FINITE ⋂ PO ⊂ image[inverse[S], TO]

Johan G. F. Belinfante
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```
In[1]:= SetDirectory["l:"]; << goedel.09may01a; << tools.m

:Package Title: goedel.09may01a            2009 May 1 at 5:45 a.m.

It is now:  2009 May 1 at 6:2

Loading Simplification Rules

TOOLS.M                                    Revised 2009 April 6

weightlimit = 40
```

introduction

Any finite partial order is a subset of a total order.

derivation

Lemma.

```
In[2]:= SubstTest[implies, subclass[u, v], subclass[image[t, u], image[t, v]],
          {t → composite[MAXIMAL[S], IMAGE[id[PO]], POWER, CART, DUP],
           u → set[fin[x]], v → V}] // Reverse

Out[2]= subclass[intersection[PO, P[cart[fin[x], fin[x]]]],
          union[image[inverse[PS], intersection[PO, P[cart[fin[x], fin[x]]]]],
           image[MAXIMAL[S], image[IMAGE[id[PO]], image[POWER, image[CART, Id]]]]]] == True

In[3]:= (% /. x → x_) /. Equal → SetDelayed
```

Theorem. A maximal partial order contained in a finite cartesian square is a total order.

```
In[4]:= SubstTest[implies, and[subclass[u, v], subclass[v, w]],
          subclass[u, w], {u → intersection[PO, P[cart[fin[x], fin[x]]],
             complement[image[inverse[PS], intersection[PO, P[cart[fin[x], fin[x]]]]]]],
           v -> image[MAXIMAL[S], image[IMAGE[id[PO]], image[POWER, image[CART, Id]]]],
           w → TO}] // Reverse

Out[4]= subclass[intersection[PO, P[cart[fin[x], fin[x]]]],
          union[TO, image[inverse[PS], intersection[PO, P[cart[fin[x], fin[x]]]]]]] == True
```

```
In[5]:= subclass[intersection[PO, P[cart[fin[x_], fin[x_]]]],
          union[TO, image[inverse[PS], intersection[PO, P[cart[fin[x_], fin[x_]]]]]]] := True
```

Corollary.

```
In[6]:= SubstTest[implies, subclass[u, v], subclass[image[t, u], image[t, v]],
          {t → inverse[S], u -> maximal[S, intersection[PO, P[cart[fin[x], fin[x]]]]], v -> TO}] //
         Reverse

Out[6]= subclass[intersection[PO, P[cart[fin[x], fin[x]]]],
          union[image[inverse[PS], intersection[PO, P[cart[fin[x], fin[x]]]]],
           image[inverse[S], TO]]] == True

In[7]:= (% /. x → x_) /. Equal → SetDelayed
```

Lemma. Every member of the finite set **intersection[PO, P[cartsq[fin[x]]]]** is a subset of a maximal member.

```
In[8]:= SubstTest[subclass, fin[t], image[inverse[S], maximal[S, fin[t]]],
          t → intersection[PO, P[cartsq[fin[x]]]]] // Reverse

Out[8]= subclass[intersection[PO, P[cart[fin[x], fin[x]]]],
          image[inverse[S], intersection[PO, complement[image[inverse[PS],
              intersection[PO, P[cart[fin[x], fin[x]]]]]], P[cart[fin[x], fin[x]]]]]] == True

In[9]:= (% /. x → x_) /. Equal → SetDelayed
```

Lemma. Every member of the set **intersection[PO, P[cartsq[fin[x]]]]** is a subset of a total order.

```
In[10]:= SubstTest[implies, and[subclass[u, v], subclass[v, w]], subclass[u, w],
           {u -> intersection[PO, P[cart[fin[x], fin[x]]]],
            v -> image[inverse[S], maximal[S, intersection[PO, P[cart[fin[x], fin[x]]]]]],
            w → image[inverse[S], TO]}] // Reverse

Out[10]= subclass[intersection[PO, P[cart[fin[x], fin[x]]]], image[inverse[S], TO]] == True

In[11]:= (% /. x → x_) /. Equal → SetDelayed
```

Lemma.

```
In[12]:= SubstTest[member, fin[t], V, t → po[fin[x]]] // Reverse

Out[12]= member[po[fin[x]], V] == True

In[13]:= member[po[fin[x_]], V] := True
```

Theorem.

```
In[14]:= SubstTest[implies, and[member[u, v], subclass[v, w]], member[u, w],
            {u → po[fin[x]], v → intersection[PO, P[cart[fin[y], fin[y]]]],
             w → image[inverse[S], TO]}] /. y → fix[po[fin[x]]] // Reverse

Out[14]= member[po[fin[x]], image[inverse[S], TO]] == True

In[15]:= member[po[fin[x_]], image[inverse[S], TO]] := True
```

Lemma. (Remove wrappers.)

```
In[16]:= SubstTest[implies, equal[x, po[fin[t]]],
           member[x, image[inverse[S], TO]], t → x] // Reverse

Out[16]= or[member[x, image[inverse[S], TO]],
           not[member[x, FINITE]], not[PARTIALORDER[x]]] == True

In[17]:= (% /. x → x_) /. Equal → SetDelayed
```

The variable x can be removed.

Theorem. Every finite partial order is a subset of a total order.

```
In[18]:= Map[equal[V, #] &,
          complement[intersection[FINITE, PO, complement[image[inverse[S], TO]]]]] // Normality]

Out[18]= subclass[intersection[FINITE, PO], image[inverse[S], TO]] == True

       subclass[intersection[FINITE, PO], image[inverse[S], TO]] := True
```

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comments

The axiom of choice implies that any partial order is a subset of a total order. (See the notebook **ims-to.nb** dated 2007 December 30). The proof used a strong form of Zorn's lemma.

```
In[19]:= implies[axch, subclass[PO, image[inverse[S], TO]]]

Out[19]= True
```

For finite sets, the axiom of choice is not needed. The strong form of Zorn's lemma in the finite case is replaced by the statement that any member of a finite set is a subset of a maximal member.