maximal elements of finite sets

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```
In[1]:= SetDirectory["l:"]; << goedel.09apr28a;<< tools.m

:Package Title: goedel.09apr28a            2009 April 28 at 2:30 p.m.

It is now:  2009 Apr 29 at 9:27

Loading Simplification Rules

TOOLS.M                                    Revised 2009 April 6

weightlimit = 40
```

summary

Any non-empty finite set has a maximal element. In this notebook a stronger result is derived which says that any element of a finite set is a subset of a maximal element.

derivation

Removing the **fin** wrapper from an existing rewrite rule yields this weak result:

Theorem. Any non-empty finite set has a maximal element.

```
In[2]:= SubstTest[implies, equal[x, fin[t]],
          or[empty[x], not[subclass[x, image[inverse[PS], x]]]], t → x] // Reverse

Out[2]= or[equal[0, x], not[member[x, FINITE]], not[subclass[x, image[inverse[PS], x]]]] == True

In[3]:= or[equal[0, x_], not[member[x_, FINITE]],
          not[subclass[x_, image[inverse[PS], x_]]]] := True
```

To show that any element **t** of a finite set **fin[x]** is a subclass of a maximal element, one can begin by applying this weak result to the set **intersection[fin[x], image[S, set[t]]]**.

Lemma.

```
In[4]:= Map[implies[member[t, fin[x]], #] &, SubstTest[or, equal[0, y],
            not[member[y, FINITE]], not[subclass[y, image[inverse[PS], y]]],
            y → intersection[fin[x], image[S, set[t]]]]] // Reverse // MapNotNot

Out[4]= or[not[member[t, fin[x]]], not[subclass[intersection[fin[x], image[S, set[t]]],
            image[inverse[PS], intersection[fin[x], image[S, set[t]]]]]]] == True

In[5]:= (% /. {t → t_, x → x_}) /. Equal → SetDelayed
```

Lemma. A normalization rule for **MAXIMAL[S]**.

```
In[6]:= MAXIMAL[S] // RelnNormality // Reverse

Out[6]= intersection[composite[complement[inverse[E]], IMAGE[inverse[PS]]], inverse[E]] ==
         MAXIMAL[S]

In[7]:= intersection[composite[complement[inverse[E]], IMAGE[inverse[PS]]], inverse[E]] :=
         MAXIMAL[S]
```

Lemma.

```
In[8]:= SubstTest[implies, and[subclass[u, v], subclass[v, w]], subclass[u, w],
          {u -> intersection[x, image[S, set[y]]],
           v -> intersection[image[S, set[y]], image[inverse[PS], x]],
           w -> image[inverse[PS], intersection[x, image[S, set[y]]]]}] // Reverse

Out[8]= or[not[subclass[intersection[x, image[S, set[y]]], image[inverse[PS], x]]],
          subclass[intersection[x, image[S, set[y]]],
           image[inverse[PS], intersection[x, image[S, set[y]]]]]] == True

In[9]:= (% /. {x → x_, y → y_}) /. Equal → SetDelayed
```

Lemma.

```
In[10]:= Map[not, SubstTest[and, implies[p1, p2], implies[p2, p3], not[implies[p1, p3]],
            {p1 → member[y, fin[x]], p2 → not[subclass[intersection[fin[x], image[S, set[y]]],
                image[inverse[PS], intersection[fin[x], image[S, set[y]]]]]],
             p3 → not[subclass[intersection[fin[x], image[S, set[y]]],
                image[inverse[PS], fin[x]]]]}]] // Reverse

Out[10]= or[not[member[y, fin[x]]], not[subclass[
             intersection[fin[x], image[S, set[y]]], image[inverse[PS], fin[x]]]]] == True

In[11]:= (% /. {x → x_, y → y_}) /. Equal → SetDelayed
```

Eliminating the variable **y** yields the main theorem.

Theorem. Every member of the finite set **fin[x]** is a subset of a maximal member of **fin[x]**.

```
In[12]:= Map[equal[V, #] &, SubstTest[class, y, or[not[member[y, t]], not[
             subclass[intersection[t, image[S, set[y]]], image[inverse[PS], t]]]], t → fin[x]]]

Out[12]= subclass[fin[x], image[inverse[S],
           intersection[complement[image[inverse[PS], fin[x]]], fin[x]]]] == True

In[13]:= subclass[fin[x_], image[inverse[S],
           intersection[complement[image[inverse[PS], fin[x_]]], fin[x_]]]] := True
```

Eliminating the **fin** wrapper yields this:

Corollary.

```
In[14]:= SubstTest[implies, equal[x, fin[t]],
          subclass[x, image[inverse[S], intersection[complement[image[inverse[PS], x]], x]]],
          t → x] // Reverse

Out[14]= or[not[member[x, FINITE]], subclass[x,
           image[inverse[S], intersection[x, complement[image[inverse[PS], x]]]]]] == True

In[15]:= or[not[member[x_, FINITE]], subclass[x_,
           image[inverse[S], intersection[x_, complement[image[inverse[PS], x_]]]]]] := True
```

A variable-free version of the main theorem is obtained by eliminating the variable **x**.

Theorem. Every member of a finite set is a subset of a maximal member.

```
In[16]:= Map[equal[V, #] &, SubstTest[class, x, implies[member[x, u], not[member[x, v]]],
          {u → FINITE, v → fix[composite[E, complement[composite[inverse[S], MAXIMAL[S]]]]]}]]

Out[16]= subclass[composite[inverse[E], id[FINITE]], composite[inverse[S], MAXIMAL[S]]] == True

In[17]:= subclass[composite[inverse[E], id[FINITE]], composite[inverse[S], MAXIMAL[S]]] := True
```

This statement can be considered to be a finite analog of a strong version of Zorn's lemma. If the axiom of choice holds, and if **x** is a set for which any chain has an upper bound, then any member of **x** is a subset of a maximal member. The following variable-free statement of that theorem was derived in the notebook **zorn-max.nb** dated 2007 November 23.

```
In[18]:= subclass[
          composite[inverse[E], id[fix[composite[inverse[IMAGE[inverse[S]]], S, UCHAINS]]]],
          composite[inverse[S], MAXIMAL[S]]]

Out[18]= axch
```

---

comments

The theorem derived in the present notebook of course does not depend on the axiom of choice being true. But if one does assume the axiom of choice, then the theorem derived in this notebook follows from the strong version of Zorn's lemma.

Theorem.   Any nonempty finite set satisfies the chain condition.

```
In[25]:= SubstTest[implies, equal[x, fin[t]],
           member[x, union[fix[composite[inverse[IMAGE[inverse[S]]], S, UCHAINS]], set[0]]],
           t → x] // Reverse

Out[25]= or[equal[0, x], not[member[x, FINITE]],
           subclass[Uchains[x], image[inverse[S], x]]] == True

In[26]:= or[equal[0, x_], not[member[x_, FINITE]],
           subclass[Uchains[x_], image[inverse[S], x_]]] := True
```

Corollary. Variable-free formulation of the statement that nonempty finite sets satisfy the chain condition.

```
In[28]:= Map[equal[V, #] &, SubstTest[class, x, implies[member[x, u], member[x, v]], {u → FINITE,
            v -> union[set[0], fix[composite[inverse[IMAGE[inverse[S]]], S, UCHAINS]]]}]]

Out[28]= subclass[FINITE,
           union[fix[composite[inverse[IMAGE[inverse[S]]], S, UCHAINS]], set[0]]] == True

In[29]:= subclass[FINITE,
           union[fix[composite[inverse[IMAGE[inverse[S]]], S, UCHAINS]], set[0]]] := True
```