set[0]] == True

In[45]:= (% /. Equal → SetDelayed)
```

Theorem. A better rewrite rule.

```
In[46]:= equal[intersection[FINITE, REGULAR, fix[BIGCUP]], set[0]]

Out[46]= True

In[48]:= intersection[FINITE, REGULAR, fix[BIGCUP]] := set[0]
```

Corollary. The only finite, regular set satisfying **U[x] = x** is the empty set.

```
In[50]:= Map[implies[#, empty[x]] &,
          SubstTest[member, x, intersection[u, v, w], {u → FINITE, v → REGULAR, w → fix[BIGCUP]}]]

Out[50]= or[equal[0, x], not[equal[x, U[x]]],
           not[member[x, FINITE]], not[member[x, REGULAR]]] == True

In[51]:= or[equal[0, x_], not[equal[x_, U[x_]]],
           not[member[x_, FINITE]], not[member[x_, REGULAR]]] := True
```